ORIENTED GRAPH SATURATION

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ABSTRACT. For graphs G and H, H is said to be G-saturated if it does not contain a subgraph isomorphic to G, but for any edge $e \in H^c$, the complement of $H, H + e$ contains a subgraph isomorphic to G. The minimum number of edges in a G-saturated graph on n vertices is denoted $sat(n, G)$. While digraph saturation has been considered with the allowance of multiple arcs and 2-cycles, we address the restriction to oriented graphs. First we prove that for any oriented graph D, there exist D-saturated oriented graphs, and hence show that $sat(n, D)$, the minimum number of arcs in a D-saturated oriented graph on n vertices, is well defined for sufficiently large n. Additionally, we determine $sat(n, D)$ for some oriented graphs D, and examine some issues unique to oriented graphs.

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1. INTRODUCTION

Let $G = (V, E)$ be a simple graph with vertex set V and edge set E, having order $n(G) = |V(G)|$ and size $e(G) = |E(G)|$. If $D = (V, A)$ is an orientation of G then its vertex set is the same as that of G and its arc set A is a collection of orderings on each element in E. We say D is an orientation of G, and $G = u(D)$ is the underlying graph of D. An oriented graph is an orientation of some simple graph. For any graph $H = (V, E)$ (digraph $F = (V, A)$), a subgraph $G = (V_1, E_1)$, denoted $G \subseteq H$, (subdigraph $D = (V_1, A_1)$) is a graph (digraph) with V_1 a subset of V and E_1 a subset of E restricted to the edges among V_1 (A_1 a similar subset of A). We call the subgraph of H that is isomorphic to G an *embedding of G in H*. Note that a subgraph (subdigraph) need not be induced. If $e \in H^c$ we refer to $H \cup \{e\}$ as $H + e$. A *source* in the digraph D is a vertex with no in-neighbors, and a *sink* in D is a vertex with no out-neighbors.

For two graphs G and H, the graph H is said to be G-saturated if there is no subgraph of H isomorphic to G, but for any edge $e \in H^c$, the graph $H+e$ contains a subgraph isomorphic to G. We define $sat(n, G)$ to be the smallest number of edges in a G-saturated graph on n vertices, and $ex(n, G)$ to be the greatest number of

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edges in a G-saturated graph on n vertices. For any $n \geq n(G)$ there is a G-saturated graph H on n vertices, so both $sat(n, G)$ and $ex(n, G)$ are well-defined.

While a number of results have been obtained regarding simple graph saturation, there has been very little work on oriented graph saturation. In [13, 14], Tuza proves the following result:

Theorem 1.1. *For every digraph F there exists a constant $c = c(F)$ such that, for every positive integer n, there exists an F-saturated digraph with n vertices and cn arcs.*

Note that Theorem 1.1 references the class of all digraphs, and does not consider F-saturated oriented graphs. Similar classes of problems are considered in [2] and [4]. In [6] the authors demonstrate that the minimum number of arcs in an oriented graph of order n and diameter 2 is bounded between $(1 - o(1))n \log n$ and $n \log n - \frac{3}{2}n$, thereby establishing a lower bound on the number of arcs in an oriented C_3-saturated oriented graph.

In [8, 9] Pikhurko examines Theorem 1.1 as applied to oriented graphs, in particular those without cycles, and proves a general asymptotic result. More recently, the authors of [15] prove a result regarding the greatest number of arcs in an oriented graph not containing a directed path of specified length. We extend this notion to show the existence of F-saturated oriented graphs for all oriented graphs F.

For an oriented graph D, let xy denote the arc with initial vertex x and terminal vertex y. We say x is *adjacent to* y and y is *adjacent from* x. If neither xy nor yx are arcs in D we say that x and y are *non-adjacent.* For a vertex $v \in D$ the *out-degree* $d^+(v)$ is the number of vertices adjacent from v, the *in-degree* $d^-(v)$ is the number of vertices adjacent to v, and the *underlying degree* is the sum of $d^+(v)$ and $d^-(v)$. Arc-extremal oriented graphs with forbidden subdigraphs are a natural extension of edge-extremal graphs with forbidden subgraphs. For oriented graphs F and D we say that F is *D-saturated* if D is not a subdigraph of F but for any non-adjacent vertices $x, y \in F$ $D \subseteq F + xy, F + yx$. Determining the existence of D-saturated oriented graphs is not as trivial as in the undirected case. To see this, we need only consider D containing a directed cycle. If F is an acyclic oriented graph with non-adjacent vertices u, v then an arc can be added between them, to produce an acyclic superdigraph.

Given a simple graph G and integer $n \geq n(G)$, the value $ex(n, G)$ has traditionally been defined to be the maximum number of edges in a G-free simple graph on n vertices. Considering the transitive tournament on n vertices, a strict extension of this definition to arcs in oriented simple graphs would mean that $ex(n, D) = \binom{n}{2}$ for any oriented graph with directed cycles. This clearly does not reflect the spirit of the study of extremal graphs and oriented graphs. In order to extend the definition to something more meaningful, we denote by $ex(n, D)$ the maximum number of arcs in a D-saturated oriented simple graph on n vertices.

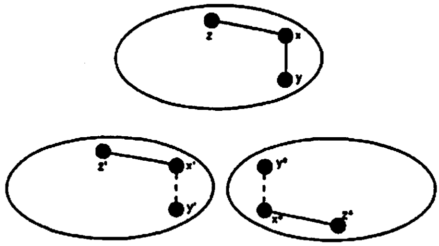

FIGURE 1. G with edges xy and xz, G' with edge $x'z'$ and non-edge $x'y'$, and G^* with edge x^*z^* and non-edge x^*y^*

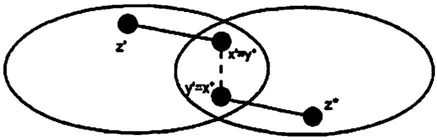

FIGURE 2. The graph $H_{G,a}$

2. EXISTENCE OF SATURATED ORIENTED GRAPHS

Consider an undirected graph G. For $x, y \in V(G)$ we call $a = xy \notin E(G)$ a *non-edge*. Define the graph $H_{G,a}$ in the following way:

Let G' and G^* be two copies of the graph $G - xy$ and, for every vertex $v \in V(G)$, label its associated vertices $v' \in G', v^* \in G^*$. Similarly, for every edge $\alpha \in E(G) - \{a\}$ label the associated edges $\alpha' \in G', \alpha^* \in G^*$, (see Figure 1).

Let $H_{G,a}$ be the graph obtained by identifying x' with y^* and x^* with y' (see Figure 2). We will show that the graph $H_{G,a}$ does not contain a subgraph isomorphic to G via induction on the order of G. If D is an oriented graph with arc a then define $H_{D,a}$ analogously.

Lemma 2.1. *If v is a cut vertex of a graph G that is not an endpoint of the edge a, a graph F is a connected subgraph of $H_{G,a}$ of the same order as G, and neither v' nor v^* are in F, then there is another vertex $w \in G$ that is not an endpoint of the edge a such that neither w' nor w^* are in F and w is not a cut vertex of G.*

Proof. Let A, B be components of $G - v$ such that $a \in A$. The component B contains at least one vertex w that is not a cut vertex of G. Each path from an endpoint of the edge a to w in G contains v, so if the embedding of F contains w' or w^* then it must also contain either v' or v^*. If neither v' nor v^* are in the embedding of F, then neither are w' nor w^*, (see Figure 3). □

Theorem 2.2. *If G is a connected graph of order at least two, then for any $a \in E(G)$ there is no subgraph in $H_{G,a}$ isomorphic to G.*

Proof. We use induction on the order of the graph G. Note that if G is 3-connected then the result is immediate, so we may assume that G is not 3-connected. If

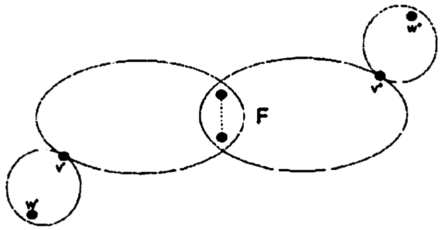

FIGURE 3. Cut vertices v', v^* and non-cut vertices w', w^* in $H_{G,a} - F$

$G \cong K_2$ then $H_{G,a}$ is a pair of isolated vertices. Since the resulting graph is edgeless, the theorem is true in the case of $n(G) = 2$. Now assume that the claim is true for every connected graph with order strictly less than k. That is, if F is a connected graph with order strictly less than k then there is no edge $f \in E(F)$ such that $F \subseteq H_{F,f}$.

Let G be a connected graph of order k with at least one edge $a = xy$ such that $G \subseteq H_{G,a}$. Note that $e(G - \{a\}) = e(G) - 1$, and $e(H_{G,a}) = 2e(G) - 2$. If every edge in $e(G) - \{a\}$ appears at most once in the embedding, then we have that $e(G) \leq e(G) - 1$. It follows that there must be an edge $\alpha \in E(G - a)$ such that both α' and α^* are in this embedding of G. We now show that there is a vertex $w \in G - \{x, y\}$ such that neither w' nor w^* is in the embedding of G.

Since $\alpha \neq a$ we must consider the case where the edges α and a are disjoint and the case in which they share an endpoint.

<u>Case 1</u>: Say that the edges α and a are disjoint, that is $\alpha = v_1 v_2$ with $v_1, v_2 \notin \{x, y\}$, and hence $n(G) \geq 4$. Since both α' and α^* are in this embedding of G, it follows that v_1', v_2', v_1^*, v_2^* are as well. The embedding of G contains $n(G) - 4$ other vertices, so at most $n(G) - 2$ remaining vertices of G appear at least once in this embedding. So, there are at least two vertices $w, z \in V(G)$ such that none of $\{w', z', w^*, z^*\}$ are in this embedding of G. Since G is connected and $\{x', y'\} = \{x^*, y^*\}$ is a cut set of $H_{G,a}$, at least one of $\{x', y'\}$ must be in this embedding of G. So, at least one of $\{w, z\}$, say w, is neither x nor y. So there is a vertex $w \in V(G) - \{x, y\}$ such that neither w' nor w^* is in the embedding of G.

<u>Case 2</u>: Say that α and a share an endpoint. Assume, without loss of generality, that v_1 is adjacent to x. Since both α' and α^* are in the embedding of G, the embedding of G includes both x' and $x^* = y'$. Since both v_2' and v_2^* are in the embedding of G then by an argument analogous to the one in the previous case there is a vertex $w \in V(G)$ such that neither w' nor w^* are in the embedding of G. The vertex w is neither x nor y since both x and y are in the embedding of G.

Note that the edge α cannot be incident to both x and y in $G - \{a\}$ since $xy = a$ is an edge in G.

So, in both cases, there exists a vertex $w \in V(G) - \{x, y\}$ such that neither w' nor w^* is in the embedding of G. By Lemma (2.1) we can assume that w is not a cut vertex of G. So, $G - w$ is connected and contains the edge a, and the embedding of G is also contained in $H_{G-w,a}$. The graph $G - w \subseteq G \subseteq H_{G-w,a}$

and $n(G-w) < k$, which contradicts our inductive assumption. Therefore, for any connected graph G and edge $a \in E(G)$ there is no subgraph of $H_{G,a}$ isomorphic to G. □

Let D be an oriented graph. We want to show that there is some integer N_D such that for every $n \geq N_D$ there is an oriented graph F on n vertices that is D-saturated.

Fact 2.3. *If there is an oriented graph H on n vertices with non-adjacent vertices x and y without a subdigraph isomorphic to D, but D is a subdigraph of both $H+xy$ and $H+yx$, then every tournament containing H also contains D. So, there is an oriented supergraph of H on the vertices of H that is D-saturated.*

Theorem 2.4. *For every oriented graph D there is some integer N_D such that for every $n \geq N_D$ there is an oriented graph H on n vertices not containing a subdigraph isomorphic to D such that for any pair of non-adjacent vertices $x, y \in V(H)$ both $H+xy$ and $H+yx$ contain subdigraphs isomorphic to D.*

Proof. First consider the case where $u(D)$ is connected. Let $N_D = 2n(D) - 2$, say $a \in A(D)$, and let $H_{D,a}$ be the oriented graph as defined above. For $n \geq 2n(D)-2$, define H to be the graph $H_{D,a} \cup \bar{K}_{n-N_D}$. By Theorem 2.2, $u(D)$ is not a subgraph of $u(H_{D,a})$, and hence D is not a subdigraph of H. However, the addition of either the arc xy or the arc yx creates a copy of D. By Fact 2.3, there is a D-saturated oriented graph on the n vertices of H.

If $u(D)$ is not connected then let J be the component of D with greatest order. Let a be an arc in J and let $H = H_{J,a} \cup (D - J)$. Since $H_{J,a}$ does not contain a copy of J, the digraph H does not contain a copy of D. As above, since the addition of either the arc xy or the arc yx creates a copy of D, by Fact 2.3 there is a D saturated oriented graph on the n vertices of H. □

Therefore, given any oriented graph D and integer $n \geq 2n(D) - 2$, the values $sat(n, D), ex(n, D)$ are well-defined. See Section 5 for further discussion of the values of n for which sat is defined.

3. Results

For each integer $m \geq 1$ define $\vec{P}_m$ to be the directed path on m vertices.

Proposition 3.1. $sat(n, \vec{P}_3) = n - 1$

Proof. The lower bound is achieved by orienting all edges of $K_{1,n-1}$ into the center vertex. In any oriented graph D containing fewer than $n-1$ arcs, the underlying graph is not connected. It is easy to see that at least two components must be oriented trees, say A_1 and A_2, which each contain a source and a sink. Any arc from a source in A_1 to a sink in A_2 would show that the oriented graph is not $\vec{P}_3$-saturated.

□

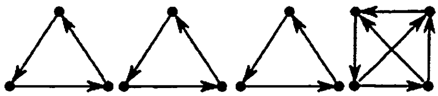

FIGURE 4. A $\vec{P_5}$-saturated oriented graph

Proposition 3.2. (1)

$$sat(n, \vec{P_4}) = \begin{cases} n & \textit{if } 3|n \\ n-1 & \textit{otherwise} \end{cases}$$

(2)

$$sat(n, \vec{P_5}) \leq \begin{cases} n & \textit{if } 3|n \\ n+2 & \textit{if } n \equiv 1 \pmod 3 \\ n+4 & \textit{if } n \equiv 2 \pmod 3 \end{cases}$$

(3)

$$sat(n, \vec{P_6}) \leq \begin{cases} n & \textit{if } 3|n \\ n+2 & \textit{if } n \equiv 1 \pmod 3 \\ n+5 & \textit{if } n \equiv 2 \pmod 3 \end{cases}$$

Proof. To see the upper bounds, consider the oriented graph composed of disjoint directed 3-cycles and possibly the following, depending on modularity:

(1) An isolated vertex or $\vec{P_2}$ for $sat(n, \vec{P_4})$
(2) One or two copies of K_4, each strongly oriented, for $sat(n, \vec{P_5})$, (see Figure 4)
(3) A strongly oriented K_4 or K_5 for $sat(n, \vec{P_6})$.

For the sharpness of $sat(n, \vec{P_4})$ we consider two cases. If $3 \nmid n$ and D is a $\vec{P_4}$-free oriented graph on n vertices with fewer than $n-1$ arcs, then as noted in the proof of Theorem 3.1, D has at least 2 components that are oriented trees. Each contains a source and a sink, and an additional arc from the source of one to the sink of another will not create a $\vec{P_4}$. If $3|n$ and D is a $\vec{P_4}$-free oriented graph that has fewer than n arcs, then the components of D are either directed cycles of length three or oriented trees. Since D contains at most $n-1$ arcs, there must be at least one component that is an oriented tree, and if only one such component exists then it cannot be an isolate since $3 \mid n$. If every source is adjacent to every sink, then the component is a star with all arcs oriented to the center or all arcs oriented from the center, which is not $\vec{P_4}$-saturated. Otherwise, an arc added from a source to a sink does not result in a $\vec{P_4}$. Therefore, D is not $\vec{P_4}$-saturated, and $sat(n, \vec{P_4})$ is as stated.

□

We now determine the exact value for the saturation number of an oriented star. Let $K_{1,(a,b)}$ be the directed star with in-degree b and out-degree a.

Theorem 3.3. *If $a > b$, then for n sufficiently large, $sat(n, K_{1,(a,b)}) = \binom{a-b-1}{2} + (a-1)(n-a+b+1)$. If $a < b$, then for n sufficiently large, $sat(n, K_{1,(a,b)}) = \binom{b-a-1}{2} + (b-1)(n-b+a+1)$.*

Proof. Refer to the directed star $K_{1,(a,b)}$ as K, and assume that $a > b$. Say D is an oriented graph on n vertices that is K-saturated. For any pair u, v of non-adjacent vertices we require that $D+uv$ and $D+vu$ both contain a copy of K. The only 2 of the 4 possible combinations that don't result in D already containing K are when either both u and v are centers of a $K_{1,(a-1,b)}$ or both are centers of a $K_{1,(a,b-1)}$. Therefore, the vertices of D can be partitioned into sets T and B, where T induces a tournament and the vertices of B have minimum out-degree at least $a-1$ and minimum underlying degree at least $a+b-1$ in D. Let B' be the oriented graph induced by B. Let $k = |T|$. If $0 \le k \le (a-b-1)$ the minimum out-degree of a vertex in B' is $(a-1-k)$. If $(a-b) \le k \le (a-1)$ then the minimum underlying degree of a vertex in B' is at least $(a+b-1-k)$ in order that every vertex in B has underlying degree at least $(a+b-1)$. If k is greater than a then D contains at least the tournament T of order k and arcs between all vertices of T and all remaining $(n-k)$ vertices of D. Therefore, the fewest possible arcs in a K-saturated oriented graph with underlying clique size k is bounded below by the function

$$e(k) = \begin{cases} \binom{k}{2} + (n-k)(a-1) & \text{if } k \le (a-b-1) \\ \frac{1}{2}\left[k(n-1) + (n-k)(a+b-1)\right] & \text{if } (a-b) \le k \le a) \\ \binom{k}{2} + k(n-k) & \text{if } k > a \end{cases}$$

For n sufficiently large, this function has as its global minimum $e(a-b-1)$. Let D' be the oriented graph consisting of a transitive tournament T on $(a-b-1)$ vertices, an oriented graph B on $(n-k)$ vertices in which every vertex has both out-degree and in-degree equal to b, and an arc from every vertex in B to every vertex in T. The oriented graph D' is K-saturated and contains precisely $e(a-b-1)$ arcs. Therefore, $sat(n, K_{1,(a,b)}) = e(a-b-1) = \binom{a-b-1}{2} + (a-1)(n-a+b+1)$.

Similarly, if $b > a$ then $sat(n, K_{1,(a,b)}) = \binom{b-a-1}{2} + (b-1)(n-b+a+1)$. □

Note that Theorem 3.3 only holds for orientations of the star in which the out-degree of the center vertex differs from its in-degree. If they are the same then we obtain a different result.

Theorem 3.4. *Let $a > 0$ be an integer. Then, for n sufficiently large $sat(n, K_{1,(a,a)}) = a(n-1)$.*

Proof. Refer to the directed star $K_{1,(a,a)}$ as K. Consider an $(a-1, a-1)$-regular oriented graph D on $n-1$ vertices, joined from a single vertex v, (see Figure 5). Every vertex other than v has degree $(a-1, a)$, and any new arc must be from one of these to another. So, D is $K_{1,(a,a)}$-saturated. Now let F be a $K_{1,(a,a)}$-saturated oriented graph on n vertices. Every vertex in F has underlying degree at least $(2a-1)$. The vertices of F can be partitioned into sets B_1, in which every vertex has out-degre at least a, and $B_2 = F - B_1$. Note that there is an arc between every vertex in B_1 and every vertex in B_2, since the addition of an arc from B_1 to B_2 cannot create a copy of K. Without loss of generality, we can assume that $|B_2| = j \le \frac{n}{2}$. If $j = 0$ then every vertex in F has out-degree

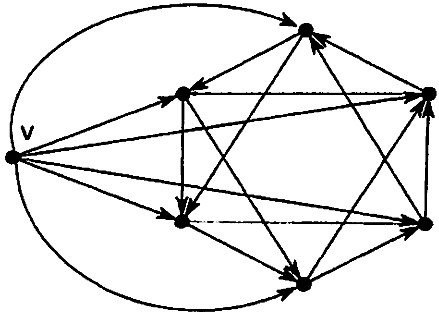

FIGURE 5. A $K_{1,(3,3)}$-saturated oriented graph

at least a and so at least one also has in-degree at least k. Therefore, $K \subseteq D$. Thus, we may assume $j \geq 1$. Each vertex in B_2 has underlying degree at least $|B_1| = (n-j)$. Therefore, the number of edges in F is bounded below by the function $f(j) = \frac{1}{2}[j(n-j) + (n-j)(2a-1)] = \frac{1}{2}(n-j)(2a-1+j)$ for $1 \leq j \leq \frac{n}{2}$. For n sufficiently large, this function has a minimum at $j = 1$. Since $f(1) = e(D)$, we know that D is the smallest K-saturated oriented graph on n vertices, and hence $sat(n, K_{1,(a,a)}) = f(1) = a(n-1)$. □

We conclude this section with another definition and a familiar construction.

Definition 3.5. *Let G, H be graphs and let $k_G(H)$ denote the number of subgraphs of H isomorphic to G. H is* strongly G-saturated *if for any edge $e \in H^c$, the number $k_G(H+e) > k_G(H)$.*

Note that any G-saturated graph is strongly G-saturated, but the converse is not necessarily true.

Theorem 3.6. *If TT_m is the transitive tournament on m vertices then*

$$sat(n, TT_m) = \binom{m-2}{2} + (n-m+2)(m-2).$$

Proof. Erdös, Hajnal, and Moon [3] showed that $sat(n, K_m) = \binom{m-2}{2} + (n-m+2)(m-2)$ and that the unique extremal graph is $K_{m-2} \vee \bar{K}_{n-m+2}$. Orient the $(m-2)$-clique to obtain TT_{m-2} and each arc from the remaining vertices to this tournament, (see Figure 6). The resulting oriented graph F is TT_m-saturated and provides an upper bound on $sat(n, TT_m)$. Now let J be a TT_m-saturated oriented graph with $x, y \in J$ non-adjacent vertices. Since $J + xy, J + yx$ both contain TT_m, the addition of xy to $u(J)$ creates a new copy of K_m. Therefore, $u(J)$ is strongly K_m-saturated. In [1], it is shown that $K_{m-2} \vee \bar{K}_{n-m+2}$ is the unique smallest strongly K_m-saturated graph on n vertices. Therefore, $u(J)$ is $K_{m-2} \vee \bar{K}_{n-m+2}$ and J has $\binom{m-2}{2} + (n-m+2)(m-2)$ arcs. Therefore,

$$sat(n, TT_m) = \binom{m-2}{2} + (n-m+2)(m-2).$$

□

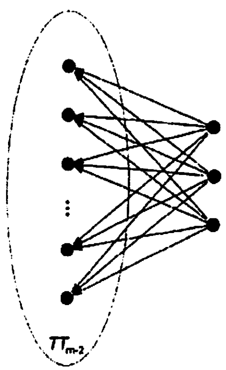

FIGURE 6. A transitive tournament-saturated oriented graph, consisting of a transitive tournament joined from a set of isolated vertices

4. SOME UPPER BOUNDS

We begin our examination of bounds on *sat* with a definition followed by an examination of path orientations.

Definition 4.1. *An oriented graph D is* hamiltonian connected *if for any pair x, y of vertices of D there is a hamiltonian path from x to y and from y to x.*

Let $k \geq 7$ be an integer. We will now construct a hamiltonian connected oriented graph of order k. Define the digraph D_k with vertices $v_0, v_1, \ldots, v_{k-1}$ and arcs v_iv_{i+1}, v_iv_{i+2}, and v_iv_{i+3} for all $0 \leq i \leq (k-1)$ with addition modulo k.

Lemma 4.2. *The digraph D_k is hamiltonian connected.*

Proof. We require that for every ordered pair of vertices in D_k there is a hamiltonian path from the first to the second. Without loss of generality, we may assume that the ordered pair is of the form (v_0, v_i) for some $1 \leq i \leq (k-1)$. If $i = (k-1)$ then the path $v_0v_1 \ldots v_{k-1}$ is hamiltonian. If $i < (k-1)$ is odd then consider the path $v_0v_2v_4 \ldots v_{i-1}v_{i+1}v_{i+2} \ldots v_{k-1}v_1v_3 \ldots v_i$. If on the other hand i is even, then the path $v_0v_1v_3 \ldots v_{i-1}v_{i+1}v_{i+2} \ldots v_{k-1}v_2v_4 \ldots v_i$ will suffice. Therefore, there is a hamiltonian path from v_0 to every other vertex in D_k, and thus D_k is hamiltonian connected. □

Because D_k is a hamiltonian connected digraph of order $k \geq 7$, so is every tournament of order k that contains D_k as a subdigraph. Therefore, Lemma 4.2 implies the existence of hamiltonian connected tournaments of all orders at least 7.

Theorem 4.3. *Let $n > m \geq 9$ be integers. Then, $n \leq sat(n, \vec{P_m}) \leq \binom{m-2}{2} + 2(n-m+2)$.*

Proof. First, note that the argument in the proof of Theorem 3.1 applies here as well, so that no $\vec{P_m}$-saturated oriented graph contains as components a pair of oriented trees. In fact, a single oriented tree is either $\vec{P_3}$-saturated, and therefore

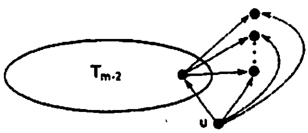

FIGURE 7. A $\vec{P_m}$-saturated oriented graph

not $\vec{P_m}$-saturated, or contains both a source and sink. Joining the source to the sink does not create a $\vec{P_m}$. Therefore, $n \leq sat(n, \vec{P_m})$.

For the upper bound consider the following oriented graph $H(m,n)$, seen in Figure 7. Let T_{m-2} be a hamiltonian connected tournament on $m-2$ vertices whose existence is guaranteed by Lemma 4.2, with vertices labelled $v_0, v_1, \ldots, v_{m-3}$. Add $n-m+1$ vertices labelled $r_0, r_1, \ldots r_{n-m}$ and adjacent from v_0. Finally, add a new vertex u adjacent to $v_0, r_0, \ldots r_{n-m}$. The graph $H(m,n)$ is $\vec{P_m}$-saturated. This can be seen by first noticing that the longest path in $H(m,n)$ has order $m-1$, then examining the graph $H(m,n)+a$ for any arc $a \in H(m,n)^c$. Let i, j be integers. If $a = r_i r_j$ then the path $v_{m-3} \ldots v_0 r_i r_j$ is a $\vec{P_m}$. If $a = v_i r_j$ then consider the path $uv_0 P_{0i} v_i r_j$ where P_{0i} is a hamiltonian path from v_0 to v_i in T_{m-2}. Alternately, if $a = r_i v_j$ then consider $r_i v_j P_{j0} r_0$, unless $i = 0$ in which case the path $r_i v_j P_{j0} r_1$ is a $\vec{P_m}$. If $a = v_i u$ then let P be the path of order $m-3$ obtained from P_{0i} by removing the vertex v_0. Then $P v_i u v_0 r_0$ is a path of order m. Finally, if $a = uv_i$ then the path $uv_i P_{i0} v_0 r_0$ suffices. Therefore, $H(m,n)$ is $\vec{P_m}$-saturated and $n \leq sat(n, \vec{P_m}) \leq \binom{m-2}{2} + 2(n-m+2)$.

□

Note that if m is 7 or 8 then $sat(n, \vec{P_m})$ is bounded above by $\frac{3}{2}n + c$ where c is a constant depending on n (mod m). This bound is achieved by a construction similar to that in Theorem 3.2 but composed of disjoint strong tournaments of order 4 along with zero, one, two, or three strong tournaments of order 5 and, in the former case, at most one directed 3-cycle.

Theorem 4.4. *Let $o(P_m)$ be an orientation of P_m, with $m = 2k, k > 8$ and odd. Then, $sat(n, o(P_m)) \leq (n \pmod k)\binom{k+1}{2} + \lfloor \frac{n}{k} \rfloor \binom{k}{2}$.*

Proof. Let $r \geq 1$ be an integer. By the *rotational tournament* ROT_{2r+1} of order $2r+1$ we mean the orientation of K_{2r+1} with vertices $\{v_0, v_1, \ldots, v_{2r}\}$ whereby for all $0 \leq i \leq 2r$, v_i is adjacent to $v_{i+1}, v_{i+2}, \ldots, v_{i+r}$, with addition considered modulo $2r+1$. Consider the oriented graph F consisting of disjoint rotational tournaments of order k. Havet and Thomassé [7] showed that every tournament on at least 8 vertices contains every orientation of a hamiltonian path. A rotational tournament is vertex transitive, and hence any oriented path contained in the tournament can be said to have as an endpoint any vertex in the tournament. So, each vertex of F is an end vertex of every orientation of every path of order at most the order of the component in which the vertex is contained. Thus, the addition of an arc between vertices of different components will yield every orientation of a path of order m.

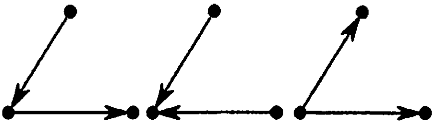

FIGURE 8. No oriented graph on 3 vertices is C_3-saturated

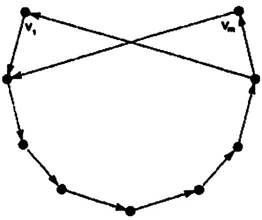

FIGURE 9. The addition of either arc v_1v_m or arc v_mv_1 generates a hamiltonian cycle

The oriented graph F is $o(P_m)$-saturated and contains $(n \pmod k)\binom{k+1}{2} + \lfloor \frac{n}{k} \rfloor \binom{k}{2}$ arcs, and therefore provides the upper bound. □

5. THE MINIMUM ORDER OF A D-SATURATED ORIENTED GRAPH

We mentioned in Section 1 that for a simple graph G, the value $sat(n, G)$ is well-defined for any $n \geq n(G)$, and that this is not necessarily the case for oriented graphs. Although any tournament not containing the oriented graph D is by definition D-saturated, we wish to consider for which values of n there is an oriented non-complete graph F that is D-saturated. Consider the case when D is a directed cycle on three vertices, denoted $\vec{C}_3$. There are, up to isomorphism, only three oriented graphs of order three and size two, (see Figure 8). For each, there is an arc whose addition creates no $\vec{C}_3$. Consequently, no oriented non-complete graph on 3 vertices is $\vec{C}_3$-saturated. However, the directed cycle on four vertices is $\vec{C}_3$-saturated.

This leads naturally to wonder for which n, the value $sat(n, D) < \binom{n}{2}$. Given an oriented graph D we refer to the smallest n such that $sat(n, D) < \binom{n}{2}$ as N_D. Our construction in section 2 of this paper gives $2n(D) - 2$ as an upper bound on N_D.

For the transitive tournament TT_n on n vertices, the construction used in Theorem 3.6 demonstrates that $N_{TT_n} = n$.

Every vertex in a strong tournament on k vertices is both the initial and terminal vertex of directed paths of order k. Therefore, for $m \geq 4$ the oriented graph on m vertices composed of an isolated vertex and a strong tournament on $m - 1$ vertices is $\vec{P}_m$-saturated. A pair of isolated vertices is $\vec{P}_2$-saturated, and three vertices with a pair of arcs that share an initial vertex is a $\vec{P}_3$-saturated graph. So, $N_{\vec{P}_m} = m$ for all $m \geq 2$.

Similarly, for $m > 3$ we can show that $N_{\vec{C_m}} = m$. Consider a directed path $v_1, v_2, \ldots v_m$ on m vertices. Add the arcs v_1v_{m-1} and v_mv_2, (see Figure 9). The resulting graph does not contain a directed m-cycle, but the addition of either v_1v_m or v_mv_1 creates one, so by Fact 2.3 there is a $\vec{C_m}$-saturated oriented graph on m vertices.

6. Further directions

Saturated simple graphs have been studied extensively. There has been a tremendous amount of interest in their structure, beginning with the work of Mantel [11] and Turán [12] with regard to maximum size and Erdös, Hajnal, and Moon's work [3] on saturated graphs of minimum size. While directed graphs have also received a great deal of attention, the intersection of these two topics has thus far been very limited in scope, and what attention it has received has been restricted to orientations of graphs containing multiple edges [4].

A possible reason for this lack of attention to oriented graph saturation is the lack of assurance that the parameters $sat(n, D)$ and $ex(n, D)$ are well-defined for all oriented graphs D. We have resolved this issue in Theorem 2.4, and thus have made oriented graph saturation a viable field of study. We have also seen that the saturation number of D can be, but is not necessarily, related to the saturation number of its underlying graph $u(D)$. While Bollobás [1] demonstrates that for integers $n \geq m$ the unique K_m-saturated graph of size $sat(n, K_m)$ is the smallest strongly K_m-saturated graph on n vertices, this is not necessarily the case for all graphs in general. Since a D-saturated oriented graph F has the property that $u(F)$ is strongly $u(D)$-saturated, a solution to the following problem will potentially resolve $sat(n, D)$ for a number of oriented graphs.

Problem 1. *For which graphs G is $sat(n, G)$ the fewest number of edges in a strongly G-saturated graph on n vertices?*

Clearly it would be advantageous to determine $sat(n, D)$ for other families of oriented graphs, including non-transitive tournaments, oriented trees, and orientations of paths. While we have established that $n \leq sat(n, \vec{P_m}) \leq \binom{m-2}{2} + 2(n - m + 2)$, we have no reason to believe that either of these bounds will be met in general. This is definitely a direction for future work in this area.

Another interesting opportunity for extending oriented graph saturation is the study of symmetric digraphs.

Definition 6.1. *A digraph D is* symmetric *if for any arc xy in D the arc yx is also in D. For a simple graph G let $s(G)$ be the symmetric digraph on the vertices of G in which xy, yx are arcs in $s(G)$ if and only if xy is an edge in G.*

For any symmetric digraph F let $f(F) = \frac{1}{2}|A(F)|$, so that f represents the number of unique pairs of adjacent vertices. Let D be an oriented graph. If $s(G)$ is a D-saturated digraph for some graph G then G is $u(D)$-saturated. The converse is not necessarily true, as can be seen in Figure 10. The graph $H = K_m \cup K_1$ is $K_{1,m}$-saturated, but $s(H)$ is not $K_{1,(m,0)}$-saturated.

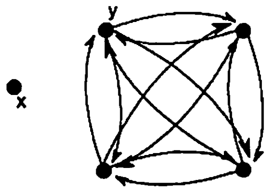

FIGURE 10. The addition of arc xy to $s(K_4 \cup K_1)$ does not create a $K_{1,(4,0)}$

Extend the definition of *sat* to symmetric digraphs in the following way. For an oriented graph D let $sat_s(n, D) = min\{f(F)\}$ over all D-saturated symmetric digraphs F on n vertices.

Problem 2. *How does* $sat_s(n, D)$ *relate to* $sat(n, D)$ *and* $sat(n, u(D))$*?*

It is clear that $sat_s(n, D) \geq sat(n, u(D)$. When is equality achieved?

7. In Memoriam

In memory of Ralph Stanton, a long-time friend to combinatorics and graph theory.

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