Embedding graphs containing K_5-subdivisions on the torus

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To Ralph G. Stanton, in memoriam.

Abstract

We simplify and further develop the methods and ideas of [A. Gagarin, W. Kocay, "Embedding graphs containing K_5-subdivisions," *Ars Combin.* 64 (2002), pp. 33-49] to efficiently test embeddability of graphs on the torus. Given a non-planar graph G containing a K_5-subdivision subgraph, we show that it is possible either to transform the K_5-subdivision into a certain type of $K_{3,3}$-subdivision, or else to reduce the toroidality testing problem for G to a small constant number of planarity checks and, eventually, rearrangements of planar embeddings. It is shown how to consider efficiently only one K_5-subdivision in the input graph G to decide whether G is embeddable on the torus. This makes it possible to detect a bigger class of toroidal and non-toroidal graphs.

1 Introduction

We use basic graph-theoretic terminology from Bondy and Murty [1], Diestel [2], and Kocay and Kreher [13]. A simple finite graph G is *embeddable* (*embeds*) on a topological surface S if it can be drawn on S with no crossing edges. Such a drawing of G on the surface S is called an *embedding*. There

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can be many different embeddings of the same graph G on a particular surface S. A *2-cell embedding* of G on S has each of its faces homeomorphic to an open disk.

A combinatorial description of a graph G embedded on a topological surface S is provided by a *rotation system* of the graph, which is a set of cyclically ordered adjacency lists for the graph vertices (for example, see Kocay and Kreher [13]). The topological surface S is usually represented by a polygon with its sides identified in pairs. For more details, see Fréchet and Fan [4], or [13].

We are interested in an efficient computer algorithm to decide whether G embeds on the torus. As shown in Myrvold and Kocay [16], this kind of algorithm can be very subtle and subject to errors. Classically, a torus embedding algorithm starts with a subgraph of G isomorphic to a subdivision of K_5 or $K_{3,3}$ and tries to extend one of its embeddings on the torus to an embedding of the whole graph G (for example, see the algorithms of Myrvold and Woodcock [20], and Juvan, Marinček, and Mohar [10]). A modification of this approach simplifying the case of an initial K_5-subdivision in G has been proposed by Gagarin and Kocay [6]. They show that non-planar graphs which do not contain a certain kind of $K_{3,3}$-subdivision are much easier to embed on the torus than by using the approach of [20] and [10] directly. Since there are six non-isomorphic embeddings of K_5 (providing numerous labelled embeddings) and only two non-isomorphic embeddings of $K_{3,3}$ (providing only twenty labelled embeddings) on the torus, the approach of [6] avoids many of the initial cases that must be considered in the algorithms of [10, 20] and provides an efficient means of handling the K_5 embeddings.

Hopcroft and Tarjan [9] described the first linear-time algorithm to check if a graph G is planar or not. If G is 2-connected, planar and not a cycle, then its planar rotation system can always be transformed into a 2-cell toroidal rotation system. Some methods to do this transformation are presented in Gagarin, Kocay, and Neilson [5] and implemented in the software *Groups&Graphs* (see Kocay [12]).

The well-known characterization of non-planar graphs in terms of forbidden subgraphs can be stated as follows:

Theorem 1 (Kuratowski [14]) *A graph G is non-planar if and only if it contains a subdivision of $K_{3,3}$ or K_5 as a subgraph.*

A short constructive proof of Theorem 1 providing a corresponding algorithm can be found in Klotz [11]. In general, a planarity testing algorithm can be modified so that, in the case of a non-planar graph G, the algorithm returns a subdivision of K_5 or $K_{3,3}$ in G (for example, see Williamson [19] and [11]). The toroidality testing problem can be reduced to testing toroidality of 2-connected graphs: an embedding on the torus can accommodate at most one non-planar 2-connected component of a graph (e.g., see Miller [15]). The vertices of degree two are clearly irrelevant to the embeddings and embeddability. Therefore, without loss of generality, it can be assumed that G is a 2-connected non-planar graph having a minimum vertex degree of at least three, and containing a subdivision of K_5 or $K_{3,3}$ as a subgraph.

The results presented in this paper simplify and extend the results of [6] and enable us to detect a bigger class of toroidal and non-toroidal graphs than the torus embedding algorithm of [6]. This is described in more detail in the conclusion. First, in Section 2, we describe the structure of non-planar graphs containing a subdivision of K_5 and no $K_{3,3}$-subdivisions of a special kind with respect to the chosen K_5-subdivision. These graphs can be uniquely decomposed into ten subgraphs corresponding to the ten edges of K_5 (however these graphs can still contain subdivisions of $K_{3,3}$ as well). Then, in Sections 3 and 4, we show how to decide on the toroidality of such a graph G by testing the planarity of its ten subgraphs resulting from the decomposition, and, whenever possible, to return one of its toroidal rotation systems.

2 Decomposition and related structural results

In this section, we briefly review notation, definitions, and results from [6]. Following [2], a K_5-subdivision is denoted by TK_5. Similarly, a $K_{3,3}$-subdivision is denoted by $TK_{3,3}$. The vertices of degree four in a TK_5 are called *corners* of the TK_5. The vertices of degree two in a TK_5 (if any) are called *inner vertices* of the TK_5. A path in a TK_5 connecting two distinct corners and having all other vertices inner vertices is called a *side* of that TK_5. Notice that two distinct sides of the same TK_5 can have at most one common corner and no common inner vertices. A side having a common corner with another side of the same TK_5 is called *adjacent* to that side. Two sides having no common corner are *non-adjacent*. Let W denote the set of corners of a TK_5. Then the set of inner vertices of the TK_5 induces the subgraph $TK_5 \setminus W$.

Let G be a 2-connected non-planar graph containing a TK_5 as a subgraph. A path P in G is a *short-cut* of the K_5-subdivision TK_5 if $P \cap TK_5 = \{u, v\}$ are the endpoints of P, u is an inner vertex of the TK_5, and v is on a side of the TK_5 distinct from the side containing u. A vertex $u \in G \setminus TK_5$ is called a 3-*corner vertex* with respect to the TK_5 if the subgraph induced by the vertices of $G \setminus (TK_5 \setminus W)$ in G contains internally vertex-disjoint paths from u to at least three different corners of the TK_5. The following proposition is shown in [6] and Fellows and Kaschube [3].

Proposition 1 *[3, 6] A non-planar graph G containing a K_5-subdivision TK_5 and a short-cut or a 3-corner vertex with respect to the TK_5 contains a $TK_{3,3}$.*

As shown in [6], it is possible to decompose graphs containing a K_5-subdivision TK_5 with no short-cuts or 3-corner vertices of the TK_5 to decide on embeddability of such graphs in the projective plane and the torus.

Proposition 2 *[3, 6] Given a graph G containing a subgraph TK_5 and no short-cuts or 3-corner vertices of the TK_5, a connected component H' of $G \setminus W$ contains inner vertices of at most one side of the TK_5. Moreover, $V(H')$ is adjacent in G to exactly two corners of the TK_5.*

Given a graph G containing a TK_5 and no short-cuts or 3-corner vertices of the TK_5, a *side component* H of the TK_5 is a subgraph of G induced by a pair of corners a and b of TK_5 and all the connected components of $G \setminus W$ adjacent to both a and b.

Corollary 1 *[6] Two side components of TK_5 in G can have at most one vertex in common, the common corner of two distinct sides of TK_5.*

Notice that side components can still contain $K_{3,3}$-subdivisions, or a part of a $TK_{3,3}$ in one side component can be completed to an entire $TK_{3,3}$ by other side components. Thus, given a graph G with a subgraph TK_5, either it is possible to find a short-cut or a 3-corner vertex of the TK_5 in G, or else the vertices and edges of $G \setminus W$ can be partitioned into ten subgraphs corresponding to the pairs of corners of TK_5 in G.

A side component H of TK_5 in G contains exactly two corners a and b of TK_5. Given a side component H with corners a and b, the *augmented side component* is H if the edge ab is in H, and $H + ab$ otherwise. The following general lemma is useful in the embedding algorithm.

Lemma 1 *[6] A graph H admits a planar embedding having vertices u and v on the same (outer) face if and only if there exists a planar embedding of the graph $H + uv$, $u, v \in V(H)$.*

3 Structure of the side components and embeddability on the torus

Let G be a 2-connected, non-planar graph containing a subdivision TK_5 as a subgraph. By Propositions 1 and 2, it is possible either to decompose G into ten side components of TK_5, or else to transform the subdivision TK_5 into a $TK_{3,3}$ in G. Then, in the case of a decomposition, it is possible to decide efficiently whether G is toroidal and to obtain an embedding of G on the torus by analyzing the side components of TK_5 in G as follows.

K_5 has six non-isomorphic embeddings on the torus shown in Fig. 1, where the torus is represented in the usual fashion as a rectangle with opposite edges identified. This has been discovered by numerous researchers, including [5], where all torus embeddings of various graphs were enumerated by computer. Some of these embeddings, namely E_1, E_4, E_5, and E_6 in Fig. 1, have a face whose boundary contains a repeated vertex or a repeated edge. Such a face is labelled F in Fig. 1. If G is embeddable on the torus, the subgraph TK_5 in G must follow one of these six embeddings of K_5 on the torus.

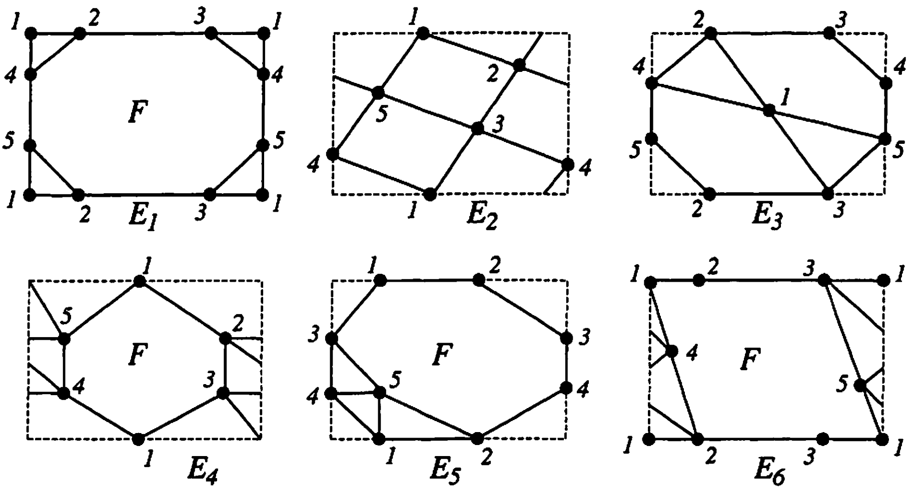

Figure 1: Six non-isomorphic embeddings of K_5 on the torus.

Suppose there are no short-cuts or 3-corner vertices of TK_5 in G. Then G is decomposable into the side components of TK_5. Let H be one of the side components of TK_5 in G. If H is planar, the corresponding augmented side component $H + ab$ can be planar or not. If H is non-planar, then clearly the corresponding augmented side component $H + ab$ is non-planar as well. The following proposition and theorem provide a characterization of toroidality for such graphs.

Proposition 3 *[6] A toroidal graph G containing a subdivision TK_5 with no short-cuts or 3-corner vertices can have at most one non-planar augmented side component. If all ten side components are planar, and at most one of them becomes non-planar when augmented, then G is toroidal.*

This implies there can be at most one non-planar side component of TK_5 for G to be toroidal. Two non-planar augmented side components of TK_5 in G imply that G is not toroidal. Therefore, it remains to distinguish between toroidal and non-toroidal graphs when there are nine planar augmented side components and a unique non-planar side component of TK_5 in G. Given a side component H with corners x and y, the subgraph $H - x$ or $H - y$ is called a *truncated side component* of TK_5 in G.

Theorem 2 *A graph G containing a subdivision TK_5 with nine planar augmented side components and a unique non-planar side component S having corners a and b is toroidal, if and only if the truncated side component $S - a$ admits a planar embedding such that all the neighbors of a in S are on the boundaries of exactly two different faces F' and F'' containing the other corner b.*

Proof: The corners a, b correspond to an edge of K_5. Let F_1, F_2 be the two faces on either side of this edge in an embedding of K_5. As shown in [6], if the embedding of TK_5 can be extended to a torus embedding of G, then S must be embedded in $F_1 \cup F_2 \cup (a, b)$, where (a, b) denotes the edge connecting a and b in the embedding of K_5. Since the side component S is non-planar, one of F_1, F_2 must be the face F with repeated vertices or edges on the boundary. Denote by F^+ the union $F_1 \cup F_2 \cup (a, b)$, where only the points a, b and all of (a, b) from the boundaries of F_1 and F_2 are included. Then at least one of the corners a and b must be a repeated vertex on the boundary of F^+. Since the repeated vertices are identified on the face boundaries, F^+ is not an open disc.

Therefore, we are interested only in the interior of the face F^+ and two corners a, b repeated on its boundary. In E_4 of Fig. 1, F^+ is homeomorphic to a disc with a pair of points (vertex 1) on its boundary identified — a subset of a cylinder. An embedding of S in F^+ would give an embedding on a cylinder, which would imply that S is planar. Some more details for this and the following three cases can be found in the proof of Proposition 4.1 on pages 40–42 in [6].

Hence, we can assume that S is embedded as one of E_1, E_5, or E_6 of Fig. 1. There is a number of ways in which this can be done. For example, in E_1, without loss of generality, we can take a to be vertex 2 because of the symmetry. Then, if b were vertex 3, identifying the repeated edges $(2, 3)$ on the boundary of F would make F^+ a cylinder. Clearly, it is not possible to have S embedded in the cylinder F^+ because S is not planar. Hence we must take b to be either vertex 4 or 5. Suppose b is vertex 4. Then F^+ consists of a disc with two pairs of points on its boundary identified. If we have an embedding of S in F^+, then, when a is deleted (vertex 2 in E_1), the result is an embedding of $S-a$ on the cylinder containing $F^+ - 2$. Denote by $N(a)$ the neighborhood of a in S. Since b appears twice on the boundary of F, the vertices of $N(a)$ must be contained in exactly two faces of the embedding containing b. Otherwise, if $N(a)$ were on the boundary of only one face of the embedding of $S - a$, S would be planar, which is not possible. By symmetry, the same holds for $S - b$. The case when b is vertex 5 is identical.

In E_5 of Fig. 1, there is a symmetry swapping 1 with 4, and 2 with 3. Therefore, without loss of generality, we can take a to be vertex 2. Then, if b were 1, 4, or 5, S would be embedded on a cylinder, which is not possible. Hence we must take b to be vertex 3. We again find that F^+ is a disc with two pairs of points identified. Hence, $S-a$ is embedded on a cylinder, and $N(a)$ is contained in exactly two faces of $F^+ - a$. Similarly for $S - b$.

In E_6 of Fig. 1, a can be vertex 1, 2, or 3. If a is 2 and b is 3, then, by identifying the repeated edges $(2, 3)$ on the boundary of F, we again have S embedded on the cylinder F^+. Therefore, one of a and b must be vertex 1. Without loss of generality, we take a to be vertex 1. The embedding E_6 has a symmetry swapping 2 with 3, 4 with 5, and fixing 1. Therefore, without loss of generality, we can take b to be vertex 2. We find that F^+ consists of a disc, with two pairs of points on its boundary identified (two copies of vertex 3 are ignored in this case), and furthermore, the interior of the triangular face with boundary $(1, 5, 2)$ attached to the disc along the edge $(1, 2)$. Note that $(1, 2)$ is *not* a repeated edge on the boundary of F. Then $F^+ - 1$ and $F^+ - 2$ are both homeomorphic to a cylinder. We again find that $N(a)$ must be contained in exactly two faces of $S - a$. This completes the proof. ∎

Clearly, Theorem 2 can be stated symmetrically with respect to the other truncated side component $S - b$ and corner a of S.

4 Algorithm for embedding graphs containing a TK_5 on the torus

The structural results presented in Sections 2 and 3 provide us with the following algorithm to test embeddability of graphs on the torus. The first three steps of Algorithm 1 are the same as the first three steps of the corresponding algorithm for the torus in [6]. However, Theorem 2 is used to simplify the algorithm of [6] and to detect a bigger class of toroidal and non-toroidal graphs.

Algorithm 1 *Embedding graphs containing a K_5-subdivision on the torus.*

Input: *a 2-connected graph G different from a cycle.*

Output: *either a toroidal rotation system of G, a $TK_{3,3}$ in G, or an indication that G is not toroidal.*

Step 1. Use a planarity testing algorithm, e.g., the algorithm of [9], to determine whether G is planar. If G is planar, then construct a planar rotation system for it. In [19], it is shown how to do this. Then transform the planar rotation system into a 2-cell toroidal rotation system by using, e.g., the method of [5], and return the 2-cell toroidal rotation system of G. If G is not planar, and a modified planarity test (e.g., the algorithm of [19]), returns a $TK_{3,3}$ in G, then return the $TK_{3,3}$ in G.

Step 2. If G is not planar, and the modified planarity test returns a TK_5 in G, then do a depth-first or breadth-first search to find either a short-cut or a 3-corner vertex of TK_5 in G. If a short-cut or a 3-corner vertex of TK_5 in G is found, return the corresponding $TK_{3,3}$ in G. If there are no short-cuts or 3-corner vertices of TK_5 in G, the depth-first or breadth-first search can be used to find the ten side components of TK_5 in G.

Step 3. If there are two or more non-planar augmented side components of TK_5 in G, return "G is not toroidal." If there is at most one non-planar augmented side component of TK_5, and the corresponding side component of TK_5 in G is planar, then return a toroidal rotation system of G. If the side component corresponding to the unique non-planar augmented side component is not planar, go to Step 4.

Step 4. There are nine planar augmented side components and exactly one non-planar side component S of TK_5 in G. Let a and b be the corners of S. Test the truncated side components $S-a$ and $S-b$ for planarity. If $S-a$ or $S-b$ is not planar, return "G is not toroidal." Otherwise we choose one of them, say $S-a$, and find a planar embedding. At this point we also determine if $S-a$ has cut-vertices, and, if so, we find all the cut-vertices and blocks. We then find all the separating pairs, thereby determining if $S-a$ has connectivity 1, 2, or at least 3.

Step 5. $S-a$ is planar and 3-connected. By a result of Whitney [18], a 3-connected graph has a unique planar embedding, up to orientation of the plane. Write $N(a)$ for the neighbourhood of a in S. We need to check if there are exactly two faces of the planar embedding of $S-a$ containing the corner b and all of $N(a)$. (This is described in detail below.) If so, we complete the cyclically ordered adjacency lists of a and each of its neighbors to obtain a toroidal rotation system of G and return the toroidal rotation system. Otherwise, return "G is not toroidal."

Since there is exactly one planar embedding of $S-a$, the facial boundaries are uniquely determined. However, note that the graph can be drawn on the plane so that any face is the outer face. Also, note that $d(v) \geq 3$ for any vertex $v \in S-a$ in this case. We want to determine if $N(a)$ is contained in the boundary of exactly two faces F, F' of $S-a$ containing b. Let $F_1, F_2, \ldots, F_m$ be the facial boundaries of $S-a$ containing b, where $m \geq 3$. In order to determine in linear time if F and F' exist, we proceed as follows. First, walk around each F_i and find the subset of $N(a)$ which it contains. Denote by S_i the subset of $N(a)$ in F_i. Now each F_i contains b. Since $S-a$ is 3-connected, each F_i intersects exactly two other of these facial boundaries in a vertex other than b. If there are five facial cycles F_i, F_j, F_k, F_p, F_q with the property that S_i, S_j, S_k, S_p, S_q are non-empty, and none is a subset of another, then F, F' do not exist. Therefore we proceed as follows.

We store four numbers i_1, i_2, i_3, i_4, of which each is either 0 or represents one of the subsets S_i. Initially each of the numbers i_1, i_2, i_3, i_4 is set to 0 representing S_0, which represents a dummy empty set. Now take each S_i in turn, from $i = 1$ to m, and compare it with the non-empty sets of $S_{i_1}, S_{i_2}, S_{i_3}, S_{i_4}$. If S_i is a subset of one of them, ignore S_i. Otherwise, if there is a non-empty S_{i_j} which is a subset of S_i, we replace i_j with i. If this occurs, we then check if any other S_{i_k} is a subset of S_i. If so, we set i_k to 0. In the remaining possibility, we take the first i_j which is 0, and set i_j to i. If this would be i_5, then F, F' do not exist. When all F_i have been considered, we have up to four non-empty subsets $S_{i_1}, S_{i_2}, S_{i_3}, S_{i_4}$, none of which is a subset of another. We simply check if some two of them contain all of $N(a)$.

Step 6. $S - a$ is planar and 2-connected, but not 3-connected. The main idea is to decompose $S - a$ into 3-connected components (e.g., using the method of Hopcroft and Tarjan [8]), and then determine if there is a planar embedding of $S - a$ with exactly two faces incident on b and containing all of $N(a)$. If such a planar embedding of $S - a$ is found, complete the embedding of $S - a$ to a toroidal rotation system of G and return the toroidal rotation system. If not, return "G is not toroidal."

Thus, we want to determine if there is a planar embedding of $S-a$ with $N(a)$ contained in the boundary of exactly two faces F and F' containing b. Let H be a graph, and X an induced subgraph of H. A *vertex of attachment* of X is a vertex $u \in V(X)$ such that u is adjacent to at least one vertex of $V(H)-V(X)$. If H is 2-connected, every induced subgraph X with $|V(X)| \geq 2$ has at least two vertices of attachment. A planar graph H which is not 3-connected can have numerous different planar embeddings. Given a planar embedding of H, a subgraph X with $|V(X)| \geq 3$ and only two vertices of attachment $\{u, v\}$ can be "flipped over," thereby changing the faces of the embedding and the planar embedding itself. The vertices of attachment form a separating pair in this case, as $H-\{u, v\}$ is disconnected. If X' is another induced subgraph of H with the same two vertices of attachment u and v, then X and X' can be interchanged, again changing the faces of the embedding and providing another planar embedding of H. It can be seen that all planar embeddings of $S - a$ can be obtained from one of them by a sequence of these operations.

We use a variation of Tutte's decomposition [17] of a 2-connected graph H into its 3-connected components. This basically consists of finding all separating pairs $\{u, v\}$ in H such that u and v both have degree at least three. Notice that the classical decomposition is unique, and not every separating pair of H is actually used for the decomposition (for an example, see Gagarin et al. [7]). However, algorithmically (e.g., see [8]) all the cycles of the unique decomposition are usually sliced into smaller cycles (triangles) which can be done in many different ways.

Given a separating pair $\{u, v\}$ with $d(u) \geq 3$ and $d(v) \geq 3$ in H, we find all the non-trivial minimal induced subgraphs of H with the vertices of attachment $\{u, v\}$. A new edge uv is added to each such subgraph of H and marked as a *virtual* edge. This can create multiple edges. The result is a reduction of H to a collection of subgraphs $X_1, X_2, \ldots, X_p$ containing virtual edges. Notice that, because of the virtual edges, the components $X_1, X_2, \ldots, X_p$ are not actual subgraphs of H: we have to account for the presence or absence of the virtual edges. Some of the X_i's may consist entirely of virtual edges. A linear-time algorithm for finding all the separating pairs and reducing H is described in [8]. Notice that if uv is a true edge of

H, then it is an edge of every X_i containing $\{u, v\}$.

At this stage, each X_i, $i = 1, \ldots, p$, is either 3-connected, or a cycle, possibly with some multiple edges. If X_i is a cycle containing a virtual edge uv which is not a true edge of H and contained in only one other component X_j, we combine them into $X_i \cup X_j$ and remove uv to obtain a larger component. *We do this amalgamation until every virtual edge which is contained in a cycle component X_i and not a true edge of H, is contained in at least three different components.*

Therefore, now all the cycles are either attached to a 3-connected component X_j, which can be seen as a subdivision of some of the edges of the original 3-connected component X_j, or else each virtual edge of a cycle X_i, which is not a true edge in H, is in at least three different components. The resulting subgraphs $X_1, X_2, \ldots, X_q$, $q \leq p$, are called the 3-*components* of H. Again, similarly to the algorithmic decomposition of [8], the decomposition here is not necessarily unique.

Lemma 2 *Each 3-component X_i, $i = 1, \ldots, q$, is 2-connected, and has a unique planar embedding.*

Proof: If X_i is 3-connected, this follows from Whitney's theorem [18]. It is also true if X_i is a cycle. Otherwise X_i has been formed by replacing one or more virtual edges uv of a 3-connected graph by a uv-path. ∎

The 3-components and separating pairs of H are related by a tree T similar to the block-cut-vertex tree for a separable graph. See [7] for details of the classical unique decomposition tree T. The vertices of the tree T, called the *Leroux tree*,[1] are the 3-components of H and the corresponding separating pairs $\{u, v\}$ in H. Each separating pair is incident on those X_i's in which it is contained. It is easy to see that the Leroux tree T is connected and has no cycles. Given a 3-component X_i or a virtual edge uv, let $\phi(X_i)$ and $\phi(uv)$ denote the vertices of T corresponding to them. Each leaf of T is $\phi(X_i)$ for some component X_i containing exactly one virtual edge, and no $\phi(uv)$'s are leaves in T.

Lemma 3 *Let X_i be a 3-component containing a virtual edge uv. Then $H \setminus E(X_i)$ contains a path connecting u to v.*

Proof: Consider $\phi(X_i)$ and $\phi(uv)$. The proof is by induction on ℓ, the length of a shortest path in T from $\phi(uv)$ to a leaf in T, avoiding $\phi(X_i)$. If

[1] After Pierre Leroux, who appears to be the first to recognize the importance of this decomposition tree for 2-connected graphs.

$\ell = 1$, then $\phi(uv)$ is incident on a leaf $\phi(X_j)$. Then X_j is 2-connected and has only one virtual edge, uv. Hence, it contains the path desired. Suppose $\ell > 1$. If there is a path P in T from $\phi(uv)$ to a leaf, whose internal vertices all have degree two, then we choose $\phi(X_j)$ on P adjacent to $\phi(uv)$, such that $\phi(X_j)$ contains a virtual edge $xy \neq uv$, and the distance on P from $\phi(xy)$ to a leaf is $\ell - 2$. Then the components from $\phi(X_j)$ to a leaf will contain an xy-path that belongs to H. Since X_j is 2-connected, it has a cycle containing uv and xy. Together, the cycle and the xy-path determine a uv-path in $H \setminus E(X_i)$.

If there is no path P with all internal vertices of degree two, choose $\phi(X_j)$ adjacent to $\phi(uv)$, such that $\phi(X_j)$ has distance $\ell - 1$ to a leaf. Then X_j contains a virtual edge xy such that $\phi(xy)$ has distance $\ell - 2$ to a leaf, and X_j has a cycle containing uv and xy. If there are no other virtual edges in this cycle, we proceed as above. If it contains another virtual edge wz, then we choose a shortest path in T from $\phi(wz)$ to a leaf, avoiding $\phi(X_j)$. As above we find an xy-path in H and a wz-path in H. Together with the cycle of X_j, we obtain a uv-path in $H \setminus E(X_i)$. If there are more virtual edges in the cycle, we proceed in the same way for each of them. ∎

Now, we begin with a planar embedding of each 3-component X_i, $i = 1, 2, \ldots, q$, of $S - a$. By Lemma 2, this is unique. Let $N_i(a)$ denote the subset of $N(a)$ contained in X_i. Consider those X_i's such that $N_i(a) \neq \emptyset$. Suppose first that $b \notin V(X_i)$. Then the Leroux tree T contains a unique shortest path P_i from $\phi(X_i)$ to some $\phi(X_j)$ for which $b \in V(X_j)$. Let uv be the virtual edge of X_j on P_i. It is contained in two facial boundaries F_1, F_2 of X_j, at least one of which must contain b. As in the proof of Lemma 3, consider all the paths in T containing $\phi(uv)$ and avoiding $\phi(X_j)$: they form a subtree (a branch) of T rooted at $\phi(uv)$ which we denote by $T_{\phi(uv)}$. All the 3-components of these paths must be embedded in the faces corresponding to F_1 and/or F_2.

Let $u_1v_1, u_2v_2, \ldots, u_kv_k$ be the sequence of virtual edges on P_i in $T_{\phi(uv)}$, where $u_1v_1 = uv$, and $u_kv_k = u'v'$ is in X_i. First, consider the case when only one of F_1, F_2, contains b on its boundary, without loss of generality, suppose it to be F_1. The virtual edges $u_1v_1, u_2v_2, \ldots, u_kv_k$ may or may not be true edges of $S - a$. If any of $u_1v_1, u_2v_2, \ldots, u_kv_k$ is an edge in $S - a$, the embedding must be such that when the virtual edges are deleted or embedded, the vertices of $N_i(a)$ can be on the facial boundary extending F_1 and still containing b. There is only one way to embed the 3-components on P_i in $T_{\phi(uv)}$ like this. This is so for every path in $T_{\phi(uv)}$ from $\phi(uv)$ to $\phi(X_\ell)$, where X_ℓ is any 3-component such that $N_\ell(a) \neq \emptyset$. Also, in this case, for every such X_ℓ, $N_\ell(a)$ must lie on a single facial boundary of X_ℓ containing the edge $u'v'$ on P_l. Then we add $N_\ell(a)$ to $N_j(a)$, and mark the edge u_1v_1 of X_j so that the vertices of $N_\ell(a)$ are associated with u_1v_1. Thus, when we later traverse F_1, the vertices of $N_\ell(a)$ are available through u_1v_1. However, if there are $\phi(X_i)$ and $\phi(X_\ell)$ in $T_{\phi(uv)}$, $X_i \neq X_\ell$, such that $N_i(a) \neq \emptyset$, $N_\ell(a) \neq \emptyset$ and $P_i \cup P_\ell$ has a vertex $\phi(u_mv_m)$ of degree three, where $m \in \{1, 2, \ldots, k\}$, then it is not possible to find two faces of $S - a$ containing all of $N(a)$.

If both F_1 and F_2 contain b in the embedding of X_j, there are some additional possibilities. Again, some or all of the virtual edges $u_1v_1, u_2v_2, \ldots, u_kv_k$ can be actual edges in $S - a$. If the virtual edges $u_1v_1, u_2v_2, \ldots, u_kv_k$ are not true edges of $S-a$ and $\phi(X_i)$ is a leaf of T, then $N_i(a)$ could be on two facial boundaries of X_i containing u_kv_k: one of them could appear in F_1, and the other in F_2. There can be several such leaves $\phi(X_\ell)$ in $T_{\phi(uv)}$ with $N_\ell(a) \neq \emptyset$ and $N_\ell(a)$ contained in two facial boundaries of X_ℓ. Also, there could be two distinct vertices $\phi(X_i)$ and $\phi(X_\ell)$ in $T_{\phi(uv)}$ such that $P_i \cup P_\ell$ has a vertex $\phi(u_mv_m)$ of degree three, where $m \in \{1, 2, \ldots, k\}$, and $N_i(a)$ and $N_\ell(a)$ are contained in a single face. In such cases, we assign one of $N_i(a), N_\ell(a)$ to F_1 and the other to F_2, by marking the two sides of u_1v_1 in X_j. In this way, for any X_i with $b \notin V(X_i)$ and $N_i(a) \neq \emptyset$, the vertices of $N_i(a)$ are assigned to a virtual edge in some X_j containing b.

Suppose now that $b \in V(X_i)$, X_i has two or more facial boundaries $F_{i_1}, F_{i_2}, \ldots$ which together contain all of $N_i(a)$, and X_i is not a cycle component. If F_{i_1} and F_{i_2} share a virtual edge uv that is not a true edge, then there is another component X_j also containing uv. By Lemma 3, $H \setminus E(X_i)$ contains a uv-path. Therefore, in every embedding of $S - a$, the vertices of $N_i(a)$ in F_{i_1} and F_{i_2} will be in different facial boundaries. We conclude that for any X_i with $b \in V(X_i)$ and $N_i(a) \neq \emptyset$, the vertices of $N_i(a)$ must be contained in either one or two (at most two) facial boundaries of X_i containing b. Furthermore, adding another 3-component X_k to the planar embedding of X_i cannot decrease the number of faces in the joint embedding. We use the method of Step 5 to determine if there are up to two facial boundaries of X_i containing b which contain all of $N_i(a)$. This works because of Lemma 2. This is done for each such X_i, $i = 1, 2, \ldots, q$.

Suppose that X_i and X_j are two distinct 3-components containing b such that each requires two facial cycles containing b to give all of $N_i(a)$ and $N_j(a)$. Let the two faces of X_i be F_{i_1}, F_{i_2}, and those of X_j be F_{j_1}, F_{j_2}. By Lemma 2, X_i and X_j both have unique planar embeddings. Therefore, X_j must be embedded inside a face of X_i incident on b. It is only feasible to embed X_i and X_j properly if F_{i_1}, F_{i_2} both contain a virtual edge bv, F_{j_1}, F_{j_2} also contain the same virtual edge, and $bv \notin S - a$, so that when X_i and X_j are both embedded, the facial boundaries can combine to give just two new facial boundaries. This determines F, F' completely, so that now we just walk along the facial boundaries of F and F' and add all the other components to determine whether they do in fact give a solution.

Suppose next that there is just one X_i containing b such that it requires two facial cycles containing b to give all of $N_i(a)$. Let the two facial cycles be F_{i_1}, F_{i_2}. There may be more than one possible choice for F_{i_1}, F_{i_2}, but the choices are easily determined by the sets $S_{i_1}, S_{i_2}, S_{i_3}, S_{i_4}$ of Step 5. It is possible that F_{i_1}, F_{i_2} share a virtual edge uv. We walk around each of F_{i_1} and F_{i_2} and add all the other components to determine whether they do in fact give a solution.

The last case is when there are several X_i's containing b which have only one facial cycle containing b which contains all of $N_i(a)$. Some of these may be cycles. By Lemma 2, each X_i has a unique embedding. Choose an X_i and let F_i be the unique facial boundary of X_i containing $N_i(a)$. Let bv and bw be the two edges of F_i incident on b. Suppose that bv is a virtual edge. Then it is contained in another component X_j, which also has a unique embedding. In X_j, bv may also be contained in the unique face F_j containing $N_j(a)$. If bv is not a true edge in $S-a$, then the outer face of the unique embedding of X_j can be chosen so that $F_i \cup F_j$ forms a single face of $X_i \cup X_j$. Let bw and bx be the two edges of $F_i \cup F_j$ incident on b. We repeat this argument; bx may be a virtual edge contained in a component X_k, with face F_k. There is only one way to extend the embedding to $X_i \cup X_j \cup X_k$ so that $F_i \cup F_j \cup F_k$ forms a single facial boundary containing $N_i(a) \cup N_j(a) \cup N_k(a)$, etc. Starting from any X_i, there is a unique way to extend it so as to contain as much of $N(a)$ as possible. If there are any remaining vertices of $N(a)$, choose a component not yet included, and extend it in the unique way. Eventually, when the components containing b and vertices of $N(a)$ on a unique face F are combined together, the unique face F will be split into at least two faces containing b and some or all of $N(a)$, which brings it to one of the previous cases. If F and F' exist, they will be found in this way, starting from any X_i for which $N_i(a) \neq \emptyset$.

Step 7. $S-a$ is planar and 1-connected, but not 2-connected. We want to determine if there is a planar embedding of $S-a$ with $N(a)$ contained in the boundary of exactly two faces of $S-a$ containing b. We first find the blocks and cut-vertices of $S-a$. A linear-time algorithm for finding the blocks and cut-vertices was discovered by Hopcroft and Tarjan in 1973, using a depth-first search. It is described in numerous textbooks, including [13]. This determines the block-cut-vertex tree decomposition of $S-a$. Each block B is either an edge, or a maximal 2-connected subgraph. If B is 2-connected, it is then reduced to its 3-connected components and separating pairs, as in Step 6. The 3-components of B may contain virtual edges, but no virtual edge is a cut-edge of $S-a$.

Given a cut-vertex v, consider the maximal induced subgraphs X with only v as vertex of attachment, such that v is not a cut-vertex in X. Call these subgraphs of $S - a$ the *1-connected components at* v. Now if b is not contained in any 3-component (this implies that b is not in a non-trivial block either), then b must be a cut-vertex not incident on any virtual edge. There is just one face incident on b, and we require it to contain all of $N(a)$ on its boundary. However, if all of $N(a)$ is on a single face boundary, then S would be planar. Therefore this case is not possible, and we can assume that b is in at least one 3-component.

Suppose first that b is not a cut-vertex of $S - a$. Then it is contained in just one block B, which must be 2-connected. Given any cut-vertex v in B, let $N_v(a)$ denote the vertices of $N(a)$ contained in all 1-connected components at v, excluding B. Note that all these 1-connected components can be embedded in any face of B incident on v. We now use the method of Step 6 to determine if all the vertices of $N(a)$ are contained in one or two faces of B containing b, where cut vertices v contribute $N_v(a)$ to this set. If there are two facial boundaries F, F', we walk around each in turn. If v is any cut-vertex encountered, then every 1-connected component X at v such that $N_v(a) \neq \emptyset$ must be embedded inside F or F'. This is only possible if in every block of X, all the vertices of $N(a)$ contained in the block are embedded on one face, the outer face. We check that this is so for F and F'.

Otherwise b is a cut-vertex. Let $B_1, B_2, \ldots, B_q$ denote the blocks containing b. For each B_i which is not an edge we use the method of Step 6 to determine if there is an embedding of B_i such that all of $N_i(a)$ is contained in one or two facial boundaries. There can be at most one B_i whose vertices $N_i(a)$ require two faces F, F'. In this case we proceed exactly as in the preceding paragraph, since each remaining block B_j must have all of $N_j(a)$ appearing on the boundary of a single face. B_j can be embedded inside either of F or F'. If every B_i requires only one face for $N_i(a)$, then there is an embedding in which all vertices of $N(a)$ can be placed on a single facial boundary, which is impossible.

For any subgraph Y of $S - a$, we can easily check if there is a planar embedding of Y with all the vertices of $N(a)$ in Y on a single face boundary and find such an embedding of Y as follows. First, add a dummy vertex a' to Y and connect a' to all the neighbors of a in Y to obtain Y'. Then run a planarity testing algorithm on Y'. Finally, if Y' is planar, remove a' from Y' to obtain a necessary planar embedding of Y (if one exists). Otherwise, there is no planar embedding of Y with all the vertices of $N(a)$ in Y on a single face boundary. ∎

When $S-a$ is 3-connected, this algorithm is straight-forward and fairly easy to program. If $S-a$ is not 3-connected, this greatly increases the number of cases that must be considered. Nevertheless, each step can be done in linear time in the number of edges of the input graph G. Since a toroidal graph G with n vertices has at most $3n$ edges (e.g., see [13]), the entire algorithm has a linear time complexity and can be implemented to run in $O(n)$ time.

5 Final remarks and conclusions

The algorithm presented in this paper improves algorithms presented in [6, 10, 20]. First, it avoids the numerous labelled embeddings derived from the six embeddings of K_5, each requiring a special treatment: it remains only to consider the cases of the two unlabelled embeddings of $K_{3,3}$ on the torus (providing only twenty labelled embeddings) for the algorithms of [10, 20]. Then, the algorithm of this paper considers the side components of only one K_5-subdivision TK_5 in the input graph G to determine if G is toroidal: this is simpler than the method of [6]. In that paper, a TK_5 and its side-components were constructed. If the side components were planar, and at most one augmented side-component was non-planar, then the graph was embedded. If there was a non-planar side component which also contained a TK_5 sharing two corners with the original TK_5, then toroidality could be determined. This required 19 side-components of the combined TK_5's. But if a $TK_{3,3}$ in a side component occurred, then the algorithm could not proceed. The current algorithm is able to detect a bigger class of toroidal and non-toroidal graphs than the algorithm of [6]. For example, a side component S containing a $TK_{3,3}$ might now be embeddable if $S-a$ is planar, and if a side component S has $S-a$ non-planar, then it is now known that G is not toroidal. This approach of excluding some $K_{3,3}$-subdivisions and doing decompositions as in Section 2 can likely be generalized to devise graph embedding algorithms for oriented and non-oriented surfaces of higher genus.

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