Enumerating Perfect Matchings in Inductively-Defined Families of Graphs

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Abstract

In this paper we look at families $\{G_n\}$ of graphs (for $n > 0$) for which the number of perfect matchings of G_n is the nth term in a sequence of generalized Fibonacci numbers. A one-factor of a graph is a set of edges forming a spanning one-regular subgraph (a perfect matching). The generalized Fibonacci numbers are the integers produced by a two-term homogeneous linear recurrence from given initial values. We explore the construction of such families of graphs, using as our motivation the *Ladder Graph* L_n; it is well-known that L_n has exactly F_{n+1} perfect matchings, where F_n is the traditional Fibonacci sequence, defined by $F_1 = F_2 = 1$, and $F_{n+1} = F_n + F_{n-1}$.

1 Introduction

Almost every student of combinatorics or graph theory runs across the pleasing result that the Fibonacci numbers count the number of perfect matchings of the ladder graph $L_n = P_n \times K_2$. More precisely, let $m(G)$ be the number of perfect matchings of G; then $m(L_n) = F_{n+1}$. This result is not easy to generalize, and we are not aware of any other examples in the literature. As a result, we will look at constructing families of graphs whose matching numbers are given by a simple recurrence. We wish to restrict ourselves to connected simple graphs for the moment. For definitions

JCMCC **80** (2012), pp. 151-156

and theorems involving graph theory, the reader is referred to standard texts on the subject, such as [2]; for definitions and theorems involving recurrences and combinatorics, the reader is referred to standard texts on combinatorics, such as [3].

The graph G_n illustrated in the following figure consists of n copies of C_4, each joined to the next by a bridge. The bridges cannot appear in any perfect matching of the graph, so their only use is to make the graphs G_n connected. It is not hard to see that G_n has 2^n perfect matchings.

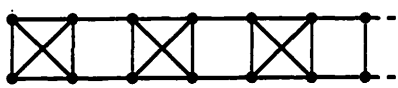
A similar construction will work for any sequence $a_n = r^n$ for any positive integer r, and this sequence of numbers satisfies the one-term linear recurrence $a_n = ra_{n-1}$. We simply replace the C_4 with a connected graph G containing exactly r perfect one-factors, and the same process works.

2 Connecting with pairs of edges

We may view the ladder graph as consisting of n copies of $G = K_2$, each joined to the next by an edge-cutset of two edges. This raises the natural question: what happens if we generalize this construction to arbitrary G in place of K_2? It is important that the graph G be connected and have one or more perfect matchings. We also assume that the two edges that join a copy of G to the next (or previous) copy of G are incident to adjacent vertices of G.

If G is C_4, we get a subsequence of the Fibonacci numbers; n copies of C_4 joined by a pair of edges gives us L_{2n}, the ladder with $2n$ rungs, and the sequence we get is $F_3, F_5, F_7, \ldots$. We explore two more cases.

If G is K_4, we get a family G_n of graphs that look like



Careful thought will reveal that the matching number of the graph satisfies the recurrence

$$m(G_n) = 3m(G_{n-1}) + m(G_{n-2}) + m(G_{n-3}) \ldots .$$

This comes from counting the matchings according to where the first pair of "connector edges" is used. This sort of recurrence is "open-ended" in the sense that the number of terms depends upon n, but many such recurrences have shorter versions. For example, the recurrence

$$T_n = \sum_{i<n} T_i, \qquad T_1 = 1$$

gives us the powers of two ($T_i = 2^{i-1}$ for $i \geq 1$), just as does the recurrence

$$T_n = 2T_{n-1}, \qquad T_1 = 1.$$

If we ignore the first pair of "connector edges" from the leftmost copy of K_4 to the next, the total number of matchings that will result is $3m(G_{n-1})$, as we choose one of the three matchings of the leftmost copy of K_4 and any of the $m(G_{n-1})$ matchings from the rest of the graph. If either of those two connector edges is used, they both must be used. There is only one choice of edge from the leftmost K_4, and we decide to ignore the second set of connector edges for the moment; this gives us $m(G_{n-2})$ ways to complete the matching. If the second set of connector edges is used but the third set is not, then we have $m(G_{n-3})$ matchings; and so on. Thus the number of matchings of G_n is given by the recurrence, as asserted earlier.

This, of course, is not a two-term recurrence, although the sequence $\{a_n = m(G_n)\}$ does satisfy the two-term recurrence $a_n = 4a_{n-1} - 2a_{n-2}$, with $a_1 = 3$ and $a_2 = 10$. The resulting sequence, known to the Online Encyclopedia of Integer Sequences as A007052 ([1]), also counts "the number of order-consecutive partitions of n." That the given two-term recurrence yields the same sequence of numbers as the original generalizes to a result that is probably well-known.

Theorem 1 *The recurrence relation $a_n = ba_{n-1} + a_{n-2} + \ldots$ describes the same sequence as $\alpha_n = (b+1)\alpha_{n-1} - (b-1)\alpha_{n-2}$, given the initial values $a_1 = \alpha_1 = 1$ and $\alpha_2 = b$.*

Proof Clearly $a_2 = b$ from the open-ended recurrence relation, and $\alpha_2 = b$ is given. It is easy to confirm that both recurrences give the same value for $a_2 = \alpha_2 = b^2 + 1$. We may proceed by induction; suppose $a_n = \alpha_n$ for $n \leq N$. Then we write

$$\alpha_{N+1} = (b+1)\alpha_N - (b-1)\alpha_{N-1} = ba_N + a_N - (b-1)a_{N-1}.$$

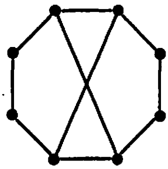
Now we replace the last a_N by the open-ended recurrence for a_N to get

$$\alpha_{N+1} = ba_N + ba_{N-1} + a_{N-2} + \cdots - (b-1)a_{N-1}.$$

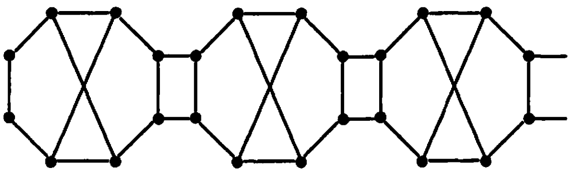
Collecting the a_{N-1} terms, we get $\alpha_{N+1} = ba_N + a_{N-1} + a_{N-2} + \ldots$, establishing that $\alpha_{N+1} = a_{N+1}$. □

3 Eight-cycle with chords

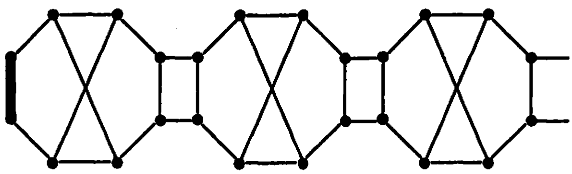
Consider the building block G:



and form a sequence with first term G and general term

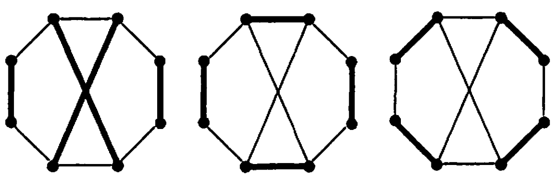


As before, G_n contains n copies of G. We think of G_{n+1} as being formed from G_n by attaching a copy of G to the left of G_n using two connector edges that join up to the ends of the "anchor edge," the edge accented in

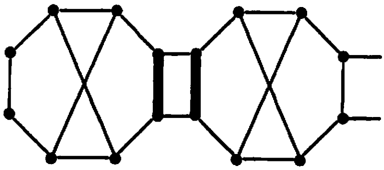


We write A_n for the set of all one-factors of G_n, and B_n for the set of "anchor factors," one-factors that contain the anchor edge. We shall define $a_n = |A_n|$ and $b_n = |B_n|$.

From the following diagram we see that $a_1 = 3$ and $b_1 = 2$:



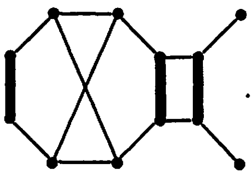
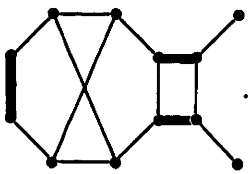
Now we count the one-factors of G_{n+1}. Factors that do not contain the connectors are the union of a factor of G with a factor of G_n, with no restrictions, so there are $3 \times a_n$ such factors. The factors that contain both connectors are in one-to-one correspondence with those that contain the two highlighted edges in



So we can multiply the number of anchor factors of G_n by the number of factors of G that contain the right-hand edge; but these are immediately seen to be the anchor factors of G, so there are $2 \times b_n$ of these factors, and

$$a_{n+1} = 3a_n + 2b_n.$$

In order to evaluate b_n in terms of earlier members of the sequences, observe that an anchor factor of G_{n+1} must look like one of the following:

 ... or  ...;

in each case there are 2 ways to complete the intersection of the factor with the left-hand G and b_n ways to complete the remainder. So

$$b_{n+1} = 4b_n.$$

Theorem 2 $a_n = 4^n - 3^{n-1}$.

Proof $b_1 = 2$ and $b_{n+1} = 4b_n$ together give $b_n = 2^{2n-1}$. So

$$a_{n+1} = 3a_n + 2b_n = 3a_n + 2^{2n} = 3a_n + 4^n.$$

Now apply induction. Case $n = 1$ is true. Assume case $n = r$. Then

$$\begin{aligned} a_{r+1} &= 3a_r + 4^r \\ &= 3(4^r - 3^{r-1}) + 4^r \\ &= 3(4^r) + 4^r - 3(3^{r-1}) \\ &= 4(4^r) - 3(3^{r-1}) \\ &= 4^{r+1} - 3^r, \end{aligned}$$

as required. □

The sequence for a_n, $3, 13, 55, 229, 943, \ldots$, is sequence A093834 in [1].

4 Conclusion

Many other small graphs can serve for G in these examples, and a variety of generalized Fibonacci sequences may be obtained. These problems are part of a more general family of matching-enumeration problems known as *domino tilings*, and these have a variety of applications in chemistry and elsewhere. Questions include whether any sequence of generalized Fibonacci numbers may be realized as the matching numbers of a family of inductively-defined graphs, and whether sequences arising from other recurrences (as for instance three-term recurrences) may be so realized.

References

[1] The Online Encyclopedia of Integer Sequences, `http://www.OEIS.org`

[2] W.D. Wallis, *A Beginner's Guide to Graph Theory*, 2nd Edition, Birkhäuser, Boston, MA, 2007.

[3] W.D. Wallis and J.C. George, *Introduction to Combinatorics*, CRC Press, New York, NY, 2011.