On Reconstructing Graphs from $n-2$ Cards

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To Ralph G. Stanton, in memoriam.

Abstract

Let G and H be graphs on $n+2$ vertices $\{u_1, u_2, \ldots, u_n, x, y\}$ such that $G - u_i \cong H - u_i$, for $i = 1, 2, \ldots, n$. Recently Ramachandran, Monikandan, and Balakumar have shown in a sequence of two papers that if $n \geq 9$, then $|\varepsilon(H) - \varepsilon(G)| \leq 1$. In this paper we present a simpler proof of their theorem, using a counting lemma.

1 Introduction

Let G and H be graphs on $n+2$ vertices $\{u_1, u_2, \ldots, u_n, x, y\}$ such that $G - u_i \cong H - u_i$, for $i = 1, 2, \ldots, n$. Recently Ramachandran and Monikandan [3] and Monikandan and Balakumar [1] have shown in a sequence of two papers that if $n \geq 9$, then $|\varepsilon(H) - \varepsilon(G)| \leq 1$. Their proof is based on determining the partial structure of the graphs. In this paper we present a simpler proof of their theorem, using a counting lemma that requires only the degrees of the vertices.

Let $d(u, G)$ denote the degree of vertex u in any graph G. The following lemma is from [2].

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Lemma 1.1 *Let $m \geq 0$ and let G and H be as above. Suppose that $\varepsilon(H) = \varepsilon(G) + m$. Then $d(u_i, H) = d(u_i, G) + m$, for $i = 1, 2, \ldots, n$.*

Proof. $d(u_i, H) = \varepsilon(H) - \varepsilon(H - u_i) = \varepsilon(G) + m - \varepsilon(G - u_i) = d(u_i, G) + m$.

Suppose now that $m \geq 2$. Choose a vertex $u_1 \in U$, and let p be an isomorphism mapping $H - u_1$ to $G - u_1$. Then p is a permutation of $V - u_1$. Let $G' = G - u_1$ and $H' = H - u_1$. Consider a cycle $(v_1, v_2, \ldots, v_k)$ of p. Is it possible that all $v_i \in U$? Let $\alpha = d(v_1, H')$. In H' there are α edges incident on v_1, hence in G' there are α edges incident on $p(v_1) = v_2$, so $d(v_2, G') = \alpha$. But $d(v_2, H) = d(v_2, G) + m$, so that $d(v_2, H') \geq \alpha + 1$. Continuing in this way, we find that $d(v_3, H') \geq \alpha + 2$, etc, until we reach $d(v_k, H') \geq \alpha + k - 1$ from which it follows that $d(v_1, H') \geq \alpha + k$, a contradiction. It follows that every cycle of p contains either x or y, so that there are at most two cycles.

Let $H[u, v]$ denote the number of edges of H joining u to v (either 0 or 1). Consider any $u \in U$, where $u \neq u_1$. Since $d(u, H) = d(u, G) + m$, we can write:

(a) $d(u, H') = d(u, G') + m$ and $H[u_1, u] = G[u_1, u]$; or
(b) $d(u, H') = d(u, G') + m - 1$ and $H[u_1, u] = 1 + G[u_1, u]$; or
(c) $d(u, H') = d(u, G') + m + 1$ and $H[u_1, u] + 1 = G[u_1, u]$.

Let there be a of the first type, b of the second type, and c of the third type. Let $h = H[u_1, x] + H[u_1, y]$ and $g = G[u_1, x] + G[u_1, y]$

Lemma 1.2 *Let $m \geq 1$. Then*

$$ma + (m-1)b + (m+1)c = mn - 2m + h - g$$

Proof. We have $a + b + c = n - 1$, since there are $n - 1$ vertices in $U - u_1$. Then $d(u_1, H) - d(u_1, G) = m = b - c + h - g$, so that $c = b - m + h - g$, from which it follows that $a + 2b = n + 1 - h + g$. Hence $ma + (m-1)b + (m+1)c = mn - 2m + h - g$.

If there is a sequence $x, v_1, v_2, \ldots, v_k$ of consecutive vertices in a cycle of p, where each $v_i \in U$, then $d(v_i, H')$ increases by $m - 1, m$, or $m + 1$ for each i. If $k = n - 1$, then starting with $d(v_1, G') =$

$d(x, H')$, we find that $d(v_k, H') = d(x, H') + ma + (m-1)b + (m+1)c = mn - 2m + h - g + d(x, H')$. If $m \geq 2$, this expression is $\geq 2n - 4 + h - g + d(x, H') > n$, if $n \geq 7$, since $h - g \geq -2$. But each $d(v_i, H') \leq n$. Hence we conclude that if $n \geq 7$, there are exactly two sequences of consecutive vertices of U in p, one starting from x, and one starting from y.

Let $x, v_1, v_2, \ldots, v_k$ be one sequence of consecutive vertices in a cycle of p, and let $y, w_1, w_2, \ldots, w_j$ be the other, where each $v_i, w_i \in U$ and $k + j = n - 1$. It is convenient to take $v_0 = x$ and $w_0 = y$. Let there be a_1, b_1, c_1 vertices of type (a),(b),(c), respectively, in $v_1, v_2, \ldots, v_k$, and a_2, b_2, c_2 in $w_1, w_2, \ldots, w_j$.

Lemma 1.3 *Let* $0 \leq i \leq k-1$. *Then*

$$d(v_{i+1}, H') = d(v_i, H') + m + G[u_1, v_{i+1}] - H[u_1, v_{i+1}]$$

Similarly, if $0 \leq i \leq j-1$, *then*

$$d(w_{i+1}, H') = d(w_i, H') + m + G[u_1, w_{i+1}] - H[u_1, w_{i+1}]$$

Proof. Since p maps H' to G' we have $d(v_{i+1}, G') = d(v_i, H')$. We also have $d(v_{i+1}, H) = d(v_{i+1}, G) + m$. But $d(v_{i+1}, H') = d(v_{i+1}, H) - H[u_1, v_{i+1}]$ and $d(v_{i+1}, G') = d(v_{i+1}, G) - G[u_1, v_{i+1}]$. The result follows.

It follows that $d(v_k, H') = d(x, H') + ma_1 + (m-1)b_1 + (m+1)c_1$. Similarly $d(w_j, H') = d(y, H') + ma_2 + (m-1)b_2 + (m+1)c_2$. Adding these gives $d(v_k, H') + d(w_j, H') = ma + (m-1)b + (m+1)c + d(x, H') + d(y, H')$, which reduces to $mn - 2m + h - g + d(x, H') + d(y, H')$.

Lemma 1.4 *If* $n \geq 9$ *then* $|\varepsilon(H) - \varepsilon(G)| \leq 2$.

Proof. We have $d(v_k, H'), d(w_j, H') \leq n$. If $m > 2$, the above formula gives $mn - 2m + h - g + d(x, H') + d(y, H') \leq 2n$, or $n \leq 2m/(m-2) - \{h - g + d(x, H') + d(y, H')\}/(m-2)$. Since $h - g \geq -2$ and $m \geq 3$, this gives $n \leq 8$, a contradiction.

At this point we take $m = 2$ and $n \geq 9$. The above formula becomes

$$2n - 4 + h - g + d(x, H') + d(y, H') = d(v_k, H') + d(w_j, H') \quad (*)$$

We now abbreviate the notation somewhat, in order to streamline the proof of the main theorem. We have $H' = H - u_1$ and $G' = G - u_1$. Here we have chosen u_1 as a vertex in U with the *largest* degree in H. Since we will be working mostly with the graphs H and H', we write $dx' = d(x, H')$ and $dy' = d(y, H')$. Similarly for dv_k' and dw_j'. Similarly, we write dx for $d(x, H)$, du_1 for $d(u_1, H)$, etc. We recall that $h = H[u_1, x] + H[u_1, y]$ and $g = G[u_1, x] + G[u_1, y]$. We write $u \longrightarrow v$ to indicate that u is adjacent to v.

Theorem 1.5 *Let G and H be graphs on $n+2$ vertices, where $n \geq 9$, such that $H - u_i \cong G - u_i$ for $i = 1, \ldots, n$. Then $|\varepsilon(H) - \varepsilon(G)| \leq 1$.*

The proof requires only the degrees of the vertices. We assume that $\varepsilon(H) = \varepsilon(G) + 2$ and obtain a contradiction. We use the inequality $(*)$ to limit the values of the parameters dx', dy', dv_k', dw_j', and compare the smallest degrees of $H - v_1$, $H - w_1$, $G - v_1$, and $G - w_1$.

Case 1. $dx' = dy' = 0$.

This implies that $dv_k', dw_j' \leq n - 2$. It then follows from $(*)$ that $2n - 4 + h - g \leq 2n - 4$, so that $h \leq g$. We observe first that if $h = 0$, then H and H'' both have at least two vertices of degree 0 (namely x and y), whereas G'' has at most one vertex of degree 0 (namely v_1 or w_1). Therefore $H'' \ncong G''$. Hence, we can assume that $h \geq 1$.

If $h = 1$, then $H - v_1$ and $H - w_1$ both have exactly one vertex of degree 0. Therefore $G - v_1$ and $G - w_1$ must each have one vertex of degree 0. It follows that $u_1 \not\longrightarrow v_1, w_1$ in G. But then $G - v_2$ has two vertices of degree 0, whereas $H - v_2$ has just one. Hence, we can assume that $h = 2$.

We now find that $H - v_1$ and $H - w_1$ both have exactly two vertices of degree 1. Therefore $G - v_1$ and $G - w_1$ also have exactly two vertices of degree 1. Therefore $u_1 \longrightarrow v_1, w_1$ in G, so that $d(v_1, G) = d(w_1, G) = 1$, and one of v_2, w_2 has degree 1 in G. Consequently $dv_1 = dw_1 = 3$, so that $dv_1', dw_1' \geq 2$. But this implies that $d(v_2, G'), d(w_2, G') \geq 2$, a contradiction.

Case 2. $dx' \neq 0$ and $dy' = 0$.

This implies that $dv'_k, dw'_j \leq n-1$. It then follows from (*) that $2n - 4 + h - g + dx' \leq 2n - 2$. If $u_1 \not\longrightarrow y$ in H, then $H - w_1$ will have one vertex of degree 0, whereas $G - w_1$ will have no vertices of degree 0. Hence we must have $u_1 \longrightarrow y$ in H, so that $h \geq 1$.

We next observe that in $H - w_1$, vertex y has degree 1. Therefore $G - w_1$ must have a vertex of degree 1, which can only be v_1. It follows that $u_1 \not\longrightarrow v_1$ in G, and that $1 = d(v_1, G') = dx'$. Therefore v_1 is adjacent to exactly one vertex z in G', and in H', x is adjacent to only $p^{-1}(z)$. Then $G - z$ has a vertex of degree 0, but $H - z$ does not, a contradiction.

Case 3. $dx' = 0$ and $dy' \neq 0$.

This is identical to Case 2, interchanging x and y, and j and k.

Case 4. $dx' \neq 0$ and $dy' \neq 0$.

We have $dx', dy' \geq 1$. Let $\delta = dx' + dy' - 2$. Then $\delta \geq 0$. Without loss of generality, we take $n \geq dv'_k \geq dw'_j$.

4.1 $dv'_k = n$. Then $du_1 \geq dv_k \geq n$. If $u_1 \longrightarrow v_k$ in H, then $du_1 = dv_k = n+1$, so that $u_1 \longrightarrow x, y$ in H, which implies that $h = 2$. If If $u_1 \not\longrightarrow v_k$ in H, then since $du_1 \geq n$, we again have $u_1 \longrightarrow x, y$ in H and $h = 2$. By (*), $2n - 4 + 2 - g + 2 + \delta = n + dw'_j$, which reduces to $g = n - dw'_j + \delta$. Now if $dw'_j = n$, then $w_j \longrightarrow x, y$ in H, so that $\delta \geq 2$, which implies that $g = \delta = 2$. If $dw'_j = n-1$, then w_j is adjacent to at least one of x, y in H, so that $\delta \geq 1$, which gives $g = 2$ and $\delta = 1$. If $dw'_j = n-2$, this gives $g = 2$ and $\delta = 0$. So g always equals 2, and dx', dy' are forced. They are either $(2,2), (2,1)$ or $(1,1)$ according as dw'_j is $n, n-1$, or $n-2$. Note that $dw'_j \not\leq n-3$, since this would give $g \geq 3$.

We now find that the two smallest degrees of $H - v_1$ and $H - w_1$ are $dx' + 1, dy' + 1$. These must also be the smallest degrees of $G - v_1$ and $G - w_1$. It follows that $u_1 \longrightarrow v_1, w_1$ in G, so that $d(v_1, G) = dx' + 1$ and $d(w_1, G) = dy' + 1$. Therefore $dv_1 = dx' + 3$ and $dw_1 = dy' + 3$, so that $dv'_1 \geq dx' + 2$ and $dw'_j \geq dy' + 2$. But then the smallest degrees of $G - v_1$ and $G - w_1$ cannot be $dx' + 1, dy' + 1$, a contradiction.

4.2 $dv'_k = n - 1$. By (*), we have $2n - 4 + h - g + 2 + \delta = n - 1 + dw'_j$, which reduces to $g = n - 1 + h + \delta - dw'_j$. If $u_1 \not\longrightarrow v_k, w_j$ in H, then since $du_1 \geq n-1$, we must have $u_1 \longrightarrow x, y$ in H, so that $h = 2$. If u_1 is adjacent to v_k but not to w_j, then $du_1 \geq n$, so that we again have $h = 2$. Then $g = n + 1 + \delta - dw'_j$. We must have $dw'_j = n - 1$, $g = 2$, and $\delta = 0$. Therefore $dx' = dy' = 1$.

If u_1 is adjacent to w_j but not to v_k, then $h \geq 1$. If $u_1 \longrightarrow v_k, w_j$ in H, then $du_1 \geq n$ and so $h \geq 1$. Then $g \geq n + \delta - dw'_j$. We either have $dw'_j = n - 1$ (which forces $h = 2$), $g = 2$, and $\delta = 0$ as in the previous paragraph, or else $dw'_j = n - 2$, $h = 1$, $g = 2$, and $\delta = 0$. Therefore $dx' = dy' = 1$.

In the situation when $h = 2$, we find that $H - v_1$ and $H - w_1$ both have two vertices of degree 2 as the smallest degrees, namely x and y. Therefore $G - v_1$ and $G - w_1$ must also have two vertices of degree two. Hence $u_1 \longrightarrow v_1, w_1$ in G, so that $d(v_1, G) = d(w_1, G) = 2$, from which $dv_1 = dw_1 = 4$ and $dv'_1, dw'_1 \geq 3$. But then $G - v_1$ and $G - w_1$ do not have two vertices of degree two, a contradiction.

In the situation when $h = 1$, $H - v_1$ and $H - w_1$ will have vertices x and y with degrees 1 and 2. Therefore $G - v_1$ and $G - w_1$ also have smallest degrees 1 and 2. It follows that $u_1 \not\longrightarrow v_1, w_1$ in G. Therefore $dv_1 = dw_1 = 3$, and $u_1 \longrightarrow v_1, w_1$ in H. Then at least one of $H - v_1$ and $H - w_1$ will have vertices v_k, w_j with largest degrees $n-1, n-2$. But since $g = 2$, the two largest degrees of $G - v_1$ and $G - w_1$ will both be $n - 1$, a contradiction.

4.3 $dv'_k = n-2$. By $(*)$, we have $2n-4+h-g+2+\delta \leq 2n-4$, which gives $g \geq 2 + h + \delta$, which requires $g = 2$, $h = 0$, and $\delta = 0$. But $du_1 \geq n - 2$, so that u_1 must be adjacent to at least one of v_k, w_j. This forces $du_1 \geq n$, so that $h \geq 1$, a contradiction.

This completes the proof of the theorem.

References

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