A census of edge-transitive planar tilings

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Dedicated to Gert Sabidussi[1]

Abstract

Recently Graves, Pisanski and Watkins have determined the growth rates of Bilinski diagrams of one-ended, 3-connected, edge-transitive planar maps. The computation depends solely on the edge-symbol $\langle p, q; k, l\rangle$ that was introduced by B. Grünbaum and G. C. Shephard in their classification of such planar tessellations. We present a census of such tessellations in which we describe some of their properties, such as whether the edge-transitive planar tessellation is vertex- or face-transitive, self-dual, bipartite or Eulerian. In particular, we order such tessellations according to the growth rate and count the number of tessellations in each subclass.

1 Introduction

We are interested in planar maps that are 3-connected and dually 3-connected and *one-ended*, i.e., the deletion of no finite subgraph leaves two or more

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infinite components. Our maps may be finite or infinite, but *locally finite*, i.e., all valences and covalences are finite. It is well known that for every such map, any automorphism of its underlying graph is extendable to a homeomorphism of the plane. Such maps are called *planar tessellations* [7] or *planar tilings.*

We are interested in the "rate of growth" of such maps. This notion may be defined is several equivalent ways. We count the number of vertices adjacent with a root vertex, and then we count the number of vertices adjacent with these vertices, and so on *ad infinitum.* Our measure of this rate of growth is the limit of the ratio D_{n+1}/D_n as n approaches infinity, when such a limit exists, where D_n is the number of of such vertices in the first n layers around the root. For growth in rooted graphs, see for instance [14].

Each edge-transitive map is also *edge-homogeneous*, i.e., there exists a 4-tuple $\langle p, q; k, \ell \rangle$ of integers ≥ 3, called the *edge-symbol* of the map, such that for each edge, p and q are the valences of its two incident vertices and k and ℓ are the covalences of its two incident faces. We will call these maps (planar edge-transitive) tessellations.

Note that finite planar tessellations tile the sphere, while infinite planar tessellations may tile the Euclidean or hyperbolic plane.

In case of planar tessellations, the class of edge-transitive tessellations coincides with the class of edge-homogenous tessellations. This is not the case for maps on surfaces other than plane or sphere. There exist maps that are edge-homogeneous and are not edge-transitive. For instance the standard quadrilateral embedding of the cartesian product of two cycles in the torus defines an edge-homogeneous map with the edge-symbol $\langle 4, 4; 4, 4 \rangle$ that is not edge-transitive unless both cycles are of the same length. It was Alen Orbanić who reminded us of this simple example.

2 Preliminaries

For the tessellations considered in this article, Grünbaum and Shephard [8] have obtained the following strong result.

Theorem 1 *[8] There exists an edge-homogeneous planar tessellation with valences p and q and covalences k and ℓ if and only if p, q, k, ℓ are integers ≥ 3 and exactly one of the following holds:*

(1) all of p, q, k, ℓ are even;

(2) $k = \ell$ even, and at least one of p, q is odd;

(3) $p = q$ even, and at least one of k, ℓ is odd;

(4) $p = q$, $k = \ell$, and all are odd.

Such a tessellation is edge-transitive.

(5) If $p = q$, then it is vertex-transitive.

(6) If $k = \ell$, then it is face-transitive.

(7) If $p = k$ and $q = \ell$ the map is self-dual.

(8) If p, q are even, the map is Eulerian,

(9) if k, ℓ are even, the map is bipartite.

Finally, the parameters p, q, k, ℓ determine the map up to homeomorphism of the plane.

For a detailed classification of all edge-transitive planar maps, see [6].

Now we can give a definition of the growth rate Γ of a rooted graph G_v. Let v be a root of graph G. Let d_i denote the number of vertices at distance i from the root v. Let $D_n = \sum_{i=0}^{n} d_i$ and

$$\Gamma(G_v) = \lim_{n\to\infty} \frac{D_{n+1}}{D_n}$$

There are all kinds of problems with the growth rate. It may happen that the limit does not exist. If it exists it may depend on the root vertex. Luckily, for edge-transitive planar tessellations the growth rate is independent of the root.

In [7] S. Graves, T. Pisanski and M. E. Watkins have determined the growth rate $\Gamma(p, q, k, \ell)$ of any tessellation with the edge-symbol $\langle p, q, k, \ell\rangle$. It turns out that the growth rate can be expressed in terms of a single parameter

$$t = ((p+q)/2 - 2)((k+\ell)/2 - 2).$$

Let us define the function

$$G(t) = (t - 2 + \sqrt{(t(t-4))})/2.$$

As shown in [7] the growth rate can be computed in a seemingly more complicated way, via the so-called *Bilinski diagrams.* The reader is referred to [7] for a definition of a Bilinski diagram of a map and its growth rate. For our purposes these concepts are not needed.

Theorem 2 *([7]) Let M be an edge-transitive map with the edge-symbol $\langle p,q;k,\ell\rangle$ or $\langle k,\ell,p,q\rangle$, $p \le q$ and $k \le \ell$ satisfying the conditions of Grünbaum and Shephard. Then*

(a) If $\Gamma(p,q,k,\ell) = \Gamma(t) = G(t)$, if either $p,q,k,\ell \ge 4$ or $p = 3, q \ge 4, k \ge 6$, even, or $p = q = 3, k = \ell \ge 6$.

(b) If $\Gamma(p,q,k,\ell) = \Gamma(t) = G(t-1)$, if $p = 3, q \ge 6, k = \ell = 4$.

There are nine finite tessellations not covered by this Theorem. There are also five Euclidean tessellations with growth rate 1. In addition to these 14 exceptional cases all other growth rates are given by the set $\{G(t) | t = 5, 6, \ldots\}$ and are irrational numbers. The smallest exponential growth rate is $(3+\sqrt{5})/2$ and there are 6 tessellations attaining this growth rate. The next one is $(2+\sqrt{3})$ with 10 tessellations.

3 Counting tessellations with a given growth rate

In this section we will count the number of planar tessellations having special properties with a given growth rate. Let a_0 denote the number of finite tessellations, a_1 the number of Euclidean tessellations (growth rate $\Gamma_1 = 1$), a_2 the number of tessellations with the smallest exponential growth rate $\Gamma_2 = G(5) = (3+\sqrt{5})/2$ and in general let a_n count the number of tessellations with the growth rate $G(n+3)$.

Before we state the result of Pisanski [12] let us recall some divisor functions. Let $\sigma_k(n)$ denote the sum of k-th powers of divisors of a positive integer n. Similarly, let $\sigma_k^o(n)$ denote the sum of k-th powers of odd divisors of the number n.

First we will establish the number of maps with a given parameter t. Let $N(t)$ denote the number of maps that exist according to Grünbaum and Shephard and have the edge-symbol $\langle p,q;k,\ell\rangle$ with parameter t.

Theorem 3 *[12] The number of maps with parameter t is given by:*

(a) $N(t) = \frac{t}{4}\sigma_0(\frac{t}{4}) + \frac{t}{2}(\sigma_0^o(t) - \sigma_{-1}^o(t)) + 2\sigma_1^o(t) + 2\sigma_1(\frac{t}{4})$, if t is divisible by 4.

(b) $N(t) = \frac{t}{2}(\sigma_0(\frac{t}{2}) - \sigma_{-1}(\frac{t}{2})) + 2\sigma_1(\frac{t}{2})$, if t is even but $\frac{t}{2}$ is odd.

(c) $N(t) = \frac{t+1}{4}\sigma_0(t) + \frac{1}{2}\sigma_1(t)$, if t is odd.

Corollary 3.1 *[12] The number of maps a_n is determined by:*

$$a_n = N(n+3), n \geq 1$$

The following Lemma, [12, Lemma 3.2], will be useful in our proofs. The subscript i in $N_i(t)$ corresponds to one of the conditions 1–4 of Theorem 1. Furthermore each of the cases 2 and 3 of this theorem can be further divided into two sub-cases 2.1, 2.2 or 3.1, 3.2, depending on whether only one of the parameters is odd, or both are odd. Double subscripts in $N_{i,j}$ correspond to these sub-cases, respectively.

Lemma 4 *The numbers of maps $N(t), N_i(t), N_{i,j}(t)$, with parameter t, is given by:*

(1a) $N_1(t) = \frac{t}{4}\sigma_0(t/4) + \frac{t}{2}(\sigma_0^o(t) - \sigma_{-1}^o(t))$, *if t is divisible by 4.*

(1b) $N_1(t) = \frac{t}{2}(\sigma_0^o(t) - \sigma_{-1}^o(t))$, *if t is even but $\frac{t}{2}$ is odd.*

(1c) $N_1(t) = \frac{t+1}{4}\sigma_0(t) - \frac{1}{2}\sigma_1(t)$, *if t is odd.*

(2.1) $N_{21}(t) = \frac{1}{2}(\sigma_1^o(t) - \sigma_0^o(t))$.

(2.2a) $N_{22}(t) = \sigma_1(\frac{t}{4}) + \frac{1}{2}(\sigma_1^o(t) + \sigma_0^o(t))$, *if t is divisible by 4.*

(2.2b) $N_{22}(t) = \frac{1}{2}(\sigma_1^o(t) + \sigma_0^o(t))$, *if t is even but $\frac{t}{2}$ is odd.*

(2.2c) $N_{22}(t) = 0$, *if t is odd.*

(3.1) $N_{31}(t) = \frac{1}{2}(\sigma_1^o(t) - \sigma_0^o(t))$.

(3.2a) $N_{32}(t) = \sigma_1(\frac{t}{4}) + \frac{1}{2}(\sigma_1^o(t) + \sigma_0^o(t))$, *if t is divisible by 4.*

(3.2b) $N_{32}(t) = \frac{1}{2}(\sigma_1^o(t) + \sigma_0^o(t))$, *if t is even but $\frac{t}{2}$ is odd.*

(3.2c) $N_{32}(t) = 0$, *if t is odd.*

(4a) $N_4(t) = 0$, *if t is divisible by 4.*

(4b) $N_4(t) = 0$, *if t is even but $\frac{t}{2}$ is odd.*

(4c) $N_4(t) = \sigma_0(t)$, *if t is odd;*

where $N_2(t) = N_{21}(t) + N_{22}(t), N_3(t) = N_{31}(t) + N_{32}(t)$, *and* $N(t) = N_1(t) + N_2(t) + N_3(t) + N_4(t) = N_1(t) + 2N_2(t) + N_4(t)$.

There are five basic subtypes of edge-transitive tessellations: v, f, s, b, e:

v	vertex-transitive
f	face-transitive
s	self-dual
b	bipartite
e	eulerian

Hence there are $2^5 = 32$ subsets of types. However, the dual of vertex-transitive tessellations are face-transitive tessellations, and the dual of Eulerian tessellations are bipartite tessellations. Also, if an edge-transitive tessellation is both vertex-transitive and self-dual, it must be face-transitive as well. A self-dual edge-transitive tessellation is bipartite if and only if it is Eulerian. If an edge-transitive tessellation is vertex-transitive, self-dual and bipartite then it must be face-transitive and Eulerian as well.

It turns out only 14 subsets of types give rise to different columns: $N, N_b = N_e, N_f = N_v, N_s, N_{be}, N_{bf} = N_{ev}, N_{bs} = N_{es} = N_{bes}, N_{bv} = N_{ef}, N_{fs} = N_{sv} = N_{fsv}, N_{fv}, N_{bef} = N_{bev}, N_{bfv} = N_{efv}, N_{bfs} = N_{bsv} = N_{efs} = N_{esv} = N_{befs} = N_{besv} = N_{bfsv} = N_{efsv} = N_{befsv}, N_{befv}$.

Now we can repeat the computation for each of the 14 subclasses of edge-transitive maps.

Proposition 5 *Let $N_b(t)$ denote the number of edge-transitive, bipartite maps with parameter t. Let $N_e(t)$ denote the number of edge-transitive, Eulerian maps with parameter t.*

$$N_b(t) = N_e(t) = \frac{t}{4}\sigma_0(t/4) + \frac{t}{2}(\sigma_0^o(t) - \sigma_{-1}^o(t)) + \sigma_1^0(t), \textit{ if } t \textit{ is divisible by } 4;$$

$$N_b(t) = N_e(t) = \frac{t}{2}(\sigma_0^o(t) - \sigma_{-1}^o(t)) + \sigma_1^o(t), \textit{ if } t \textit{ is even but } \tfrac{t}{2} \textit{ is odd;}$$

$$N_b(t) = N_e(t) = \frac{t+1}{4}\sigma_0(t) - \frac{1}{2}\sigma_1(t) + \frac{1}{2}(\sigma_1^o(t) - \sigma_0^o(t)), \textit{ if } t \textit{ is odd.}$$

Proof. Note that the maps counted by $N_b(t)$ are precisely the maps of Grünbaum-Shephard type 1 or 2. Since we know their number we obtain the result. Clearly, Eulerian tessellations are the duals of bipartite tessellations, hence $N_b(t) = N_e(t)$. QED.

Proposition 6 *Let $N_v(t)$ denote the number of edge-transitive and vertex-transitive maps with parameter t. Let $N_f(t)$ denote the number of edge-transitive and face-transitive maps with parameter t.*

Then

$$N_v(t) = N_f(t) = (\sigma_1(t) + \sigma_0^o(t))/2, \textit{ if } t \textit{ is not divisible by } 4;$$

$$N_v(t) = N_f(t) = (\sigma_1^o(t) + \sigma_0^o(t) + 2\sigma_1(t/2))/2, \textit{ if } t \textit{ is divisible by } 4.$$

Proof. To compute $N_v(t)$, consider $t = ab$ with $a = p - 2$ and $b = (k + l - 4)/2$. Both a and b are divisors of t in all cases but when exactly one of k, l is odd. In the latter case we obtain all $N_{31}(t)$ possible maps.

So assume that $k + l$ is even. Denote by A the number of maps with $k + l$ even and a odd, and by B the number of maps with $k + l$ even and a even. If a is odd then k and l must be equal by Theorem 1 and are exactly determined with a. This yields $A = \sigma_0^o(t)$. If a is even (which can only be the case if t is even as well) we obtain $b = t/a$ possible choices for k. If t is even but not divisible by 4 then t/a, a even, corresponds exactly to all odd divisors of t, yielding $B = \sigma_1^o(t) = (\sigma_1(t) - \sigma_1^o(t))/2$. If t is divisible by 4 then $a = 2a_0$ where a_0 is a divisor of $t/2$. Therefore

$$B = \sum_{a_0 | \frac{t}{2}} \frac{t/2}{a_0} = \frac{1/2}{\sum_{a_0 | \frac{t}{2}}} \frac{t}{a_0} = \sum_{c | \frac{t}{2}} c = \sigma_1(t/2),$$

where $c = t/a$. Summarizing the results we obtain the assertion. Obviously, $N_f(t) = N_v(t)$. QED.

Proposition 7 *Let $N_s(t)$ denote the number of edge-transitive, self-dual maps with parameter t. Then*

$$N_s(t) = \lceil \sqrt{t}/2 \rceil, \text{ if } t \text{ is a perfect square.}$$
$$N_s(t) = 0, \text{ if } t \text{ is not a perfect square.}$$

Proof. Theorem 1 implies $p = k$ and $q = l$. Hence $t = ((p+q)/2 - 2)^2$ and $N_s(t) = 0$ unless t is a perfect square. If $t = a^2$ with $a \in \mathbb{N}$ then we can choose p s.t. $p \leq q$ in $\lceil \sqrt{t}/2 \rceil$ many ways. QED.

Proposition 8 *Let $N_{be}(t)$ denote the number of edge-transitive, bipartite and Eulerian maps with parameter t. These are exactly the edge-transitive maps with all parameters p, q, k, l being even and*

$$N_{be}(t) = N_1(t).$$

Note that each vertex-transitive map is either face-transitive or Eulerian. Also, each vertex-transitive map that is self-dual is face-transitive.

Proposition 9 *Let $N_{ev}(t)$ denote the number of edge-transitive, vertex-transitive, Eulerian maps with parameter t, and let $N_{bf}(t)$ denote the number of edge-transitive, face-transitive, bipartite maps with parameter t. Then*

$$N_{ev}(t) = N_{bf}(t) = \sigma_1(t/2) + (\sigma_1^o(t) - \sigma_0^o(t))/2, \text{ if } t \text{ is even;}$$
$$N_{ev}(t) = N_{bf}(t) = (\sigma_1^o(t) - \sigma_0^o(t))/2, \text{ if } t \text{ is odd.}$$

Proof. In the vertex-transitive, Eulerian case, $p = q$ even. Either exactly one of k, l is odd (we have N_{31} maps of this kind) or $k = l$. In the latter case t is even and p, q, k, l are determined by a choice of an even divisor $a = p - 2$ of t.

Proposition 10 *Let $N_{fv}(t)$ denote the number of edge-transitive maps that are both vertex- and face-transitive with parameter t. Then*

$$N_{fv}(t) = \sigma_0(t).$$

Proof. We have $p = q$, $k = l$ and hence $t = (p-2)(k-2)$. This means that p, q, k and l are exactly described by a choice of a divisor $a = p - 2$ of t. QED.

Proposition 11 *Let $N_{bv}(t)$ denote the number of edge-transitive, vertex-transitive, bipartite maps with parameter t, and let $N_{ef}(t)$ denote the number of edge-transitive, face-transitive, Eulerian maps with parameter t. Then*

$$N_{bv}(t) = N_{ef}(t) = (\sigma_1(t/2) + \sigma_0^o(t))/2, \textit{ if } t \textit{ is divisible by } 4;$$
$$N_{bv}(t) = N_{ef}(t) = (\sigma_1^o(t) + \sigma_0^o(t))/2, \textit{ if } t \textit{ is even but } \tfrac{t}{2} \textit{ is odd;}$$
$$N_{bv}(t) = N_{ef}(t) = 0, \textit{ if } t \textit{ is odd.}$$

Proof. In the bipartite, vertex-transitive case, k and l are even and $p = q$. Either p, q, k and l are all even or $p = q$ are odd and $k = l$ are even. Denote the number of the former maps by X and the number of the latter maps by Y. Then $Y = 0$ if t is odd and $Y = \sigma_0^o(t)$ otherwise.

If p, q, k, l are all even, then $a = p - 2$ is even, and for every even divisor a of t we have $\lfloor t/2a \rfloor$ possibilities for k. If t is even but not divisible by 4 then $b = t/a$ must be odd and $X = (\sigma_1^o(t) - \sigma_0^o(t))/2$. For t divisible by 4, let X_e denotes the number of such maps with a and $b = t/a$ both even, and X_o denotes the number of such maps with a even, b odd. Then $X_e = (\sigma_1(t/2) - \sigma_1^o(t/2))/2$ and $X_o = (\sigma_1^o(t/2) - \sigma_0^o(t/2))/2$ and $X = X_e + X_o = (\sigma_1(t/2) - \sigma_0^o(t/2))/2$. Observe that for t divisible by 4 we have $\sigma_0^o(t) = \sigma_0^o(t/2)$ to obtain the desired formulas. QED.

Proposition 12 *Let $N_{bes}(t)$ denote the number of edge-transitive, bipartite, Eulerian and self-dual maps with parameter t. Then*

$$N_{bes}(t) = \lfloor \sqrt{t}/2 \rfloor, \textit{ if } t \textit{ is a perfect square.}$$
$$N_{bes}(t) = 0, \textit{ if } t \textit{ is not a perfect square.}$$

Proof. In this case, $p = k$ and $q = l$ with all p, q, k and l even. As in the proof of Proposition 7, $N_{bes} = 0$ if t is not a perfect square. If t is a perfect square we have $\lfloor\sqrt{t}/2\rfloor$ possibilities to choose an even p. QED.

Proposition 13 *Let $N_{bev}(t)$ denote the number of edge-transitive, vertex-transitive, bipartite and Eulerian maps with parameter t. Then*

$$N_{bev}(t) = \sigma_1(t/4) + (\sigma_1^o(t) - \sigma_0^o(t))/2, \text{ if } t \text{ is divisible by } 4;$$
$$N_{bev}(t) = (\sigma_1^o(t) - \sigma_0^o(t))/2, \text{ if } t \text{ is even but } \tfrac{t}{2} \text{ is odd;}$$
$$N_{bev}(t) = 0, \text{ if } t \text{ is odd.}$$

Proof. Here all parameters p, q, k, l are even and $p = q$. Hence $a = p - 2$ is an even divisor of t and $N_{bev} = 0$ if t is odd. If t is even, each $b = t/a$ yields $\lfloor b/2\rfloor$ possible choices for an even parameter k. If t is even but not divisible by 4, then b is odd and can be chosen as any odd divisor of t. Each such b yields $(b-1)/2$ possible choices for an even k. Therefore $N_{bev}(t) = (\sigma_1^o(t) - \sigma_0^o(t))/2$. Finally, assume that t is divisible by 4. Denote by X_e the number of maps with b even and by X_o the number of maps with b odd. To compute X_e write $b = 2b_0$ and obtain

$$X_e = \sum_{b_0|t/4} b_0 = \sigma_1(t/4).$$

To compute X_o, write $a = 4a_0$. Since b is odd we obtain

$$X_o = \sum_{b|t, b \text{ odd}} \lfloor\frac{b}{2}\rfloor = \sum_{b|t, b \text{ odd}} \lfloor\frac{t - 4a_0}{8a_0}\rfloor = \sum_{b|t, b \text{ odd}} \frac{b-1}{2} = \frac{1}{2}(\sigma_1^o(t) - \sigma_0^o(t)).$$

QED.

Proposition 14 *Let $N_{bfv}(t)$ denote the number of edge-transitive, vertex-transitive, face transitive, bipartite maps with parameter t. Then*

$$N_{bfv}(t) = \sigma_0(t) - \sigma_0^o(t).$$

Proof. In this case $t = (p-2)(k-2)$ with k even. The number of such maps equals the number of even divisors of t. QED.

Proposition 15 *Let $N_{fsv}(t)$ denote the number of edge-transitive, self-dual maps that are vertex-, and face-transitive with parameter t. Then*

$$N_{fsv}(t) = 1, \text{ if } t \text{ is a perfect square;}$$

$$N_{fsv}(t) = 0, \text{ if } t \text{ is not a perfect square.}$$

Proof. We have $t = (p-2)^2$. Therefore $N_{fsv}(t) = 0$ if t is not a perfect square, and we have only one option for p in the perfect square case. QED.

Proposition 16 *Let $N_{befv}(t)$ denote the number of edge-transitive, vertex-transitive, face-transitive, bipartite and Eulerian maps with parameter t. Then*

$$N_{befv}(t) = \sigma_0(t/4) \text{ if } t \text{ is divisible by } 4;$$

$$N_{befv}(t) = 0, \text{ if } t \text{ is not divisible by } 4.$$

Proof. We have $t = (p-2)(k-l)$ with both p, k even. Hence t is divisible by 4 and $t = 4a_0b_0$ where a_0 is an arbitrary divisor of $t/4$. QED.

Proposition 17 *Let $N_{befsv}(t)$ denote the number of edge-transitive, vertex-transitive, face-transitive, bipartite, Eulerian and self-dual maps with parameter t. Then $N_{bfs} = N_{bsv} = N_{efs} = N_{esv} = N_{befs} = N_{besv} = N_{bfsv} = N_{efsv} = N_{befsv}$ and*

$$N_{befsv}(t) = 1, \text{ if } t \text{ is an even perfect square;}$$

$$N_{befsv}(t) = 0, \text{ if } t \text{ is not an even perfect square,}$$

and all such maps are Eulerian.

Proof. In this case, $t = (p-2)^2$, and p is even. Therefore t is an even perfect square, and for any even perfect square t, parameter p can be chosen in exactly one way. QED.

t	N	N_v	N_{vf}	N_s	N_{vfs}	N_b	N_{be}	N_{bes}	N_{bv}	N_{ev}	N_{bvf}	N_{bev}	N_{bevf}	N_{bevfs}
36	107	47	9	3	1	81	55	3	44	21	6	18	3	1
37	38	20	2	0	0	18	0	0	18	0	0	0	0	0
38	58	31	4	0	0	38	18	0	29	11	2	9	0	0
39	68	30	4	0	0	38	12	0	26	0	0	0	0	0
40	104	46	8	0	0	80	56	0	44	22	6	20	4	0
41	42	22	2	0	0	20	0	0	20	0	0	0	0	0
42	116	50	8	0	0	84	52	0	46	18	4	14	0	0
43	44	23	2	0	0	21	0	0	21	0	0	0	0	0
44	90	43	6	0	0	66	42	0	41	19	4	17	2	0
45	108	42	6	0	0	66	30	0	36	0	0	0	0	0
46	70	37	4	0	0	46	22	0	35	13	2	11	0	0
47	48	25	2	0	0	23	0	0	23	0	0	0	0	0
48	152	63	10	0	0	120	88	0	61	31	8	29	6	0
49	66	30	3	4	1	36	9	3	27	0	0	0	0	0
50	106	48	6	0	0	75	44	0	45	17	3	14	0	0
51	88	38	4	0	0	50	16	0	34	0	0	0	0	0
52	106	50	6	0	0	78	50	0	48	22	4	20	2	0
53	54	28	2	0	0	26	0	0	26	0	0	0	0	0
54	148	62	8	0	0	108	68	0	58	22	4	18	0	0
55	92	38	4	0	0	54	20	0	34	0	0	0	0	0
56	144	61	8	0	0	112	80	0	59	29	6	27	4	0
57	98	42	4	0	0	56	18	0	38	0	0	0	0	0
58	88	46	4	0	0	58	28	0	44	16	2	14	0	0
59	60	31	2	0	0	29	0	0	29	0	0	0	0	0
60	228	86	12	0	0	180	132	0	82	38	8	34	4	0
61	62	32	2	0	0	30	0	0	30	0	0	0	0	0
62	94	49	4	0	0	62	30	0	47	17	2	15	0	0
63	148	55	6	0	0	93	44	0	49	0	0	0	0	0
64	144	64	7	4	1	112	80	4	63	32	6	31	5	1
65	108	44	4	0	0	64	24	0	40	0	0	0	0	0
66	180	74	8	0	0	132	84	0	70	26	4	22	0	0
67	68	35	2	0	0	33	0	0	33	0	0	0	0	0
68	138	64	6	0	0	102	66	0	62	28	4	26	2	0
69	118	50	4	0	0	68	22	0	46	0	0	0	0	0
70	188	74	8	0	0	140	92	0	70	26	4	22	0	0

4 A Census of edge-transitive, one-ended, 3-connected planar maps and their growth rates.

In the following tables the parameter μ is defined by

$$\mu = 1/p + 1/q + 1/k + 1/\ell$$

This parameter may distinguish between finite, Euclidean and hyperbolic case, however, it is not directly related to the growth rate Γ.

t	a	b	p	q	k	ℓ	μ	Γ
1	1	1	3	3	3	3	1.3333	0.0000
2	1	2	3	3	4	4	1.1667	0.0000
2	2	1	4	4	3	3	1.1667	0.0000
3	1	3	3	3	5	5	1.0667	0.0000
3	1.5	2	3	4	4	4	1.0833	0.0000
3	2	1.5	4	4	3	4	1.0833	0.0000
3	3	1	5	5	3	3	1.0667	0.0000
4	2	2	3	5	4	4	1.0333	0.0000
4	2	2	4	4	3	5	1.0333	0.0000
4	1	4	3	3	6	6	1.0000	1.0000
4	2	2	4	4	4	4	1.0000	1.0000
4	4	1	6	6	3	3	1.0000	1.0000
5	2.5	2	3	6	4	4	1.0000	1.0000
5	2	2.5	4	4	3	6	1.0000	1.0000
5	1	5	3	3	7	7	0.9524	2.6180
5	2	2.5	4	4	4	5	0.9500	2.6180
5	2.5	2	4	5	4	4	0.9500	2.6180
5	5	1	7	7	3	3	0.9524	2.6180
6	3	2	3	7	4	4	0.9762	2.6180
6	2	3	4	4	3	7	0.9762	2.6180
6	1	6	3	3	8	8	0.9167	3.7321
6	1.5	4	3	4	6	6	0.9167	3.7321
6	2	3	4	4	4	6	0.9167	3.7321
6	2	3	4	4	5	5	0.9000	3.7321
6	3	2	4	6	4	4	0.9167	3.7321
6	3	2	5	5	4	4	0.9000	3.7321
6	4	1.5	6	6	3	4	0.9167	3.7321
6	6	1	8	8	3	3	0.9167	3.7321

t	a	b	p	q	k	ℓ	μ	Γ
7	3.5	2	3	8	4	4	0.9583	3.7321
7	2	3.5	4	4	3	8	0.9583	3.7321
7	1	7	3	3	9	9	0.8889	4.7913
7	2	3.5	4	4	4	7	0.8929	4.7913
7	2	3.5	4	4	5	6	0.8667	4.7913
7	3.5	2	4	7	4	4	0.8929	4.7913
7	3.5	2	5	6	4	4	0.8667	4.7913
7	7	1	9	9	3	3	0.8889	4.7913
8	4	2	3	9	4	4	0.9444	4.7913
8	2	4	4	4	3	9	0.9444	4.7913
8	1	8	3	3	10	10	0.8667	5.8284
8	2	4	3	5	6	6	0.8667	5.8284
8	2	4	4	4	4	8	0.8750	5.8284
8	2	4	4	4	5	7	0.8429	5.8284
8	2	4	4	4	6	6	0.8333	5.8284
8	4	2	4	8	4	4	0.8750	5.8284
8	4	2	5	7	4	4	0.8429	5.8284
8	4	2	6	6	4	4	0.8333	5.8284
8	4	2	6	6	3	5	0.8667	5.8284
8	8	1	10	10	3	3	0.8667	5.8284
9	4.5	2	3	10	4	4	0.9333	5.8284
9	2	4.5	4	4	3	10	0.9333	5.8284
9	1	9	3	3	11	11	0.8485	6.8541
9	1.5	6	3	4	8	8	0.8333	6.8541
9	2	4.5	4	4	4	9	0.8611	6.8541
9	2	4.5	4	4	5	8	0.8250	6.8541
9	2	4.5	4	4	6	7	0.8095	6.8541
9	3	3	4	6	4	6	0.8333	6.8541
9	4.5	2	4	9	4	4	0.8611	6.8541
9	3	3	5	5	5	5	0.8000	6.8541
9	4.5	2	5	8	4	4	0.8250	6.8541
9	4.5	2	6	7	4	4	0.8095	6.8541
9	6	1.5	8	8	3	4	0.8333	6.8541
9	9	1	11	11	3	3	0.8485	6.8541
10	5	2	3	11	4	4	0.9242	6.8541
10	2	5	4	4	3	11	0.9242	6.8541
10	1	10	3	3	12	12	0.8333	7.8730
10	2.5	4	3	6	6	6	0.8333	7.8730
10	2	5	4	4	4	10	0.8500	7.8730
10	2	5	4	4	5	9	0.8111	7.8730
10	2	5	4	4	6	8	0.7917	7.8730
10	2	5	4	4	7	7	0.7857	7.8730
10	2.5	4	4	5	6	6	0.7833	7.8730
10	5	2	4	10	4	4	0.8500	7.8730
10	5	2	5	9	4	4	0.8111	7.8730

t	a	b	p	q	k	ℓ	μ	Γ
10	4	2.5	6	6	3	6	0.8333	7.8730
10	4	2.5	6	6	4	5	0.7833	7.8730
10	5	2	6	8	4	4	0.7917	7.8730
10	5	2	7	7	4	4	0.7857	7.8730
10	10	1	12	12	3	3	0.8333	7.8730
11	5.5	2	3	12	4	4	0.9167	7.8730
11	2	5.5	4	4	3	12	0.9167	7.8730
11	1	11	3	3	13	13	0.8205	8.8875
11	2	5.5	4	4	4	11	0.8409	8.8875
11	2	5.5	4	4	5	10	0.8000	8.8875
11	2	5.5	4	4	6	9	0.7778	8.8875
11	2	5.5	4	4	7	8	0.7679	8.8875
11	5.5	2	4	11	4	4	0.8409	8.8875
11	5.5	2	5	10	4	4	0.8000	8.8875
11	5.5	2	6	9	4	4	0.7778	8.8875
11	5.5	2	7	8	4	4	0.7679	8.8875
11	11	1	13	13	3	3	0.8205	8.8875
12	6	2	3	13	4	4	0.9103	8.8875
12	2	6	4	4	3	13	0.9103	8.8875
12	1	12	3	3	14	14	0.8095	9.8990
12	2	6	3	5	8	8	0.7833	9.8990
12	2	6	3	5	8	8	0.7833	9.8990
12	3	4	3	7	6	6	0.8095	9.8990
12	2	6	4	4	4	12	0.8333	9.8990
12	2	6	4	4	5	11	0.7909	9.8990
12	2	6	4	4	6	10	0.7667	9.8990
12	2	6	4	4	7	9	0.7540	9.8990
12	2	6	4	4	8	8	0.7500	9.8990
12	3	4	4	6	4	8	0.7917	9.8990
12	3	4	4	6	6	6	0.7500	9.8990
12	4	3	4	8	4	6	0.7917	9.8990
12	6	2	4	12	4	4	0.8333	9.8990
12	3	4	5	5	6	6	0.7333	9.8990
12	6	2	5	11	4	4	0.7909	9.8990
12	4	3	6	6	3	7	0.8095	9.8990
12	4	3	6	6	4	6	0.7500	9.8990
12	4	3	6	6	5	5	0.7333	9.8990
12	6	2	6	10	4	4	0.7667	9.8990
12	6	2	7	9	4	4	0.7540	9.8990
12	6	2	8	8	3	5	0.7833	9.8990
12	6	2	8	8	3	5	0.7833	9.8990
12	6	2	8	8	4	4	0.7500	9.8990
12	12	1	14	14	3	3	0.8095	9.8990

In the table one can represent the presence or absence of each of the symbols (v, f, s, e, b) by a binary vector (with 1 signifying the presence and 0 absence of a symbol). In general there are 32 such vectors but not all of them are possible. Here we present the list of possible tuples and the smallest map realizing it.

Note that for edge-transitive tessellations, the following hold: vertex-transitive implies face-transitive or Eulerian; self-dual implies (vertex-transitive and face-transitive) or (Eulerian and bipartite); bipartite implies (Eulerian or face-transitive); bipartite and vertex-transitive imply Eulerian; face-transitive and Eulerian imply vertex-transitive or bipartite. This yields the following 13 pure types: $M \equiv 0$, M_{be}, $M_{bf} = M_{ev}$, M_{fv}, $M_{bef} = M_{bev}$, M_{bes}, $M_{bfv} = M_{efv}$, M_{fsv}, M_{befv}, $M_{befsv} = N_{befsv}$; where for example M_{be} denotes the number of edge-transitive, bipartite and Eulerian tessellations that are neither vertex-transitive nor face-transitive nor self-dual. Minimal cases of each of the pure types are given in the table below.

t	a	b	p	q	k	ℓ	μ	Γ	Type
1	1	1	3	3	3	3	1.3333	0.0000	M_{fsv}
2	1	2	3	3	4	4	1.1667	0.0000	M_{bfv}
2	2	1	4	4	3	3	1.1667	0.0000	M_{evf}
3	1	3	3	3	5	5	1.0667	0.0000	M_{fv}
3	1.5	2	3	4	4	4	1.0833	0.0000	M_{bf}
3	2	1.5	4	4	3	4	1.0833	0.0000	M_{ev}
4	2	2	4	4	4	4	1.0000	1.0000	M_{befsv}
6	2	3	4	4	4	6	0.9167	3.7321	M_{bev}
6	3	2	4	6	4	4	0.9167	3.7321	M_{bef}
8	2	4	4	4	6	6	0.8333	5.8284	M_{befv}
9	3	3	4	6	4	6	0.8333	6.8541	M_{bes}
12	3	4	4	6	4	8	0.7917	9.8990	M_{be}

Proposition 18 *1. The number of pure bipartite, Eulerian tessellations is*

$M_{be}(t) = (t+1)\sigma_0(t)/4 - \sigma_1(t)/2$, *if t is odd, t is not a perfect square;*

$M_{be}(t) = (t+1)\sigma_0(t)/4 - \sigma_1(t)/2 - \sqrt{t}/2 + 1/2$, *if t is an odd perfect square;*

$M_{be}(t) = t(\sigma_0^o(t) - \sigma_{-1}^o(t))/2 - \sigma_1^o(t) + \sigma_0^o(t)$, *if t is even, but $t/2$ is odd;*

$M_{be}(t) = (t/4+1)\sigma_0(t/4) + (t/2+1)\sigma_0^o(t) - t/2\sigma_{-1}^o(t) - \sigma_1^o(t) - 2\sigma_1(t/4)$, *if t is divisible by 4, but t is not a perfect square;*

$M_{be}(t) = (t/4+1)\sigma_0(t/4) + (t/2+1)\sigma_0^o(t) - t/2\sigma_{-1}^o(t) - \sigma_1^o(t) - 2\sigma_1(t/4) - \sqrt{t}/2 + 1$, *if t is an even perfect square;*

2. The number of bipartite, face-transitive tessellations is

$$M_{bf}(t) = (\sigma_1(t) - \sigma_0(t))/2, \text{ if } t \text{ is odd;}$$

$$M_{bf}(t) = \sigma_1(t/2) - \sigma_0(t) + \sigma_0^o(t), \text{ if } t \text{ is even, but } t/2 \text{ is odd;}$$

$$M_{bf}(t) = \sigma_1(t/2) + \sigma_0(t/4) - \sigma_1(t/4) - \sigma_0(t) + \sigma_0^o(t), \text{ if } t \text{ is divisible by } 4.$$

3. *The number of vertex-transitive, face-transitive tessellations is*

$$M_{fv}(t) = 0, \text{ if } t \text{ is even;}$$

$$M_{fv}(t) = \sigma_0(t), \text{ if } t \text{ is odd, but } t \text{ is not a perfect square;}$$

$$M_{fv}(t) = \sigma_0(t) - 1, \text{ if } t \text{ is an odd perfect square.}$$

4. *The number of bipartite, Eulerian, face-transitive tessellations is*

$$M_{bef}(t) = 0, \text{ if } t \text{ is odd;}$$

$$M_{bef}(t) = (\sigma_1^0(t) - \sigma_0^o(t))/2, \text{ if } t \text{ is even, but } t/2 \text{ is odd;}$$

$$M_{bef}(t) = \sigma_1(t/4) + (\sigma_1^0(t) - \sigma_0^o(t))/2, \text{ if } t \text{ is divisible by } 4, t \text{ is not a perfect square.}$$

$$M_{bef}(t) = \sigma_1(t/4) + (\sigma_1^0(t) - \sigma_0^o(t))/2 - 1, \text{ if } t \text{ is an even perfect square.}$$

5. *The number of bipartite, Eulerian, self-dual tessellations is*

$$M_{bes}(t) = \lceil \frac{\sqrt{t} - 2}{2} \rceil.$$

6. *The number of bipartite, face-transitive, vertex-transitive tessellations is*

$$M_{bfv}(t) = 0, \text{ if } t \text{ is odd;}$$

$$M_{bfv}(t) = \sigma_0^o(t), \text{ if } t \text{ is even.}$$

7. *The number of face-transitive, self-dual, vertex-transitive tessellations is*

$$M_{fsv}(t) = 1, \text{ if } t \text{ is an odd perfect square;}$$

$$M_{fsv}(t) = 0, \text{ if } t \text{ is not an odd perfect square.}$$

8. *The number of bipartite, Eulerian, face-transitive, vertex-transitive tessellations is*

$$M_{befv}(t) = 0, \text{ if } t \text{ is not divisible by } 4;$$

$$M_{befv}(t) = \sigma_0(t) - \sigma_0^o(t) - 1, \text{ if } t \text{ is divisible by } 4, \text{ but } t \text{ is not a perfect square;}$$

$$M_{befv}(t) = \sigma_0(t) - \sigma_0^o(t) - 2, \text{ if } t \text{ is an even perfect square.}$$

9. *The number of bipartite, Eulerian, face-transitive, self-dual, vertex-transitive tessellations is*

$$M_{befsv} = N_{befsv}(t) = 1, \text{ if } t \text{ is an even perfect square;}$$

$$M_{befsv} = N_{befsv}(t) = 0, \text{ if } t \text{ is not an even perfect square.}$$

Proof. These formulas can be derived from the formulae for the N's using the inclusion-exclusion principle. For example, $M_{be} = N_{be} - N_{bef} - N_{bes} - N_{bev} + N_{befs} + N_{befv} + N_{besv} - N_{befsv} = N_{be} - 2N_{bef} - N_{bes} + N_{befs} + N_{befv}$ and the result follows. QED.

The table of M's for $t \leq 70$ is given below.

t	N	$M_{ve} = M_{fb}$	M_{vf}	M_{be}	$M_{vfe} = M_{vfb}$	M_{vfs}	$M_{vbe} = M_{fbe}$	M_{sbe}	M_{bevf}	M_{bevfs}
1	1	0	0	0	0	1	0	0	0	0
2	2	0	0	0	1	0	0	0	0	0
3	4	1	2	0	0	0	0	0	0	0
4	5	1	0	0	1	0	0	0	0	1
5	6	2	2	0	0	0	0	0	0	0
6	10	2	0	0	2	0	1	0	0	0
7	8	3	2	0	0	0	0	0	0	0
8	12	3	0	0	1	0	1	0	2	0
9	14	5	2	0	0	1	0	1	0	0
10	16	4	0	0	2	0	2	0	0	0
11	12	5	2	0	0	0	0	0	0	0
12	26	6	0	2	2	0	3	0	2	0
13	14	6	2	0	0	0	0	0	0	0
14	22	6	0	0	2	0	3	0	0	0
15	28	10	4	4	0	0	0	0	0	0
16	28	7	0	0	1	0	4	1	2	1
17	18	8	2	0	0	0	0	0	0	0
18	40	10	0	4	3	0	5	0	0	0
19	20	9	2	0	0	0	0	0	0	0
20	42	10	0	4	2	0	6	0	2	0
21	38	14	4	6	0	0	0	0	0	0
22	34	10	0	0	2	0	5	0	0	0
23	24	11	2	0	0	0	0	0	0	0
24	64	14	0	10	2	0	9	0	4	0
25	35	14	2	2	0	1	0	2	0	0
26	40	12	0	0	2	0	6	0	0	0
27	48	18	4	8	0	0	0	0	0	0
28	58	14	0	6	2	0	9	0	2	0
29	30	14	2	0	0	0	0	0	0	0
30	84	20	0	16	4	0	10	0	0	0
31	32	15	2	0	0	0	0	0	0	0
32	64	15	0	6	1	0	11	0	4	0
33	58	22	4	10	0	0	0	0	0	0
34	52	16	0	0	2	0	8	0	0	0
35	60	22	4	12	0	0	0	0	0	0

t	N	$M_{ve} = M_{fb}$	M_{vf}	M_{be}	$M_{vfe} = M_{vfb}$	M_{vfs}	$M_{vbe} = M_{fbe}$	M_{sbe}	M_{bevf}	M_{bevfs}
36	107	23	0	20	3	0	15	2	2	1
37	38	18	2	0	0	0	0	0	0	0
38	58	18	0	0	2	0	9	0	0	0
39	68	26	4	12	0	0	0	0	0	0
40	104	22	0	20	2	0	16	0	4	0
41	42	20	2	0	0	0	0	0	0	0
42	116	28	0	24	4	0	14	0	0	0
43	44	21	2	0	0	0	0	0	0	0
44	90	22	0	10	2	0	15	0	2	0
45	108	36	6	30	0	0	0	0	0	0
46	70	22	0	0	2	0	11	0	0	0
47	48	23	2	0	0	0	0	0	0	0
48	152	30	0	36	2	0	23	0	6	0
49	66	27	2	6	0	1	0	3	0	0
50	106	28	0	16	3	0	14	0	0	0
51	88	34	4	16	0	0	0	0	0	0
52	106	26	0	12	2	0	18	0	2	0
53	54	26	2	0	0	0	0	0	0	0
54	148	36	0	32	4	0	18	0	0	0
55	92	34	4	20	0	0	0	0	0	0
56	144	30	0	30	2	0	23	0	4	0
57	98	38	4	18	0	0	0	0	0	0
58	88	28	0	0	2	0	14	0	0	0
59	60	29	2	0	0	0	0	0	0	0
60	228	44	0	68	4	0	30	0	4	0
61	62	30	2	0	0	0	0	0	0	0
62	94	30	0	0	2	0	15	0	0	0
63	148	49	6	44	0	0	0	0	0	0
64	144	31	0	20	1	0	26	3	4	1
65	108	40	4	24	0	0	0	0	0	0
66	180	44	0	40	4	0	22	0	0	0
67	68	33	2	0	0	0	0	0	0	0
68	138	34	0	16	2	0	24	0	2	0
69	118	46	4	22	0	0	0	0	0	0
70	188	44	0	48	4	0	22	0	0	0

Remark 19 *There are nine edge-transitive finite planar maps and there are exactly 5 edge-transitive tessellations in the Euclidean plane, depicted in Figure 4.*

p	q	k	ℓ	μ	Polyhedron
3	3	3	3	1.3333	tetrahedron
3	3	4	4	1.1667	cube
4	4	3	3	1.1667	octahedron
3	3	5	5	1.0667	dodecahedron
3	4	4	4	1.0833	rhombic dodecahedron
4	4	3	4	1.0833	cuboctahedron
5	5	3	3	1.0667	icosahedron
3	5	4	4	1.0333	rhombic triacontahedron
4	4	3	5	1.0333	icosidodecahedron

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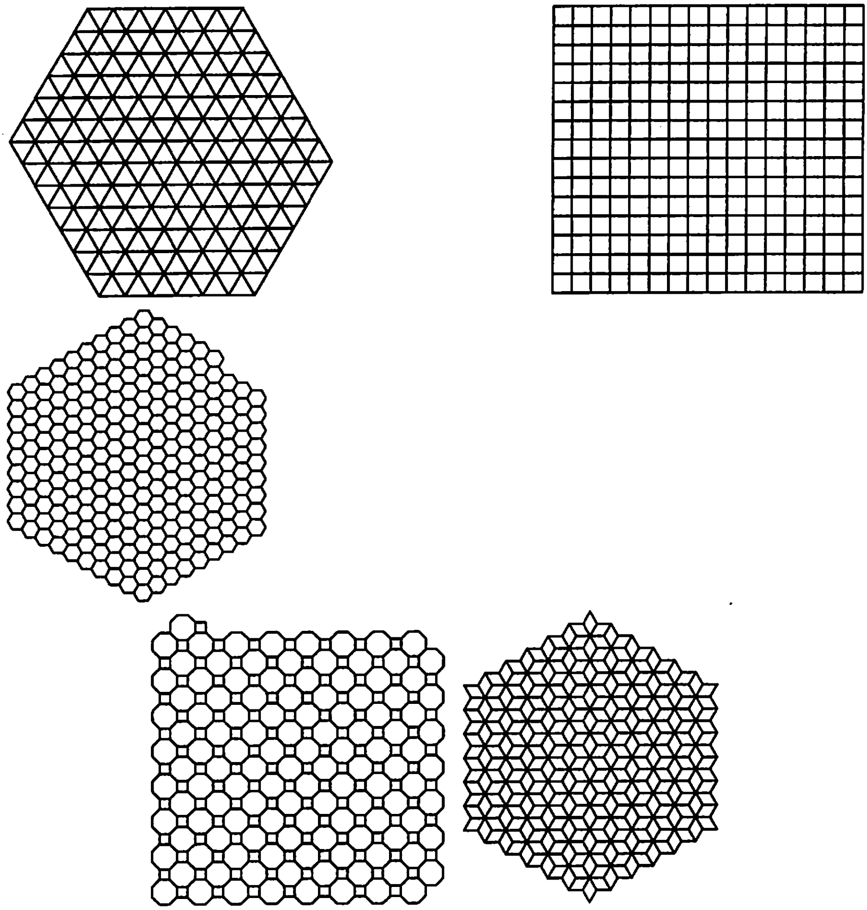

Figure 1: The five Euclidean plane edge-transitive tessellations with respective edge symbols: $\langle 6,6;3,3\rangle$, $\langle 4,4;4,4\rangle$, $\langle 3,3;6,6\rangle$, $\langle 3,3;4,8\rangle$, $\langle 4,8;3,3\rangle$.

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