The Equivalence Chain of a Graph

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Abstract

Let $G = (V, E)$ be a graph. A subset S of V is called an *equivalence set* if every component of the induced subgraph $\langle S \rangle$ is complete. In this paper starting with the concept of equivalence set as seed property, we form an inequality chain of six parameters, which we call the *equivalence chain* of G. We present several basic results on these parameters and problems for further investigation.

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1 Introduction

By a graph $G = (V, E)$ we mean a finite, undirected and connected graph with neither loops nor multiple edges. The order and size of G are denoted by n and m respectively. For graph theoretic terminology we refer to Chartrand and Lesniak [5].

One of the major areas in graph theory is the study of domination and related subset problems such as independence, irredundance, covering and matching. An excellent treatment of fundamentals of domination in graphs is given in the book by Haynes et al. [12]. Surveys of several advanced topics in domination are given in the book edited by Haynes et al. [13].

Let $G = (V, E)$ be a graph. Let $v \in V$. The open neighborhood of v denoted by $N(v)$ and the closed neighborhood of v denoted by $N[v]$ are defined by $N(v) = \{u \in V : uv \in E\}$ and $N[v] = N(v) \cup \{v\}$. A subset S of

V is said to be an *independent set* if no two vertices in S are adjacent. A subset S of V is called a *dominating set* if every vertex in $V - S$ is adjacent to at least one vertex in S. A subset S of V is called an *irredundant set* if for every vertex $v \in S$ there exists a vertex w such that $N[w] \cap S = \{v\}$. The above concepts of independence, domination and irredundance lead to the following six parameters:

$$\begin{aligned} i(G) &= min\{|S| : S \text{ is a maximal independent set in } G\}, \\ \beta_0(G) &= max\{|S| : S \text{ is an independent set in } G\}, \\ \gamma(G) &= min\{|S| : S \text{ is a dominating set in } G\}, \\ \Gamma(G) &= max\{|S| : S \text{ is a minimal dominating set in } G\}, \\ ir(G) &= min\{|S| : S \text{ is a maximal irredundant set in } G\} \text{ and} \\ IR(G) &= max\{|S| : S \text{ is an irredundant set in } G\}. \end{aligned}$$

These parameters are respectively called *independent domination number, independence number, domination number, upper domination number, irredundance number* and *upper irredundance number.* We see that the maximality condition for an independent set is the definition of dominating set and the minimally condition for a dominating set is the definition of irredundant set. Cockayne et al. [6] established the following inequality chain which is now known as the domination chain.

Theorem 1.1. [6] *For any graph G, $ir(G) \leq \gamma(G) \leq i(G) \leq \beta_0(G) \leq \Gamma(G) \leq IR(G)$.*

Definition 1.2. Given an integer sequence $1 \leq a \leq b \leq c \leq d \leq e \leq f$, if there exists a graph G such that $a = ir(G), b = \gamma(G), c = i(G), d = \beta_0(G), e = \Gamma(G)$ and $f = IR(G)$, then (a, b, c, d, e, f) is called a *domination sequence.*

Cockayne et al. [7] completely characterized the domination chain.

Theorem 1.3. [7] *A sequence a, b, c, d, e, f of positive integers is a domination chain if and only if :*

1. $a \leq b \leq c \leq d \leq e \leq f$,
2. $a = 1$ *implies that* $c = 1$,
3. $d = 1$ *implies that* $f = 1$ *and*
4. $b \leq 2a - 1$.

The domination chain has been extended further by introducing two more parameters called the *external redundance number* $er(G)$ and the *upper external redundance number* $ER(G)$, leading to the following chain:

$$er(G) \leq ir(G) \leq \gamma(G) \leq i(G) \leq \beta_0(G) \leq \Gamma(G) \leq IR(G) \leq ER(G).$$

This is called the *extended domination chain* of G and we refer to [8] for further details.

An *equivalence graph* is a vertex disjoint union of complete graphs. An *equivalence covering* of a graph G is a family of equivalence subgraphs of G such that every edge of G is an edge of at least one member of the family. The *equivalence covering number* of G is the cardinality of a minimum equivalence covering of G. The equivalence covering number was first studied in [9]. Interesting bounds for the equivalence covering number in terms of maximal degree of the complement were obtained in [2]. The computation of the equivalence covering number of split graphs was considered in [4].

An important concept which uses equivalence graph is subcoloring studied in [14, 1, 11]. A *subcoloring* of G is a partition of its vertex set into subsets $X_1, X_2, \ldots, X_k$, where for each $i \leq k$ the induced subgraph $\langle X_i \rangle$ is an equivalence graph. The order of a minimum subcoloring is called the *subchromatic number* of G. The notion of subchromatic number is a natural generalization of the well studied chromatic number since for any independent set S, the induced subgraph $\langle S \rangle$ is trivially an equivalence graph.

The concept of equivalence graph also arises naturally in the study of domination in claw-free graphs, as shown in the following theorem which was proved in [10].

Theorem 1.4. [10] *Let D be a minimal dominating set of vertices in a $K_{1,3}$-free graph. Then D is a collection of disjoint complete subgraphs.*

Motivated by these observations, we have introduced the concept of equivalence set in [3].

Definition 1.5. [3] Let $G = (V, E)$ be a graph. A subset S of V is called an *equivalence set* if every component of the induced subgraph $\langle S \rangle$ is complete.

The concept of an equivalence set is a natural generalization of the concept of independence, since every independent set is obviously an equivalence set. Haynes et al. [12, Page 286] have suggested that almost any property such as vertex cover, packing, $\langle S \rangle$ is acyclic etc. can be used as a seed property to generate an inequality chain. In this paper we use the concept of equivalence as seed property and form an inequality chain of six parameters which we call the *equivalence chain* of G. We present several basic results on these parameters and problems for further investigation.

2 The Equivalence Chain

The concept of equivalence set is obviously a hereditary property and hence an equivalence set S is maximal if and only if it is 1-maximal.

Definition 2.1. [3] The *equivalence number* $\beta_{eq}(G)$ and the *lower equivalence number* $i_{eq}(G)$ are defined by

$$\beta_{eq}(G) = \max\{|S| : S \text{ is an equivalence set of } G\} \text{ and}$$
$$i_{eq}(G) = \min\{|S| : S \text{ is a maximal equivalence set of } G\}.$$

Let S be an equivalence set in G and let $v \in V - S$. If v is not dominated by S, then $S \cup \{v\}$ is also an equivalence set and hence it follows that any maximal equivalence set is a dominating set.

Proposition 2.2. *Let S be an equivalence set in G. Then S is a maximal equivalence set if and only if for every $v \in V - S$, there exist two vertices $u, w \in S$ such that the induced subgraph $\langle\{u, v, w\}\rangle$ is isomorphic to P_3.*

Proof. Let S be a maximal equivalence set and let $v \in V - S$. Then $S \cup \{v\}$ is not an equivalence set and hence $\langle S \cup \{v\}\rangle$ contains a component which is not complete. This component contains P_3 as an induced subgraph and $v \in V(P_3)$. Conversely, if S is an equivalence set in G satisfying the given condition, then for every $v \in V - S$, $\langle S \cup \{v\}\rangle$ contains a component which is not complete. Hence $S \cup \{v\}$ is not an equivalence set. Thus S is a maximal equivalence set. □

Definition 2.3. A subset $S \subseteq V$ is said to be an *eq-dominating set* of G if for every $v \in V - S$, there exist two vertices $u, w \in S$ such that the induced subgraph $\langle\{u, v, w\}\rangle$ is isomorphic to P_3.

Clearly *eq*-domination is a super hereditary property and hence an *eq*-dominating set S is minimal if and only if S is 1-minimal.

Definition 2.4. The *eq-domination number* $\gamma_{eq}(G)$ and the *upper eq-domination number* $\Gamma_{eq}(G)$ are defined by

$$\gamma_{eq}(G) = \min\{|D| : D \text{ is a minimal } eq\text{-dominating set of } G\} \text{ and}$$
$$\Gamma_{eq}(G) = \max\{|D| : D \text{ is a minimal } eq\text{-dominating set of } G\}.$$

Proposition 2.5. *Every maximal equivalence set S of a graph G is a minimal eq-dominating set of G.*

Proof. It follows from Proposition 2.2 that S is an *eq*-dominating set of G. Now let $v \in S$ and let C be the component of $\langle S\rangle$ which contains v. Let $u, w \in S - \{v\}$. If both u, w are in C, then $\langle\{u, v, w\}\rangle$ is isomorphic to K_3 and in other cases $\langle\{u, v, w\}\rangle$ is isomorphic to $K_2 \cup K_1$ or $\overline{K}_3$. Thus $S - \{v\}$ is not an *eq*-dominating set of G, so that S is a minimal *eq*-dominating set of G. □

Corollary 2.6. *For any graph G, $\gamma_{eq}(G) \leq i_{eq}(G) \leq \beta_{eq}(G) \leq \Gamma_{eq}(G)$.*

Proposition 2.7. *Let S be an eq-dominating set of G. Then S is a minimal eq-dominating set if and only if for every $u \in S$, there exists a vertex $v \in V - (S - \{u\})$ such that in the induced subgraph $\langle (S - \{u\}) \cup \{v\} \rangle$, the component containing v is complete.*

Proof. Let S be a minimal *eq*-dominating set of G and let $u \in S$. Then $S - \{u\}$ is not an *eq*-dominating set and hence there exists a vertex $v \in V-(S-\{u\})$ such that for any two vertices $u', u'' \in S$, the induced subgraph $\langle \{u', u'', v\} \rangle$ is not P_3. Now, let G_1 be the component of $\langle (S - \{u\}) \cup \{v\} \rangle$ that contains v. If there exists a vertex $w \in V(G_1)$ such that w is not adjacent to v, let $P = (v, u_1, u_2, \ldots, w)$ be a shortest $v - w$ path in G_1. Then $\langle \{v, u_1, u_2\} \rangle \cong P_3$, which is a contradiction. Thus v is adjacent to every vertex in G_1. Now if there exist two nonadjacent vertices v_1, v_2 in G_1, then $\langle \{v, v_1, v_2\} \rangle \cong P_3$, which is again a contradiction. Hence G_1 is complete. The converse is obvious. □

Definition 2.8. A subset $S \subseteq V$ is said to be an *eq-irredundant set* of G if for each vertex $u \in S$, there exists $v \in V - (S - \{u\})$ such that in the induced subgraph $\langle (S - \{u\}) \cup \{v\} \rangle$, the component containing v is complete.

Clearly *eq*-irredundance is a hereditary property and hence an *eq*-irredundant set S is maximal if and only if S is 1-maximal.

Proposition 2.9. *Every minimal eq-dominating set is a maximal eq-irredundant set.*

Proof. Let S be a minimal *eq*-dominating set. By proposition 2.7, S is *eq*-irredundant. Suppose S is not maximal *eq*-irredundant. Then there exists a vertex $u \in V - S$ such that $S_1 = S \cup \{u\}$ is *eq*-irredundant. Hence there exists $v \in V - (S_1 - \{u\}) = V - S$ such that in the induced subgraph $\langle S \cup \{v\} \rangle$, the component containing v is complete. Hence for any two vertices $x, y \in S$ the induced subgraph $\langle \{v, x, y\} \rangle$ is not a path, so that S is not an *eq*-dominating set of G, which is a contradiction. □

Definition 2.10. The *eq-irredundance number* $ir_{eq}(G)$ and the *upper eq-irredundance number* $IR_{eq}(G)$ are defined by

$ir_{eq}(G) = \min\{|D| : D$ is a maximal *eq*-irredundant set of $G\}$ and
$IR_{eq}(G) = \max\{|D| : D$ is a maximal *eq*-irredundant set of $G\}$.

Since every minimal *eq*-dominating set is a maximal *eq*-irredundant set, we have the following inequality chain:

Theorem 2.11. *For any graph G, we have*

$$ir_{eq}(G) \le \gamma_{eq}(G) \le i_{eq}(G) \le \beta_{eq}(G) \le \Gamma_{eq}(G) \le IR_{eq}(G).$$

This inequality chain is called the equivalence chain of G.

Definition 2.12. Given an integer sequence $2 \le a \le b \le c \le d \le e \le f$, if there exists a graph G such that $a = ir_{eq}(G), b = \gamma_{eq}(G), c = i_{eq}(G), d = \beta_{eq}(G)$, $e = \Gamma_{eq}(G)$ and $f = IR_{eq}(G)$, then (a, b, c, d, e, f) is called an *equivalence sequence.*

Example 2.13.

1. For any integer $n \ge 2$ the sequence $\pi = (n, n, n, n, n, n)$ is an equivalence sequence, since for the graph $G = K_n$, we have $ir_{eq} = \gamma_{eq} = i_{eq} = \beta_{eq} = \Gamma_{eq} = IR_{eq} = n$. We observe that the sequence π is the domination sequence of the graph $G = \overline{K}_n$.

2. The sequence $(2, 2, 2, b, b, b)$ is an equivalence sequence, since for the graph $G = K_{a,b}$, $2 \le a \le b$, we have $ir_{eq} = \gamma_{eq} = i_{eq} = 2$ and $\beta_{eq} = \Gamma_{eq} = IR_{eq} = b$. Also the above sequence is the domination sequence of the graph $G = K_{2,b}$.

Observation 2.14. Since $ir_{eq}(G) \ge 2$ for any graph G with $n \ge 2$, it follows that any domination sequence (a, b, c, d, e, f) with $a = 1$ is not an equivalence sequence. Hence the following problem naturally arise.

Problem 2.15. *If $\pi = (a, b, c, d, e, f)$ is an equivalence chain of a graph G, does there exist a graph G_1 such that π is a domination sequence of G_1?*

We now proceed to obtain an extension of the equivalence chain.

Definition 2.16. Let $G = (V, E)$ be a graph, $S \subseteq V$ and $u \in S$. A vertex v is said to be an *eq-private neighbor* of u with respect to S if in the induced subgraph $\langle (S - \{u\}) \cup \{v\} \rangle$, the component containing v is complete.

Observation 2.17. The set of all *eq*-private neighbors of u with respect to S is denoted by $pn_{eq}[u, S]$. It follows from the definition that S is an *eq*-irredundant set if and only if every vertex $u \in S$ has at least one *eq*-private neighbor, or equivalently $pn_{eq}[u, S] \ne \emptyset$ for all $u \in S$. Now for any subset S of V, let $pn_{eq}(S) = \{v \in S : pn_{eq}[v, S] \ne \emptyset\}$. The *eq-private neighbor count* of S is defined by $pnc_{eq}(S) = |pn_{eq}(S)|$. Thus S is *eq*-irredundant if and only if $pnc_{eq}(S) = |S|$. Also an *eq*-irredundant set S is maximal if and only if $S \cup \{u\}$ is not *eq*-irredundant for every $u \in V - S$. Hence there exists at least one $w \in S \cup \{u\}$ such that w does not have an *eq*-private neighbor with respect to S. Thus if we add any vertex in $V - S$ to a maximal *eq*-irredundant set S, the *eq*-private neighbor count will not increase.

Thus an *eq*-irredundant set S is maximal if and only if $pnc_{eq}(S \cup \{w\}) \le pnc_{eq}(S)$ for all $w \in V - S$.

Definition 2.18. A subset $S \subseteq V$ is said to be an *external eq-redundant* if for every $v \in V - S$ such that $pnc_{eq}(S \cup \{v\}) \leq pnc_{eq}(S)$. The *external eq-redundance number* $er_{eq}(G)$ and *upper external eq-redundance number* $ER_{eq}(G)$ are defined to be the minimum and maximum cardinalities of a minimal external eq-redundant set S in G.

Lemma 2.19. *Every maximal eq-irredundant set S is a minimal eq-external redundant set.*

Proof. It follows from Observation 2.17 that S is external eq-irredundant. Now let S_1 be a proper subset of S and let $w \in S - S_1$. Since eq-irredundance is a hereditary property, it follows that S_1 and $S_1 \cup \{w\}$ are both eq-irredundant. Hence $pnc_{eq}(S_1 \cup \{w\}) = |S_1| + 1 = pnc_{eq}(S_1) + 1$, so that $pnc_{eq}(S_1 \cup \{w\}) > pnc_{eq}(S_1)$. Thus S_1 is not external eq-redundant and hence S is a minimal external eq-redundant set. □

As a consequence we have the following:

Theorem 2.20. *For any graph G, we have $er_{eq}(G) \leq ir_{eq}(G) \leq \gamma_{eq}(G) \leq i_{eq}(G) \leq \beta_{eq}(G) \leq \Gamma_{eq}(G) \leq IR_{eq}(G) \leq ER_{eq}(G)$. This inequality chain is called the extended equivalence chain of G.*

3 Complexity Results

In this section we prove that the decision problems corresponding to the parameters $ir_{eq}, \gamma_{eq}, i_{eq}, \beta_{eq}, \Gamma_{eq}$ and IR_{eq} are NP-complete. The reduction is from 3-SAT.

3-SAT
INSTANCE: A set $X = \{x_1, x_2, \ldots, x_r\}$ of variables and a set $C = \{C_1, C_2, \ldots, C_s\}$ of 3-element sets called clauses, where each clause C_i contains three distinct occurrences of either a variable x_i or its complement x_i'.
QUESTION: Does C have a satisfying truth assignment?

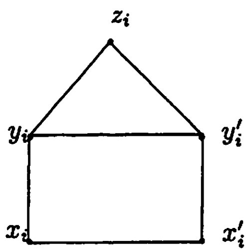
EQUIVALENCE SET (EQ)
INSTANCE: A graph G and a positive integer k.
QUESTION: Does G have a maximal equivalence set S with $|S| \leq k$?

***eq*-DOMINATING SET (EQD)**
INSTANCE: A graph G and a positive integer k.
QUESTION: Does G have an eq-dominating set S with $|S| \leq k$?

eq-**IRREDUNDANT SET (EQIR)**
INSTANCE: A graph G and a positive integer k.
QUESTION: Does G have a maximal *eq*-irredundant set S with $|S| \leq k$?

Theorem 3.1. *The decision problems EQ, EQD, EQIR sets are NP-complete.*

Proof. Clearly EQ, EQD and EQIR are in NP. We first prove EQ is NP-complete. Given an instance C of 3-SAT, we construct an instance G of EQ as follows: for each literal x_i we take a copy of G_i with its vertices labeled as in Figure 1. For each clause C_j we add a vertex C_j to the three literals which are in the clause C_j. Let $k = 2r$.



G_i
Figure 1

Suppose C has a satisfying truth assignment. Let D be the set consisting of all literals x_i or x'_i which are assigned the value true along with the corresponding vertex y_i or y'_i. Clearly each component $\langle D \rangle$ is K_2 and hence D is an equivalence set in G. Since each clause C_j contains a true literals say x_i, it follows that $\langle \{x_i, y_i, C_j\} \rangle \cong P_3$. Also each vertex of G_i which is not in D along with a corresponding copy of K_2 in $\langle D \rangle$ forms a P_3. Thus D is a maximal equivalence set in G and $|D| = 2r = k$.

Conversely, suppose G contains a maximal equivalence set D with $|D| \leq 2r$. Since D is a maximal equivalence set, it follows that D contains at least two vertices from each G_i and D contains no clause vertex C_j. Since $|D| \leq 2r$, it follows that D contains exactly two vertices from each G_i. If $D \cap V(G_i) = \{x_i, x'_i\}$, then $D \cup \{z_i\}$ is an equivalence set, which is a contradiction. Also if both $x_i, x'_i \notin D$, then $|D \cap V(G_i)| = 3$, which again a contradiction. Hence exactly one of x_i or x'_i is in D. Now for each variable x_i, we assign the value True if $x_i \in D$ and the value False otherwise. Now if clause C_j does not contain any true literal, then C_j is not adjacent to any vertex in D and hence $D \cup \{C_j\}$ is an equivalence set, which contradicts the maximality of D. Thus each C_j contains a true literal.

We now prove EQD is NP-complete. By using the same reduction given in the proof of EQ. If C has a satisfying truth assignment, then the set D

constructed in the proof of EQ is a maximal equivalence set and hence it follows from Proposition 2.5 that is a minimal eq-dominating set.

Conversely, suppose G has a eq-dominating set with $|D| \leq 2r$. Since D is a eq-dominating set, it follows that D contains at least two vertices from each G_i. Since $|D| \leq 2r$, it follows that D contains exactly two vertices from each G_i and D contains no clause vertices. If $D \cap V(G_i) = \{x_i, x_i'\}$, $\{x_i', y_i\}$, $\{x_i, y_i'\}$ or $\{y_i, y_i'\}$, then z_i is not eq-dominated by D, which is a contradiction. Thus $D \cap V(G_i) = \{x_i, y_i\}$ or $\{x_i', y_i'\}$ and hence D contains exactly one of x_i or x_i' and the truth assignment as given in the proof of EQ gives a satisfying truth assignment for C.

We now prove EQIR is NP-complete. By using the same reduction given in the proof of EQ. If C has a satisfying truth assignment, then the set D constructed in the proof of EQ is a maximal equivalence set and hence it follows from Proposition 2.5 and Proposition 2.9 that is a maximal eq-irredundant set. The proof for the converse is similar to that of the proof of EQD. □

Corollary 3.2. *Given a graph G, the problems of deciding whether $ir_{eq}(G) = \gamma_{eq}(G) = i_{eq}(G)$ or $ir_{eq}(G) = \gamma_{eq}(G)$ or $\gamma_{eq}(G) = i_{eq}(G)$ are NP-complete.*

UPPER EQUIVALENCE SET (UEQ)
INSTANCE: A graph G and a positive integer k.
QUESTION: Does G have a maximal equivalence set S with $|S| \geq k$?

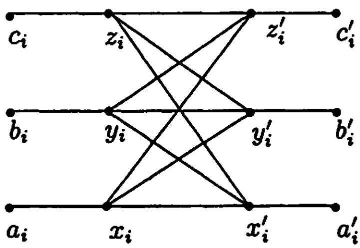
UPPER eq-DOMINATING SET (UEQD)
INSTANCE: A graph G and a positive integer k.
QUESTION: Does G have a minimal eq-dominating set S with $|S| \geq k$?

UPPER eq-IRREDUNDANT SET (UEQIR)
INSTANCE: A graph G and a positive integer k.
QUESTION: Does G have a maximal eq-irredundant set S with $|S| \geq k$?

Theorem 3.3. *The decision problems UEQ, UEQD, UEQIR sets are NP-complete.*

Proof. Clearly the decision problems UEQ, UEQD, UEQIR are in NP. We first prove the theorem for UEQ is NP-complete.

Given an instance C of 3-SAT, we construct an instance G of EQ as follows: for each literal x_i we take a copy of G_i with its vertices labeled as in Figure 2. For each clause C_j we add a vertex C_j and join it to the three literals which it contains. Let $k = 9r$.



G_i

Figure 2

We claim that C has a satisfying truth assignment if and only if G has a maximal equivalence set D with $|D| \geq 9r$. Suppose C has a satisfying truth assignment. Let D be the set consisting of all literals x_i or x'_i which are assigned the value true along with the corresponding vertices $\{y_i, z_i\}$ or $\{y'_i, z'_i\}$ and $\{a_i, b_i, c_i, a'_i, b'_i, c'_i\}$. Clearly each component in $\langle D \rangle$ is K_2 or K_1 and hence D is an equivalence set in G. Since each clause C_j contains a true literal say x_i, it follows that $\langle \{x_i, a_i, C_j\} \rangle \cong P_3$. Also each vertex of G_i which is not in D along with a corresponding copy of K_2 in $\langle D \rangle$ forms a P_3. Thus D is a maximal equivalence set in G and $|D| = 9r = k$.

Conversely, suppose G contains a maximal equivalence set D with $|D| \geq 9r$. Clearly D contains at most 9 vertices from each G_i. Since $|D| \geq 9r$, it follows that D contains exactly 9 vertices from each G_i and D contains no clause vertex C_j. Thus $|D| = 9r$. If both x_i, x'_i are not in D or if both x_i and x'_i are in D, then $|D \cap V(G_i)| < 9$, which is a contradiction. Hence exactly one of x_i or x'_i is in D. Now for each variable x_i, we assign the value True if $x_i \in D$ and the value False otherwise. Since D is a maximal equivalence set it follows that C_j is adjacent to a vertex of D and hence contains a true literal.

The same reduction also shows that UEQD, UEQIR are NP-complete. □

Corollary 3.4. *Given a graph G, the problems of deciding whether $\beta_{eq}(G) = \Gamma_{eq}(G) = IR_{eq}(G)$ or $\beta_{eq}(G) = \Gamma_{eq}(G)$ or $\Gamma_{eq}(G) = IR_{eq}(G)$ are NP-complete.*

4 Conclusion and Scope

In this paper, starting from the concept of equivalence set we have constructed two inequality chains of parameters namely, the equivalence chain and the extended equivalence chain. These parameters might have relationships to other parameters studied in the literature. We mention a few such problems for further investigation.

Problem 4.1. *We have eight eq-parameters in the extended equivalence chain and the corresponding eight parameters in the extended domination chain. Does there exist any relation between an eq-parameter and the corresponding classical parameter?*

Problem 4.2. *It has been proved that for many classes of graphs, including bipartite graphs, chordal graphs, circular arc graphs, cographs and permutation graphs, just to name a few,* $\beta_0(G) = \Gamma(G) = IR(G)$ [12, Page 81]. *Is a similar result true for the upper eq-parameters?*

Problem 4.3. *Obtain a characterization for an integer sequence* $2 \leq a \leq b \leq c \leq d \leq e \leq f$ *to be an equivalence sequence of a graph.*

Problem 4.4. *We have proved that the decision problems corresponding to the various eq-parameters are NP-complete. Develop efficient algorithms for computing these parameters for special classes of graphs.*

Problem 4.5. *Is every equivalence sequence a domination sequence?*

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