On the Maximum Degree of 3_t-Critical Graphs

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Abstract

We give general lower bounds and upper bounds on the maximum degree $\Delta(G)$ of a 3_t-critical graph G in terms of the order of G. We also establish tighter sharp lower bounds on $\Delta(G)$ in terms of the order of G for several families of 3_t-critical graphs, such as crown-graphs, claw-free graphs, and graphs with independence number $\alpha(G) = 2$.

Keywords: total domination, edge-critical, maximum degree, maximum order

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1 Introduction

In this paper we will focus our study on *simple*, *loopless*, *undirected* graphs. For a graph G, $V(G)$ and $E(G)$ will denote the *vertex set* and the *edge set* of G, respectively. Therefore, a *vertex of* G is an *element of* $V(G)$, even if, for simplicity, we occasionally write $v \in G$ rather than $v \in V(G)$.

Given vertices u and v of a graph G, we say that u is a *neighbor* of v (or is *adjacent* to v) if the edge uv is in $E(G)$. The *open neighborhood* $N(v)$ of a vertex v is the set of all the neighbors of v, while the *closed neighborhood* of v is defined by $N[v] = N(v) \cup \{v\}$. Finally, if $B \subseteq V(G)$, the set of

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vertices of B adjacent to a vertex v of G will be denoted $N_B(v)$. Clearly $N_B(v) = N(v) \cap B$.

For a given vertex v, the cardinality of $N(v)$ is known as the *degree* of v, and is denoted by $\deg(v)$. The *minimum degree* and *maximum degree* of G are defined by

$$\delta(G) = \min_{v \in V(G)} \deg(v), \qquad \Delta(G) = \max_{v \in V(G)} \deg(v),$$

respectively. The *order* of G, denoted by $|G|$ is the number of vertices of G, while the *size* of G is the number of edges of G. Finally, the *independence number* $\alpha(G)$ is the maximum number of pairwise nonadjacent vertices in G.

For sets S, $X \subseteq V(G)$, we say that S *dominates* X, and write $S \succ X$, if every vertex in $X \setminus S$ is adjacent to at least one vertex in S, while we say that S *totally dominates* X, and write $S \succ_t X$, if every vertex in $X \cup S$ is adjacent to at least one vertex in S. If $S = \{s\}$ or $X = \{x\}$, we also write $s \succ_t X$, $S \succ x$, etc, while, we say that the *edge* uv *dominates* X if $uv \in E(G)$ and $\{u, v\} \succ X$.

A *total dominating set* in a graph G is any $S \subseteq V(G)$ such that $S \succ_t V(G)$. Note that graphs with isolated vertices have no total dominating sets, while every graph G with no isolated vertices has a total dominating set, since one could trivially choose $S = V(G)$. In the latter case it makes sense to define the *total domination number* $\gamma_t(G)$ as the minimum cardinality of a total dominating set of G. Moreover, since for a disconnected graph with no isolated vertices, the total domination number is just the sum of the total domination numbers of its components, for the rest of our discussion we will assume G is a connected graph. Finally, note that for every graph G, $\gamma_t(G) \geqslant 2$. Henning [7], gives a survey of recent results on total domination.

If G is not a clique, and if $\gamma_t(G+e) < \gamma_t(G) = k$, for any natural number $k \geq 3$ and for any edge $e \notin E(G)$, then G is said to be γ_t-*critical.* Van der Merwe, Haynes, and Mynhardt [9] initiated the study of γ_t-critical graphs. Although they characterized several families of 3_t-critical graphs, much is unknown about γ_t-critical graphs, and in particular 3_t-critical graphs.

It has been shown [9] that, by adding an edge e to a graph G, the total domination number cannot drop by more than 2, that is, $\gamma_t(G) - 2 \leqslant \gamma_t(G+e) \leqslant \gamma_t(G)$. Graphs for which $\gamma_t(G+e) = \gamma_t(G) - 2$ for each $e \notin E(G)$ are called *supercritical.* It was proven in [6] that a graph is 4_t-supercritical if and only if it consists of the union of two cliques K_r, K_s with $r, s \geqslant 2$. If G is either 3_t-critical or 4_t-supercritical, for any $e \notin E(G)$

we necessarily have $\gamma_t(G+e)=2$. This fact has an interesting application in the study of *diameter* 2*-critical* graphs, namely, those graphs of diameter 2 for which the removal of any edge would increase the diameter. Indeed, Hanson and Wang [3] showed the following important relationship.

Theorem 1 *A graph is diameter* 2*-critical if and only if its complement is either* 3_t*-critical or* 4_t*-supercritical.*

The diameter of G will be denoted by diam(G). In excess of 30 years ago, Murty and Simon [1] posed the following conjecture.

Conjecture 1 *If G is a diameter 2-critical graph with order n and size m, then $m \leqslant \lfloor n^2/4 \rfloor$, with equality if and only if G is the complete bipartite graph* $K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$.

In view of Theorem 1, Conjecture 1 can be rephrased in terms of the complement of a graph, that is:

Conjecture 2 *If G is a graph with order n and size m, then*

(i) if G is 3_t-critical, then $m > \lceil n(n-2)/4 \rceil$;

(ii) if G is 4_t-supercritical, then $m = \lceil n(n-2)/4 \rceil$.

Condition (*ii*) has been proved in [3]. Recently, Condition (*i*) was settled for the two following families of 3_t-critical graphs.

Theorem 2 [4] *If G is a 3_t-critical graph of diameter 3, order n, and size m, then $m > \lceil n(n-2)/4 \rceil$.*

Theorem 3 [4] *If G is a 3_t-critical claw-free graph of order n and size m, then $m > \lceil n(n-2)/4 \rceil$.*

Note that a graph is *claw-free* if it does not contain the complete bipartite graph $K_{1,3}$ as an induced subgraph. To settle the above conjectures completely, it will be useful to further study properties of 3_t-critical graphs. In this paper we establish the maximum degree $\Delta(G)$ for some families of 3_t-critical graphs G, in terms of the order of G. In Section 2, we give general upper and lower bounds on the maximum degree for 3_t-critical graphs in terms of the order of G. For the remainder of the paper, we establish sharp upper and lower bounds on the maximum degree in terms of the order for specific families of graphs. In particular, in Section 3, we study 3_t-critical graphs G with $\alpha(G) = 2$ and in Section 4, we study crown graphs, while in Section 5, we study claw-free graphs.

2 General bounds for $\Delta(G)$

In this section we present some general upper bounds and lower bounds for $\Delta(G)$ in terms of the order n of a 3_t-critical graph G.

We start by presenting a characterization of 3_t-critical graphs that will be useful in our study. If $u, v, w \in V(G)$ and $\{u, v\} \succ_t V(G) \setminus \{w\}$, we will denote this by $uv \mapsto w$. Note that $uv \mapsto w$ and $\gamma_t(G) = 3$ imply that $uv \in E(G)$ while $uw, vw \notin E(G)$.

Proposition 4 [9] *If $\gamma_t(G) = 3$, then G is 3_t-critical if and only if for any pair of nonadjacent vertices u and v, either*

1. *$\{u, v\} \succ V(G)$, or*
2. *$uw \mapsto v$ for some $w \in N(u)$, or*
3. *$vw \mapsto u$ for some $w \in N(v)$.*

Lemma 5 [9] *If G is a 3_t-critical graph, then $|G| \geqslant 5$. Furthermore, if $|G| = 5$ then $G = C_5$, the cycle on five vertices.*

Cockayne et al. [2] showed that if a graph G is connected and $\Delta(G) < |G| - 1$, then $\gamma_t(G) \leqslant |G| - \Delta(G)$. This result leads to the following observation.

Observation 6 [9] *Any graph G with $\gamma_t(G) = 3$ has $\Delta(G) \leqslant |G| - 3$. More generally, $\Delta(G) \leqslant |G| - \gamma_t(G)$.*

Although there are many examples of 3_t-critical graphs G with $\Delta(G) = |G| - 3$, only one of them is regular, as the following proposition shows.

Proposition 7 *If G is a 3_t-critical $(|G| - 3)$-regular graph, then $G = C_5$.*

Proof. The complement $\overline{G}$ is 2-regular, and by Theorem 1, has diameter 2. Therefore $\overline{G} = C_n$ for some $n \leqslant 5$. In view of Lemma 5, we can conclude that $n = 5$ and that $G = C_5$. □

Theorem 8 *If G is a 3_t-critical graph with $|G| > 5$, then $\Delta(G) \geqslant \lceil \frac{1}{2}|G| \rceil$.*

Proof. Let $n = |G|$ and $k = \Delta(G)$. If $\alpha(G) = 2$, by [9] $\mathrm{diam}(G) = 2$, so that any two nonadjacent vertices u, v share at least one neighbor. If $n \geqslant 2k+2$, or if $n = 2k+1$ but $\deg(v) < k$, u and v cannot have common neighbors, a contradition. Thus, let G be k-regular with $n = 2k+1$, which implies $k > 3$. Let u and v be nonadjacent and let w be their only common neighbor. Partition $V(G) \setminus \{u, v, w\}$ into disjoint sets $A = N(u) \setminus \{w\}$, $B = N(v) \setminus \{w\}$. It is easy to see that the subgraphs induced by A and by B are cliques of cardinality $k-1$. Vertex w is nonadjacent to some $a_1 \in A$ (otherwise $\deg(w) \geqslant k+1$). If w is adjacent to some $a_2 \in A$, then w and a_1 share two neighbors (u and a_2), a contradiction. Thus, w is nonadjacent to any vertex in A and, with similar reasoning, to any vertex in B, which implies $\deg(w) = 2 < k$, again a contradiction.

We may now assume $\alpha(G) \geqslant 3$, and, by way of contradiction, suppose that $n \geqslant 2k+1$. Since $\alpha(G) \geqslant 3$, let u and w be nonadjacent vertices such that $\{u, w\} \not\succ V(G)$. By Proposition 4, and without loss of generality, we may assume there is a vertex $v \in V$ such that $uv \mapsto w$.

We partition $V \setminus \{u, v, w\}$ into sets

$$A = N(u) \setminus N(v), \qquad B = N(v) \setminus N(u), \qquad C = N(u) \cap N(v).$$

It is easy to verify that $C = \emptyset$, $|A| = |B| = k-1$, and therefore $n = 2k+1$, which also implies $k \geqslant 3$. Since $\deg(w) \leqslant k < 2k-2 = |A|+|B|$, we may assume, without loss of generality, that there is an $a \in A$ that is nonadjacent to w. Since v is nonadjacent to w, $\{a, v\} \not\succ V(G)$ and by Proposition 4, there is a vertex x such that $ax \mapsto v$ or $vx \mapsto a$. If $ax \mapsto v$, then $x \in A$. But then $\deg(a) + \deg(x) \geqslant 2k+1$, since $u \in (N(a) \cap N(x))$, a contradiction. Assume then that $vx \mapsto a$. Then $x \in B$ and $x \succ A \setminus \{a\}$. Also, x is nonadjacent to any vertex in $B \setminus \{x\}$. Then $\{u, x\} \not\succ V(G)$ and by Proposition 4, there is a vertex y such that $xy \mapsto u$ or $vy \mapsto x$. If $xy \mapsto u$, then $y = w$ since x is nonadjacent to any vertex in $B \setminus \{x\}$. But then $\{x, y\} \not\succ_t a$ and hence, $xy \not\mapsto u$. If $uy \mapsto x$, then $y = a$ since $x \succ A \setminus \{a\}$. But then $\{u, y\} \not\succ_t w$, a contradiction. Hence, $n \leqslant 2k$. □

Combining Observation 6 and Theorem 8, we have the following theorem.

Theorem 9 *If G is a 3_t-critical graph, then* $\lceil \frac{1}{2}|G| \rceil \leqslant \Delta(G) \leqslant |G| - 3$.

3 3_t-critical graphs G with $\alpha(G) = 2$.

In this section we show that the additional condition $\alpha(G) = 2$ allows us to improve the general lower bound provided by Theorem 8. This family of 3_t-critical graphs is characterized in [9], where it is also established that $\mathrm{diam}(G) = 2$. In particular, since when $\alpha(G) = 2$, every pair of nonadjacent vertices dominates G, by Proposition 4 we have

Lemma 10 *If $\gamma_t(G) = 3$ and $\alpha(G) = 2$, then G is 3_t-critical.*

Another straightforward yet important fact that will be used several times is the following.

Lemma 11 *If G is a 3_t-critical graph with $\alpha(G) = 2$, then two vertices are adjacent if and only if they share a common nonneighbor.*

In particular, if u and v are nonadjacent vertices, we have

$$n = \deg(u) + \deg(v) - |N(u) \cap N(v)| + 2. \tag{1}$$

Theorem 12 *If G is a 3_t-critical graph with $\alpha(G) = 2$, then $\Delta(G) \geqslant \lceil \frac{3}{5}|G| - 1 \rceil$.*

Proof. Let $n = |G|$ and $k = \Delta(G)$. Among all pairs of nonadjacent vertices in G, let (u, v) be a pair maximizing $\deg(u) + \deg(v)$. Note that $\alpha(G) = 2$ implies that $\deg(u) + \deg(v)$ is maximum exactly when $|N(u) \cap N(v)|$ is maximum. We then partition $V(G) \setminus \{u, v\}$ into the three subsets

$$A = N(u) \setminus N(v), \qquad B = N(v) \setminus N(u), \qquad C = N(u) \cap N(v).$$

Furthermore

$$|A| + |C| = \deg(u) \leqslant k, \qquad |B| + |C| = \deg(v) \leqslant k, \tag{2}$$

and that, by Lemma 11, any two vertices in A are necessarily adjacent, having v as a common nonneighbor. A similar reasoning applies to B, so that both A and B are complete.

Since, as noted above, G has diameter 2, C is necessarily nonempty. Thus, let $c_1 \in C$. Since $\{c_1, v\} \succ V(G) \setminus A$, while $\{c_1, v\} \not\succ V(G)$, there must be a vertex $a_1 \in A$ that is nonadjacent to c_1. Similarly, there is a vertex $b_1 \in B$ that is nonadjacent to c_1. Since every pair of nonadjacent vertices dominates G, $\{a_1, c_1\} \succ V(G)$ and $\{b_1, c_1\} \succ V(G)$. Since a_1 and v are nonadjacent, by the maximality of $\deg(u) + \deg(v)$, it follows $\deg(a_1) \leqslant \deg(u)$. Therefore, since $a_1 \succ A \setminus \{a_1\}$, we have

$$|N_B(a_1)| = \deg(a_1) - (|A| - 1) - 1 - |N_C(a_1)| \leqslant \deg(a_1) - |A| \leqslant \deg(u) - |A| = C.$$

Also, since $v \in N(b_1) \cap N(c_1)$, we have $N_B(c_1) \subsetneq N(b_1) \cap N(c_1)$, so that

$$|N_B(c_1)| < |N(b_1) \cap N(c_1)| \leqslant |N(u) \cap N(v)| = |C|.$$

Finally, since $\{a_1, c_1\} \succ B$, we have

$$|B| \leqslant |N_B(a_1)| + |N_B(c_1)| \leqslant 2|C| - 1. \tag{3}$$

From (2) and (3), we then obtain

$$\begin{aligned} n &= |A| + |B| + |C| + 2 \\ &= \frac{3(|A| + |C|) + 2(|B| + |C|) + |B| - 2|C| + 6}{3} \\ &\leqslant \frac{5k+5}{3} = \frac{5}{3}(k+1), \end{aligned}$$

which completes the proof. □

By combining Proposition 6 with Theorem 12 and Theorem 8 we can determine a range of possible values of $\Delta(G)$ in terms of the order n, as presented in Table 1.

At this point a natural question is whether, according to Table 1, for any possible pair of values (n, k) there exist 3_t-critical graphs with order n and $\Delta(G) = k$. In general, we do not have a definitive answer, as, for instance, we currently have no examples for which $(n, k) = (12, 6)$. Note that, if in this last case an example exists, by Proposition 12 it must necessarily satisfy $\alpha(G) > 2$. For the case $\alpha(G) = 2$, however, a positive answer is provided in Theorem 15.

We say that vertices u and v of G are *twin vertices* (or *duplicate vertices*) if $N[u] = N[v]$. A graph G' is said to be obtainable from G by *vertex duplication* if G' has twin vertices u and v such that $G = G' - v$.

The proof of the following is a simple routine and is left to the reader.

n	$\lceil\frac{1}{2}n\rceil$	$\lceil\frac{3}{5}n-1\rceil$	$n-3$	k
5	-	2	2	2
6	3	3	3	3
7	4	4	4	4
8	4	4	5	4,5
9	5	5	6	5,6
10	5	5	7	5,6,7
11	6	6	8	6,7,8
12	6	7	9	6,7,8,9

Table 1: Possible values for $k = \Delta(G)$ in terms of the order n.

Lemma 13 *Let G be a graph with $\alpha(G) = 2$, and let G' be obtainable from G by vertex duplication. Then*

i. $\alpha(G') = 2$;

ii. *G' is 3_t-critical if and only if G is 3_t-critical.*

We observe that a similar result does not hold when $\alpha(G) > 2$. This fact will be discussed in Section 5 and involves the definition of *duplicable vertices* (see Figure 5 for an example).

A graph G is said to be a *5-cycle-type graph* if G can be obtained from the 5-cycle C_5 through a sequence of vertex duplications. In other words, a 5-cycle-type graph may be viewed as a 5-cycle in which each vertex has been replaced by a clique of suitable size. In particular, if the sizes of these cliques are $c_1, \ldots, c_5$, we will write $G = C(i_1, i_2, i_3, i_4, i_5)$.

The following statement can be proved by direct inspection.

Proposition 14 *If $G = C(i_1, i_2, i_3, i_4, i_5)$ is a 5-cycle-type graph, then*

i. $|G| = \sum_{j=1}^{5} i_j$,

ii. $\Delta(G) = \max_j(i_{j-1} + i_j + i_{j+1}) - 1$,

iii. $\alpha(G) = 2$,

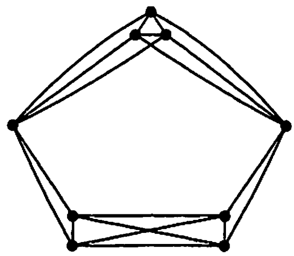

Figure 1: The graph $C(3,1,2,2,1)$.

iv. G is 3_t-critical,

where indices have to be considered modulo 5.

Theorem 15 *Let $n, k \in \mathbb{N}$ such that $n \geqslant 5$ and $\lceil \frac{3n}{5} - 1 \rceil \leqslant k \leqslant n-3$. Then there exists a 3_t-critical graph G of order n with $\Delta(G) = k$ and $\alpha(G) = 2$.*

Proof. Using Proposition 14, it is easy to show that $G = C(n-4p, \lfloor p \rfloor, \lceil p \rceil, \lceil p \rceil, \lfloor p \rfloor)$ where $p = \frac{n-k-1}{2}$ satisfies all of the requirements. This check is left to the reader. □

Another interesting property of 5-cycle-type graphs is that they exhaust the class of 3_t-critical graphs with $\alpha(G) = 2$ and $\Delta(G) = |G| - 3$, as stated in the following theorem.

Theorem 16 *If G is a 3_t-critical graph with $\alpha(G) = 2$ and $\Delta(G) = |G| - 3$, then G is a 5-cycle-type graph.*

Proof. Let u be a vertex of maximum degree in G, so that $V(G) \setminus N(u)$ consists of exactly two vertices, say v and w. We first note that $N(v) \cap N(w) = \varnothing$. Indeed, if $x \in N(v) \cap N(w)$, we have $\{u, x\} \succ_t V(G)$, which is a contradiction. We can then partition $V(G)$ in the three subsets

$$A = N(v) \setminus \{w\}, \qquad B = N(w) \setminus \{v\}, \qquad D = V(G) \setminus (A \cup B \cup \{v\} \cup \{w\}).$$

Since all the vertices in $A \cup D$ share the common nonneighbor w, by Lemma 11 we conclude that $A \cup D$ is a clique. Similarly for $B \cup D$. In particular, this fact implies that A, B, and D are all nonempty. Indeed, if $A = \varnothing$, then $B \cup D$ would be a clique of order $n-2$, which, because of $\Delta(G) = n-3$, would easily imply that G is disconnected.

In addition, for any $x \in A$, $y \in B$ we have $x \succ A \cup D \cup \{v\}$, $y \succ B \cup D \cup \{w\}$, which implies $\{x, y\} \succ V(G)$, and therefore x and y are nonadjacent. Thus we conclude that $N[x] = A \cup D \cup \{v\}$ for any $x \in A$, that is, all vertices in A are twins. Similarly, all vertices in B are twins and the same holds for all vertices in D are twins. Since by removing all duplicates we are down to a 5-cycle, we conclude that G is a 5-cycle-type graph. □

Another interesting class of 3_t-critical graphs with $\alpha(G) = 2$ is the subclass of circulant graphs $C_{(n)} = Ci_n\left(1, 2, \ldots, \frac{n-2}{3}\right)$, for any $n \equiv 2 \pmod 3$. These graphs will be considered again in the next section.

We observe that, since $C_{(5)} = C_5$, we can extend this class of circulant graphs by vertex duplication to contain the class of 5-cycle-type graphs. However, there are still examples of 3_t-critical graphs G with $\alpha(G) = 2$ that cannot be obtained from any circulant graph of the form $C_{(n)}$ through vertex duplication, as, for instance, the complement of the Petersen graph.

Figure 2: The graphs $C_{(8)}$ and the complement of Petersen graph

4 Crown graphs

In this section we continue investigating how additional conditions on the structure of a 3_t-critical graph G may improve the general lower bound provided by Theorem 8. We here study the class of crown graphs defined in [9]. A graph G is said to be a *crown graph* if G is 3_t-critical and for any pair of nonadjacent vertices u and v in G, there exist vertices x and y in G such that $ux \mapsto v$ and $vy \mapsto u$. We note that if G is a 3_t-critical graph with nonadjacent vertices u and w in G, and $uv \mapsto w$, for some vertex v, then we have

$$|G| = \deg(u) + \deg(v) - |N(u) \cap N(v)| + 1. \tag{4}$$

Lemma 17 *If G is a crown graph, then $|G| \leqslant \Delta(G) + \frac{1}{2}\delta(G) + 2$.*

Proof. Let $n = |G|$, $k = \Delta(G)$, $\delta = \delta(G)$, and let u be a vertex of degree δ. Let A be the set of neighbors of u, and let B be the set of nonneighbors of u. Then $|B| = n - \delta - 1$, so that we can write $B = \{b_1, \ldots, b_{n-\delta-1}\}$. For each $b_i \in B$ there is at least one distinct vertex $a_i \in A$ such that $ua_i \mapsto b_i$. Let b_r and b_s be vertices in B such that $b_r b_s \mapsto u$. Note that $ua_i \mapsto b_i$ implies that a_i, $i \neq r, s$, is adjacent to both b_r and b_s. Thus, b_r and b_s have at least $n-\delta-3$ common neighbors in A, that is $|N(b_r) \cap N(b_s)| \geqslant n-\delta-3$. By (4) we have

$$n \leqslant \deg(b_r) + \deg(b_s) - (n - \delta - 3) + 1 \leqslant 2k - n + \delta + 4,$$

which yields $n \leqslant k + \frac{1}{2}\delta + 2$. □

Since $\delta(G) \leqslant \Delta(G)$, Lemma 17 can be adapted to provide a better lower bound for $\Delta(G)$ in terms of the order n in the case of crown graphs.

Corollary 18 *If G is a crown graph, then $\Delta(G) \geqslant \left\lceil \frac{2}{3}(|G| - 2) \right\rceil$.*

When equality holds in Corollary 18, then Lemma 17 yields $\delta(G) = \Delta(G)$ so that we have the following interesting result.

Corollary 19 *If G is a crown graph with $\Delta(G) = \left\lceil \frac{2}{3}(|G| - 2) \right\rceil$, then G is regular.*

We can also establish a lower bound for $\delta(G)$ in terms of $|G|$, namely:

Lemma 20 *If G is a crown graph, then $\delta(G) \geqslant \left\lceil \frac{1}{2}(|G| - 1) \right\rceil$.*

Proof. Let u be an arbitrary vertex in G, and define A as the set of neighbors of u and B as the set of nonneighbors of u. As noted in the proof of Lemma 17, for every vertex $b \in B$ there is a distict vertex $a \in A$ such that $ua \mapsto b$. Therefore, $|A| \geqslant |B|$, which implies $\deg(u) \geqslant \frac{1}{2}(|G| - 1)$. □

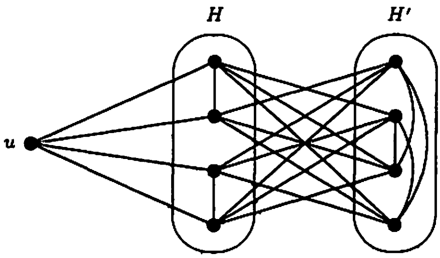

Figure 3: Crown graph with $\delta(G) = \frac{1}{2}(|G| - 1)$

The inequality in Lemma 20 is sharp for $|G|$ arbitrarily large, as shown by the following construction.

For any $r \geqslant 1$ define $V(G) = \{u, x_1, \ldots, x_r, y_1, \ldots, y_r\}$. Define the subgraph H of G induced by $\{x_1, \ldots, x_r\}$ in any way as long as no edge in H dominates H (see Figure 3 for an example). Define the subgraph H' induced by $\{y_1, \ldots, y_r\}$ by requiring $(y_i, y_j) \in E(H')$ if and only if $(x_i, x_j) \notin E(H)$. We then have $H' \sim \overline{H}$. Connect all the vertices in H with all the vertices in H' except for the perfect matching (x_i, y_i), $i = 1, \ldots, r$. Finally, connect u with all the vertices in H. Simple inspection shows that G is a crown graph. Since $|G| = 2r + 1$ and $\delta(G) = r$, then $\delta(G) = \frac{1}{2}(|G| - 1)$.

When, in Lemma 20, equality is attained, the result of Corollary 18 can be further improved as follows.

Corollary 21 *If G is a crown graph with $\delta(G) = \frac{1}{2}(|G| - 1)$, then $\Delta(G) \geqslant \left\lceil \frac{3|G|-7}{4} \right\rceil$.*

Proof. From Lemma 17 we have

$$|G| \leqslant \Delta(G) + \frac{1}{4}(n-1) + 2$$

which is equivalent to $\Delta(G) \geqslant \frac{3|G|-7}{4}$. □

By combining Proposition 6 with Corollary 18 and Corollary 21, we can determine a range of possible values of $\Delta(G)$ in terms of the order n as presented in Table 2.

The upper bound provided by Observation 6 is sharp within the class of crown graphs and for $|G|$ arbitrarily large. Indeed, for every odd number

n	$\lceil\frac{2(n-2)}{3}\rceil$	$\lceil\frac{3n-7}{4}\rceil$	$n-3$	k
5	2	2	2	2
6	3	3	3	3
7	4	4	4	4
8	4	5	5	4,5
9	5	5	6	5,6
10	6	6	7	6,7
11	6	7	8	6,7,8
12	7	8	9	7,8,9

Table 2: Possible values for $k = \Delta(G)$ and order n for crown graphs.

$n \geqslant 5$ a crown graph with order n and $\Delta(G) = n-3$ can be constructed by defining $V(G) = \{u, x_1, \ldots, x_r, y_1, \ldots, y_r, v, w\}$, where $r = \frac{n-3}{2}$. Define the subgraph H of G induced by $\{x_1, \ldots, x_r\}$, and define the subgraph H' induced by $\{y_1, \ldots, y_r\}$ by requiring $(y_i, y_j) \in E(H')$ if and only if $(x_i, x_j) \notin E(H)$. We then have $H' = \overline{H}$ (see Figure 4 for an example). Connect all the vertices in H with all the vertices in H' except for the perfect matching (x_i, y_i), $i = 1, \ldots, r$. Connect u with all the vertices in H and H', connect v with w and with all the vertices in H, and connect w with all the vertices in H'. The graph G can be shown to be 3_t-critical with order n and $\Delta(G) = n-3$.

The lower bound provided in Corollary 18 is also sharp for n arbitrarily large. Indeed, it is sufficient to observe that the circulant graphs $C_{(n)}$ defined in Section 3 are crown graphs with $\Delta(G) = \frac{2(n-2)}{3}$.

Finally, also the lower bound given in Corollary 21 is sharp for n arbitrarily large. Indeed, it is sufficient to modify the construction presented after Lemma 20 by requiring H to be a $\lceil\frac{r-2}{2}\rceil$-regular graph. This construction leads to a graph G with $n = 2r+1$, $\Delta(G) = \lceil\frac{3r-2}{2}\rceil$, that yields $\Delta(G) = \lceil\frac{3n-7}{4}\rceil$.

5 Claw-free graphs

A graph G is said to be *claw-free* if it does not contain $K_{1,3}$ as an induced subgraph. As 5-cycle-type graphs are claw-free, in view of Theorem 15 we

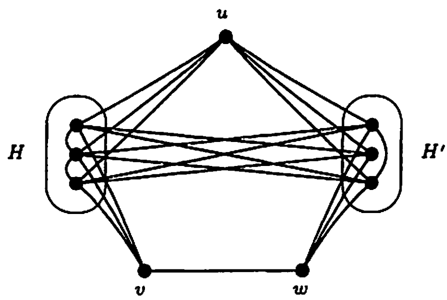

Figure 4: A crown graph with $\Delta(G) = |G| - 3$.

do not expect claw-freeness to lead to any significant improvement in lower and upper bounds for $\Delta(G)$, in general. However, as long as $\alpha(G) > 2$, the lower bound may be improved as shown in the following result.

Theorem 22 *If G is a 3_t-critical claw-free graph with $\alpha(G) > 2$, then* $\Delta(G) \geqslant \lceil \frac{2}{3}(|G| - 2) \rceil$.

Proof. We can write $\Delta(G) = 2m + r$, for $r \in \{0, 1\}$. Note that, if, by way of contradiction, $\Delta(G) < \lceil \frac{2}{3}(|G| - 2) \rceil$, we then have $|G| \geqslant 3m + r + 3$.

Since $\alpha(G) > 2$, G has nonadjacent vertices u and w such that $\{u, w\} \not\succ V(G)$, and by Proposition 4, without loss of generality, we may assume that $uv \mapsto w$ for some vertex v. We then partition $V(G) \setminus \{u, v, w\}$ into sets

$$A = N(u) \setminus N(v), \qquad B = N(v) \setminus N(u), \qquad C = N(u) \cap N(v).$$

The proof proceeds now by steps as follows.

1. $|A| + |C| \leqslant 2m + r - 1$; $|B| + |C| \leqslant 2m + r - 1$.
   Indeed $|A| + |C| = \deg(u) - 1 \leqslant k - 1 = 2m + r - 1$. Similarly for $|B| + |C|$.

2. $|A| \geqslant m + 1$; $|B| \geqslant m + 1$; $|C| \leqslant m + r - 2$.
   Indeed, by step 1 we can write $|A| = n - (|B| + |C|) - 3 \geqslant (3m + r + 3) - (2m + r - 1) - 3 = m + 1$. The computations for $|B|$ and for $|C|$ are along the same line.

3. *If $xy \mapsto z$ for $x, y, z \in V(G)$, then x and y have at most $m + r - 2$ common neighbors.*

This fact comes directly from $|C| \leqslant m+r-2$, with x, y, z in place of u, v, w.

4. *A and B are complete graphs (cliques).*

   If $a_1, a_2 \in A$ and $a_1a_2 \notin E(G)$, then $\{u, v, a_1, a_2\}$ induces a claw. A similar reasoning applies for B.

5. *There exist $a \in A$ and $x \in C$ such that $aw \notin E(G)$, $x \succ (A \setminus \{a\}) \cup \{w\}$, $w \succ (A \setminus \{a\}) \cup \{x\}$, and $vx \mapsto a$.*

   Actually, this can be claimed without loss of generality. Indeed, since, by step 2, $\deg(w) \leqslant 2m+r < 2m+2 \leqslant |A|+|B|$, without loss of generality, we can assume that there is a vertex $a \in A$ that is nonadjacent to w. Note that $\{a, v\} \not\succ V(G)$, since neither a nor v is adjacent to w. Then by Proposition 4, there is a vertex x such that $ax \mapsto v$ or $vx \mapsto a$.

   - If $ax \mapsto v$, then $x \in A$. Since, by step 4, $A \cup \{u\}$ is complete, then a and x have at least $|A|-2+1 \geqslant m$ common neighbors, which, by step 3, is a contradiction.
   - If $vx \mapsto a$, then $x \in B$ or $x \in C$. If $x \in B$, then since B is complete, with a similar reasoning as above, we conclude that x and v have at least m common neighbors in B, which is again a contradiction.

   Thus $vx \mapsto a$ and $x \in C$. In particular $x \succ (A \setminus \{a\}) \cup \{w\}$. Finally, if w is nonadjacent to a vertex $a' \in A \setminus \{a\}$, then $\{x, a', w, v\}$ induces a claw. Therefore w must be adjacent to every vertex in $A \setminus \{a\}$.

6. *There exist $b \in B$ and $x' \in C$ such that $bw \notin E(G)$, $x' \succ (B \setminus \{b\}) \cup \{w\}$, $w \succ (B \setminus \{b\}) \cup \{x'\}$, and $ux' \mapsto b$.*

   As shown in step 5, w has at least $m+1$ neighbors in $A \cup C$. Since $\deg(w) \leqslant 2m+r$, then w has at most m neighbors in B. So there exists $b \in B$ with $bw \notin E(G)$. Using similar reasoning to step 5 for set A, we can show that there exists $x' \in C$ such that $ux' \mapsto b$, $x' \succ (B \setminus \{b\}) \cup \{w\}$, and $w \succ (B \setminus \{b\}) \cup \{x'\}$.

7. $x = x'$.

   It suffices to observe that, by steps 5 and 6, $w \succ (A \setminus \{a\}) \cup (B \setminus \{b\}) \cup \{x, x'\}$. If x and x' are distinct, then $\deg(w) \geqslant 2m+2 > 2m+r = \Delta(G)$.

8. $\deg(x) \geqslant 2m+3$, *that is, a contradiction.*

   Since $x = x'$, by steps 5 and 6 we now have $x \succ (A \setminus \{a\}) \cup (B \setminus \{b\}) \cup \{u, v, w\}$, and by step 2 we conclude that $\deg(x) \geqslant 2m+3$.

□

In Theorem 24 we will show that the lower bound in Theorem 22 is actually sharp. To facilitate this, we need to extend some terminology introduced in Section 3 about duplicate vertices. In particular we say that a vertex u of a 3_t-critical graph G is *duplicable* if, for any $v \notin N(u)$ either $\{u,v\} \succ V(G)$ or $ux \mapsto v$ for some vertex x. When $\alpha(G) = 2$, or when G is a crown graph, every vertex is automatically duplicable. However, observe that in the graph presented in Figure 5 only vertices u, v, and w are duplicable.

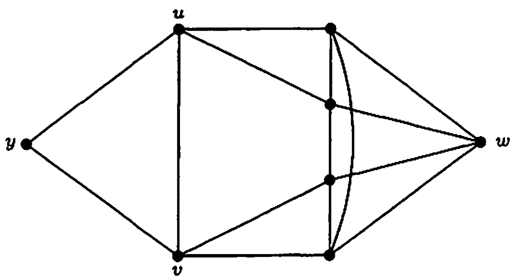

Figure 5: A 3_t-critical graph with nonduplicable vertices.

By direct inspection one can easily prove the following.

Lemma 23 *Let G be a 3_t-critical graph and let u be a duplicable vertex of G. Then the graph G' obtained from G by duplicating u is 3_t-critical. Furthermore, u is still duplicable in G'.*

For claw-free graphs we have the following theorem, similar to Theorem 15 obtained in the case $\alpha(G) = 2$.

Theorem 24 *Let $n, k \in \mathbb{N}$ such that $n \geqslant 8$ and $\lceil \frac{2}{3}(n-2) \rceil \leqslant k \leqslant n-3$. Then there exists a 3_t-critical claw-free graph G of order n with $\Delta(G) = k$ and $\alpha(G) > 2$.*

Proof. We first consider the case $\lceil \frac{2}{3}(n-2) \rceil \leqslant k \leqslant n-4$. We already observed that, if G is the graph in Figure 5, then G is 3_t-critical, $\alpha(G) > 2$, with u, v, and w as duplicable vertices. We then define a graph G' obtained through vertex duplication by replacing u, v, and w by cliques K_r, K_s, and K_t respectively, where $r = \lceil \frac{k-2}{2} \rceil$, $s = \lfloor \frac{k-2}{2} \rfloor$, and $t = n-k-3$. In particular, the condition $\lceil \frac{2}{3}(n-2) \rceil \leqslant k \leqslant n-4$ implies $r \geqslant s \geqslant t \geqslant 1$.

By Lemma 23, G' is 3_t-critical. In addition G' is claw-free and $\alpha(G') > 2$. We can see that $n = r + s + t + 5$. Moreover, vertex y has degree $r + s$; vertices in $K_r \cup K_s$ have degree $r+s+2 = k$; the four vertices inducing the 4-cycle have degree either $r + t + 2$ or $s + t + 2$; vertices in K_t have degree $t+3 = n - k \leqslant k$ (since $k \geqslant \lceil \frac{2}{3}(n-2) \rceil \geqslant \frac{n}{2}$ when $n \geqslant 8$). Since $r \geqslant s \geqslant t$, the maximum degree $\Delta(G)$ is attained by the vertices in $K_r \cup K_s$ and is equal to k.

In order to address the case $k = n-3$, we first observe that the graph G in Figure 6 is a claw-free 3_t-critical graph with $\alpha(G) > 2$. In addition, $\Delta(G) = |G| - 3$. Since u, v, and w are duplicable vertices, by vertex duplicating (any of) these vertices a suitable number of times, we can attain a graph G' of order n which is still 3_t-critical, claw-free, and satisfies $\alpha(G') > 2$ and $\Delta(G') = |G'| - 3$. □

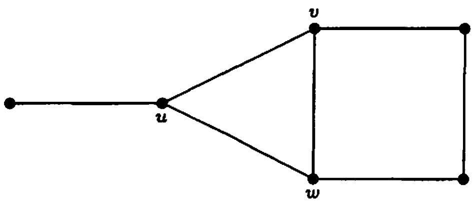

Figure 6: Claw-free 3_t-critical graph with $\alpha(G) = 3$, and $\Delta(G) = |G| - 3$.

6 Future developments

While for the case $\alpha(G) = 2$, as well for crown graphs and for claw-free graphs, we established sharp lower bounds on $\Delta(G)$, we still lack a sharp lower bound for the general case, as we think that the bound $\lceil \frac{1}{2}|G| \rceil$ provided in Theorem 8 is not sharp, starting, probably, from $|G| = 12$.

It is also somehow bizarre that in the general case, the additional condition $\alpha(G) = 2$ yields a tighter lower bound (Theorem 12), while, within the class of claw-free graphs, the condition $\alpha(G) > 2$ is the one that provides the tighter lower bound (Theorem 22). This fact may be due, again, to the lack of a sharp lower bound for the general case, which, at this point we really think should be studied and possibly determined.

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