Rainbow Cycles in Cube Graphs

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Dedicated to Ralph G. Stanton

ABSTRACT. A graph G is called rainbow with respect to an edge coloring if no two edges of G have the same color. Given a host graph H and a guest graph $G \subseteq H$, an edge coloring of H is called G-anti-Ramsey if no subgraph of H isomorphic to G is rainbow. The anti-Ramsey number $f(H, G)$ is the maximum number of colors for which there is a G-anti-Ramsey edge coloring of H. In this note we consider cube graphs Q_n as host graphs and cycles C_k as guest graphs. We prove some general bounds for $f(Q_n, C_k)$ and give the exact values for $n \leq 4$.

1. Introduction

A graph G is called rainbow with respect to an edge coloring if no two edges of G have the same color, that is, the edges of G are totally multicolored. Given a host graph H and a guest graph $G \subseteq H$, an edge coloring of H is called G-anti-Ramsey if no subgraph of H isomorphic to G is rainbow. The anti-Ramsey number $f(H, G)$ is the maximum number of colors for which there is a G-anti-Ramsey edge coloring of H. Equivalently, any edge coloring of H with at least $\mathrm{rb}(H, G) = f(H, G) + 1$ colors contains a rainbow copy of G. This number $\mathrm{rb}(H, G)$ is called rainbow number of G with respect to H.

The function $f(H, G)$ was introduced by Erdős, Simonovits, and Sos [6]. In many of the papers on this function complete host graphs $H \cong K_n$ are considered. The function $f(K_n, G)$ is completely determined for, e.g., complete graphs [4, 10], cycles [6, 10], and matchings [5] as guest graphs. There are partial results if G is complete bipartite or a tree, for example. There are also some results for complete bipartite host graphs H (see, i.e., [2, 9]).

In this note we consider cube graphs Q_n as host graphs. The cube graph Q_n has 2^n vertices to which all $(0, 1)$-sequences of length n are assigned and two vertices are adjacent if and only if the assigned sequences differ in exactly one position. All 2^{n-1} edges for which the assigned sequences differ in the same position determine one of the n parallel classes of edges.

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In the case that also the guest graphs are cube graphs Q_k then partial results on $f(Q_n, Q_k)$ can be found in [1, 3]. Here we consider guest graphs C_k in host graphs Q_n where k has to be even since odd cycles are no subgraphs of cube graphs.

2. General Results

At first we determine some general lower bounds for $f(Q_n, C_k)$ by giving constructions for colorings of Q_n without any rainbow copy of C_k.

Theorem 1: If $n \geq 3$ then

$$f(Q_n, C_k) \geq (s-1)2^{n-1} + n - s + 1$$

where $s = \lceil \log_2 k \rceil$.

Proof: Color all edges of the parallel classes in which the assigned sequences of the vertices differ at position i, $1 \leq i \leq n - s + 1$, with color i. All $(s-1)2^{n-1}$ edges of the remaining $s-1$ parallel classes obtain pairwise different colors greater then $n - s + 1$ implying that the total number of colors of this coloring is $(s-1)2^{n-1} + n - s + 1$.

Since any cycle contains an even number of edges of each parallel class, rainbow cycles in Q_n cannot contain an edge of a color $1, 2, \ldots, n - s + 1$.

The components of the subgraph of Q_n induced by the edges of the remaining colors are cube graphs $Q_{n-(n-s+1)} \cong Q_{s-1}$. Since the order of Q_{s-1} is $2^{s-1} = 2^{\lceil \log_2 k \rceil - 1} < k$, the components Q_{s-1} cannot contain cycles C_k as subgraphs, and therefore there is no rainbow C_k in Q_n. □

In the next theorem the lower bound for $f(Q_n, C_k)$ depends on $t(Q_n, C_k)$, where the vertex Turán number $t(H, G)$ is defined as the minimum number of vertices which have to be removed from H such that the remaining graph does not contain a copy of G as subgraph.

Theorem 2: It holds

$$f(Q_n, C_k) \geq n2^{n-1} - (n-1)t(Q_n, C_k).$$

Proof. Let T be a set of $t(Q_n, C_k)$ vertices such that each copy of C_k in Q_n contains a vertex of T. We start with a coloring of the edges of Q_n with pairwise different colors, that is, with $n2^{n-1}$ colors. Then we recolor a minimum number of edges of this coloring such that at the end each vertex of T is incident to equally colored edges. This can be achieved by recoloring at most $n-1$ of the incident edges for each vertex of T. □

We provide an upper bound for $t(Q_n, C_k)$.

Lemma 1: It holds

$$t(Q_n, C_k) \leq 2^{n-1} - \frac{k}{2} + 1.$$

Proof. Since Q_n is bipartite any cycle C_k contains $k/2$ vertices of each partition set. The cardinality of each partition set is 2^{n-1}. The removal of $2^{n-1} - k/2 + 1$ vertices of one partition set results in a subgraph with less than $k/2$ vertices in this partition set and therefore without a cycle C_k. □

Combining Theorem 2 and Lemma 1 gives

Corollary 1: It holds

$$f(Q_n, C_k) \geq 2^{n-1} + \frac{1}{2}(n-1)(k-2).$$

Analogously to the vertex Turán number the edge Turán number $t'(H, G)$ is defined as the minimum number of edges which have to be removed from H such that the remaining graph does not contain a copy of G as subgraph.

Theorem 3: It holds

$$f(H, G) \leq q(H) - t'(H, G)$$

where $q(H)$ is the size of H.

Proof. Let c be an edge coloring of H with $f(H, G)$ colors such that there is no rainbow copy of G. Define T' to be a set of all edges of H reduced by one edge of each of the $f(H, G)$ colors. Thus, $|T'| = q(H) - f(H, G)$. If the edges of T' are removed from H then the remaining graph has edges of pairwise different colors and therefore does not contain a copy of G since otherwise this copy would be rainbow, contradicting the definition of the coloring c. This implies $t'(H, G) \leq |T'|$ which completes the proof. □

In the following we summarize some properties of cube graphs.

Propositions:

1. Cube graphs Q_n are bipartite with partition sets of, say, black and white vertices.

2. Q_n is hamiltonian for $n \geq 2$.

3. In Q_n there exists a hamiltonian path between any pair of a black and a white vertex (see [8, Lemma 1]).

The following result proves that a cube graph Q_n remains hamiltonian after removing any $n - 2$ edges.

Lemma 2 [7]: It holds

$$t'(Q_n, C_{2^n}) = n - 1.$$

Proof. Removing $n-1$ edges from Q_n incident to the same vertex results in a non-hamiltonian graph, that is, in a graph without a subgraph C_{2^n}.

Now consider a graph obtained by removing any $n-2$ edges from Q_n. Let r, $1 \leq r \leq n-2$, be the number of different parallel classes to which the removed edges belong. Therefore, at most $n-2-(r-1) = n-r-1$ edges belong to the same parallel class. The removal of all the edges of these r parallel classes results in 2^r disjoint subcubes Q_{n-r}. Contraction of the vertices of each of these Q_{n-r}s in Q_n to a single vertex yields a Q_r.

In the case $r \geq 2$ choose a hamiltonian cycle C, which exists in Q_r by Proposition 2, and an arbitrary orientation of C. For each oriented edge uv of C consider the two copies Q and Q' of Q_{n-r} corresponding to u and v, respectively. There are 2^{n-r} parallel edges between Q and Q' and therefore 2^{n-r-1} edges from the black vertices of Q to the white vertices of Q' in Q_n. There remains at least one of these edges after the removal of the $n-2$ edges of Q_n since $n-r-1 < 2^{n-r-1}$. For each oriented edge of C choose one of those remaining edges of Q_n. Therefore, in each Q_{n-r} there exists a black vertex and a white vertex incident to such a chosen edge. Since there exists a hamiltonian path between these pairs of black and white vertices in each Q_{n-r} according to Proposition 3, there is a hamiltonian cycle in Q_n.

In the case $r = 1$ we obtain just two subcubes $Q \cong Q' \cong Q_{n-1}$ by the removal of the corresponding parallel class. Note that, by the same argument as before, there is one edge from a white vertex of Q to a black vertex of Q', and vice versa. These edges together with the corresponding hamiltonian paths in Q and Q' form a hamiltonian cycle in Q_n. □

Using the preceding lemma we can solve the "hamiltonian case" $k = 2^n$ completely.

Corollary 2. It holds

$$f(Q_n, C_{2^n}) = n2^{n-1} - n + 1.$$

Proof. Theorem 3 and Lemma 2 imply that $f(Q_n, C_{2^n}) \leq n2^{n-1} - (n-1)$ since $q(Q_n) = n2^{n-1}$. Coloring all edges incident to the same vertex with one color and all other edges with additional pairwise distinct colors proves the lower bound. □

3. Exact results for small cube graphs Q_n

The cube graph Q_1 does not contain a cycle, and Q_2 as host graph is completely solved by Corollary 2: $f(Q_2, C_4) = 3$. The following theorem covers the case $n = 3$.

Theorem 4: It holds

$$f(Q_3, C_4) = 8, \quad f(Q_3, C_6) = 9, \quad \text{and} \quad f(Q_3, C_8) = 10.$$

Proof. The value of $f(Q_3, C_8)$ is determined in Corollary 2 and that of $f(Q_3, C_4)$ in [1, Theorem 3] (note that $C_4 \cong Q_2$).

Theorem 1 proves $f(Q_3, C_6) \geq 9$.

We prove equality by showing that each coloring of Q_3 with 10 colors contains a rainbow cycle C_6.

Consider an arbitrary set of $i+1$ edges of Q_3 and count the cycles C_6 containing at least two of these edges. The maximum number of C_6s over all $(i+1)$-sets is denoted by $g(i)$.

To determine upper bounds for $g(1)$ and $g(2)$ observe that there are 16 distinct copies of C_6 in in Q_3. Since Q_3 has 12 edges there are $16 \cdot 6/12 = 8$ copies of C_6 that contain a certain edge implying $g(1) \leq g(2) \leq 8$. Since for any pair of edges there exists a C_6 containing just one of these edges, it follows $g(1) < 8$.

An edge coloring of Q_3 with 10 colors contains either two pairs of equally colored edges or a triple of edges of the same color. Therefore, the number of non-rainbow copies of C_6 is at most $g(1) + g(1) < 16$ or $g(2) < 16$, respectively, that is, in both cases there remains at least one rainbow copy in Q_3. □

Note that, considering the indices of $g(i)$ in the preceeding proof, $1+1$ and 2 are all possible partitions of $12 - 10$ where 12 is the number of edges of Q_3 and 10 the number of colors. This idea will also be used in the following determination of $f(Q_4, C_k)$.

Theorem 5: It holds

$$f(Q_4, C_4) = 18, \quad f(Q_4, C_6) = 20, \quad f(Q_4, C_8) = 21, \quad f(Q_4, C_{10}) = 25,$$
$$f(Q_4, C_{12}) = 25, \quad f(Q_4, C_{14}) = 26, \quad \text{and} \quad f(Q_4, C_{16}) = 29.$$

Proof: The value of $f(Q_4, C_k)$ for $k = 4$ is determined in [1, Theorem 4] and for $k = 16$ in Corollary 2.

In the remaining cases we first prove lower bounds for $f(Q_4, C_k)$.

In the case $k = 6$ deletion of the four white vertices in Figure 1, which implies deletion of all double-line edges, too, results in a graph without a cycle C_6 implying $t(Q_4, C_6) \leq 4$. Therefore, $f(Q_4, C_6) \geq 20$ by Theorem 2.

For $k = 8$ color all double-line edges in Figure 2 with one color and all the other edges with additional pairwise distinct colors. This coloring uses 21 colors. Removal of the 12 double-line edges results in a graph with two components not containing a cycle C_8. Therefore, any C_8 in Q_4 contains two of the double-line edges and thus is not rainbow.

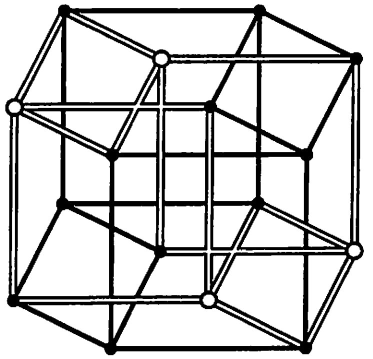

Figure 1: $f(Q_4, C_6) \geq 20$.

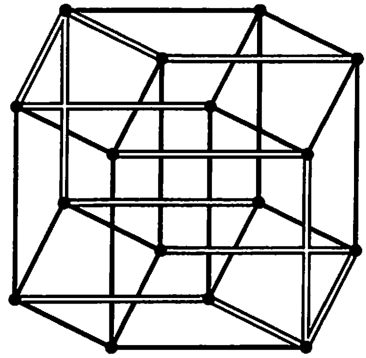

Figure 2: $f(Q_4, C_8) \geq 21$.

For $k = 10$ and $k = 12$ we have $f(Q_4, C_k) \geq 25$ by Theorem 1, and Corollary 1 gives $f(Q_4, C_{14}) \geq 26$.

To prove the corresponding upper bounds we generalize the method of the proof of Theorem 4.

Extending the definition of $g(i)$ we consider an edge coloring of Q_4 containing $i_j + 1$ edges of color j for $j = 1, \ldots, t$ and all other edges of additional pairwise distinct color. This coloring uses $32 - (i_1 + i_2 + \ldots + i_t)$ colors. The maximum number of non-rainbow C_ks over all such colorings is denoted by $g(i_1, i_2, \ldots, i_t)$.

To prove $f(Q_4, C_k) < z$ it suffices to show that $g(i_1, i_2, \ldots, i_t)$ is smaller than the number of copies of C_k in Q_4 for all partitions $(i_1, i_2, \ldots, i_t)$ with $i_1 + i_2 + \ldots + i_t = 32 - z$.

Using the fact that $g(i_1, \ldots, i_t) \leq g(i_1, \ldots, i_r) + g(i_{r+1}, \ldots, i_t)$ reduces the number of cases to be considered. Note that the order of arguments in function g is not important.

We used a computer to establish the respective upper bounds. □

4. Concluding remarks

Applying the results of Theorem 1 and Corollary 2 yields the lower bounds for $f(Q_5, C_k)$ in Table 1.

k	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32
$f(Q_5, C_k) \geq$	20	35	35	50	50	50	50	65	65	65	65	65	68	72	76

Table 1: Lower bounds for $f(Q_5, C_k)$.

Note that Corollary 2 proves $f(Q_5, C_{32}) = 76$ and that $f(Q_5, C_4) = 43$ (see [3, Theorem 1]).

The following result provides a general upper bound for $f(H, G)$.

Theorem 6: Let F, G, and H be graphs with $G \subseteq F \subseteq H$. Then

$$f(H,G) \leq f(H,F) + f(F,G) - q(F) + 1.$$

Proof: An edge coloring of H with $f(H,F)+1$ colors assures the existence of a rainbow copy of F in H, that is, this copy has $q(F)$ colors. Reducing the number of colors in H by $q(F) - f(F,G) - 1$ results in a coloring of H using $f(H,F) - q(F) + f(F,G) + 2$ colors with a copy of F in H using at least $f(F,G) + 1$ colors. Thus, there exists a rainbow copy of G in F and therefore also in H. This proves that in any edge coloring of H with $f(H,F) + f(F,G) - q(F) + 2$ colors there is a rainbow copy of G. □

Setting $H \cong Q_5$, $G \cong C_k$, and $F \cong Q_3, Q_4$ gives the upper bounds in Table 2. Note that $f(Q_5, Q_3) = 68$ (see [3]) and $f(Q_5, Q_4) = 76$ (see [1]).

k	6	8	10	12	14	16
$f(Q_5, C_k) \leq$	65	66	70	70	71	74

Table 2: Upper bounds for $f(Q_5, C_k)$.

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