An Alternative Proof of Hadamard's Determinant Theorem

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I. Introduction

An n by n matrix $H = H(h_{ij})$ with all its entries $h_{ij} = +1$ or -1 is called a *Hadamard matrix* of order n if

$$HH^T = nI,$$

where H^T is the transpose of H.

In [4] by showing *Hadamard's inequality*:

$$|\det A|^2 \leq \prod_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^2$$

for a complex matrix $A = A(a_{ij})$ of order n, Hadamard showed that if $|a_{ij}| \leq 1$ for $i, j = 1, 2, \ldots, n$, then

$$|\det A| \leq n^{n/2}.$$

The bound is attained if and only if $AA^* = nI$ with $|a_{ij}| = 1$, for $i, j = 1, 2, \ldots, n$. In particular, when all the entries of A are real, the bound is attained by A if and only if A is a Hadamard matrix. There have since been many different proofs of Hadamard's inequality; see, for example, [1], [2], [3], [5], and [6].

The purpose of this paper is to present a different, simple and elementary proof of Hadamard's determinant theorem by showing a weaker inequality:

$$|\det A|^2 \leq \left(\frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^2\right)^n.$$

Note that Hadamard's inequality is not used here although it implies our inequality by the geometric arithmetic mean inequality.

II. Hadamard's determinant theorem

Let A be a complex matrix of order n and let A^* be the conjugate transpose of A. A is called *unitary* if $A^*A = AA^* = I$.

Theorem. (Hadamard's Determinant Theorem) *For any complex matrix* $H = H(h_{ij})$ *of order* n *with complex entries* $|h_{ij}| \leq 1$ *for* $i, j = 1, 2, \ldots, n$,

$$|\det H| \leq n^{n/2},$$

and the equality holds if and only if $HH^* = nI$ *with* $|h_{ij}| = 1$, *for* $i, j = 1, 2, \ldots, n$.

Proof: From the well-known result of matrix theory, since HH^* is nonnegative definite, there exists a unitary matrix U, such that

$$HH^* = U\text{diag}(\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2)U^* \tag{1}$$

where σ_i^2's are the eigenvalues of HH^*, and the sum of all the eigenvalues of HH^* is equal to the sum of all the diagonal entries of HH^*, we have

$$\sum_{i=1}^{n} \sigma_i^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} |h_{ij}|^2. \tag{2}$$

Since a geometric average is not greater than an arithmetic average, we have

$$\prod_{i=1}^{n} \sigma_i^2 \leq \left(\frac{1}{n} \sum_{i=1}^{n} \sigma_i^2\right)^n \tag{3}$$

and the equality holds if and only if $\sigma_1^2 = \sigma_2^2 = \ldots = \sigma_n^2$.

Thus, by (1), (2) and (3),

$$|H|^2 = \prod_{i=1}^{n} \sigma_i^2 \leq \left(\frac{1}{n} \sum_{i=1}^{n} \sigma_i^2\right)^n = \left(\frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} |h_{ij}|^2\right)^n \leq n^n.$$

So

$$|\det H| \leq n^{n/2}$$

and equality holds if and only if $\sigma_1^2 = \sigma_2^2 = \ldots = \sigma_n^2$ and $|h_{ij}| = 1$, for $i, j = 1, 2, \ldots, n$.

But this condition implies by (2) that $\sigma_1^2 = \sigma_2^2 = \ldots = \sigma_n^2 = n$ (i.e. $HH^* = nI$) and $|h_{ij}| = 1$, for $i, j = 1, 2, \ldots, n$. ∎

In particular, we have

Hadamard's real determinant theorem. *For any real matrix* $H = H(h_{ij})$ *of order* n *with real entries* $|h_{ij}| \leq 1$ *for* $i, j = 1, 2, \ldots, n$,

$$|\det H| \leq n^{n/2},$$

and equality holds if and only if H *is a Hadamard matrix.*

References

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