# Generalizing Clatworthy Group Divisible Designs II

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This paper is dedicated to Ralph Stanton, a visionary; he was always encouraging young mathematicians, and was a catalyst for the establishment of combinatorics as a substantive branch of mathematics.

**Abstract**

Clatworthy described the eleven group divisible designs with three groups, block size four, and replication number at most 10. With these in mind one might ask: Can each of these designs be generalized in natural ways? In two previous papers the existence of natural generalizations of four of these designs were settled. Here we essentially settle the existence of natural generalizations of five of the remaining seven Clatworthy designs.

# 1 Introduction

An ordered pair $(V, B)$, where $V$ is a set of $mn$ elements called symbols and $B$ is a collection of $k$-subsets of $V$ called blocks is

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said to be a group divisible design $GDD(n, m, k; \lambda_1, \lambda_2)$ where $V$ is partitioned into $m$ sets of size $n$, each element of which is called a group, and if each pair of symbols occurring in the same group appears together in precisely $\lambda_1$ blocks, while each pair of symbols occurring in different groups appears together in exactly $\lambda_2$ blocks. Symbols occuring in the same or different groups are known as first or second associates respectively. A restricted version of this original definition with $\lambda_1 = 0$ is more commonly used as the definition of $GDD$ in the milieu of combinatorial designs; in this setting, a $GDD(n, m, k; 0, \lambda)$ is more commonly known as a $(k, \lambda)$-$GDD$ of type $n^m$. The existence of a $GDD(n, m, 3; \lambda_1, \lambda_2)$ was completely settled by Fu, Rodger, and Sarvate [5, 6]. The most difficult and novel constructions were required when the number of groups, $m$, was less than $k$, namely when $m = 2$ [5]. The existence of $GDDs$ when $m < k$ is, in general, a difficult case to solve. Indeed, when $k = 4$, little is known about the existence of such $GDDs$. For example, when $k = 4$ Henson, Hurd, and Sarvate [9, 11, 12] have existence results for $GDDs$ that are necessary and sufficient for small values of $m$ and $n$, and are then used to construct some infinite families of $GDDs$. They also consider a restricted version of the problem in which the number of symbols in each block in any group has the same parity as in any other group. Hurd, Mishra, and Sarvate [10] have some results when $k = 5$ and $m = 6$.

Clatworthy's table from 1973 [2] lists all eleven $GDDs$ with three groups and block size four that have replication number at most 10. (See Table 1). These can be generalized in natural ways. The existence of such generalizations for four of these eleven Clatworthy $GDDs$ has been studied in two previous papers, as described in the following results.

Henson and Sarvate [8] generalized one of these 11 designs, namely $R127$, proving the following result.

| name | $n$ | $m$ | $k$ | $\lambda_1$ | $\lambda_2$ |
|---|---|---|---|---|---|
| S1 | 2 | 3 | 4 | 2 | 1 |
| S2 | 2 | 3 | 4 | 4 | 2 |
| S3 | 2 | 3 | 4 | 6 | 3 |
| S4 | 2 | 3 | 4 | 8 | 4 |
| S5 | 2 | 3 | 4 | 10 | 5 |
| R96 | 2 | 3 | 4 | 4 | 5 |
| R104 | 3 | 3 | 4 | 3 | 1 |
| R105 | 3 | 3 | 4 | 6 | 2 |
| R111 | 4 | 3 | 4 | 2 | 3 |
| R117 | 5 | 3 | 4 | 1 | 2 |
| R127 | 8 | 3 | 4 | 2 | 1 |

Table 1: Clatworthy's Table

**Theorem 1.1.** *There exists a* $GDD(n,3,4;2,1)$ *if and only if* $n \equiv 2 \ (mod\ 6)$.

Rodger and Rogers [13] generalized three of these 11 designs, namely, $R96$, $S2$, and $S4$, proving the following results.

**Theorem 1.2.** *There exists a* $GDD(n,3,4;4,5)$ *if and only if* $n \equiv 2 \ (mod\ 6)$.

**Theorem 1.3.** *There exists a* $GDD(n,3,4;4,2)$ *if and only if* $n \equiv 2 \ (mod\ 3)$*, except possibly if* $n = 11$.

**Corollary 1.4.** *There exists a* $GDD(n,3,4;8,4)$ *if and only if* $n \equiv 2 \ (mod\ 3)$*, except possibly if* $n = 11$.

In this paper we study the existence of generalizations of five of the remaining seven designs in the Clatworthy table, namely $S1$ ,$S5$, $S3$, $R104$, and $R105$ (see Theorem 3.1, Corollary 4.1, Theorem 5.2, Theorem 6.2 and Theorem 7.1).

## 2 Preliminaries

Throughout this paper we will use the following design constructed by Brouwer, Schrijver, and Hanani [1] (for a more general setting see also [16] and Theorem 4.6 in [7] on page 256).

**Theorem 2.1.** *Necessary and sufficient conditions for the existence of a $(4, \lambda)$-GDD of type $m^u$ are:*

1. $u \geq 4$,
2. $\lambda(u-1)m \equiv 0 \pmod 3$, *and*
3. $\lambda u(u-1)m^2 \equiv 0 \pmod{12}$,

*with the exception of $(m, u, \lambda) \in \{(2,4,1), (6,4,1)\}$, in which case no such GDD exists.*

It is also fruitful to describe these designs as graph decompositions, each symbol being represented by a vertex. Let $G(n, 3; \lambda_1, \lambda_2)$ be the graph with vertex set $\mathbb{Z}_n \times \mathbb{Z}_3$ in which $(u, i)$ is joined to $(v, j)$ with

1. $\lambda_1$ edges if $i = j$, and
2. $\lambda_2$ edges if $i \neq j$.

Then a $GDD(n, 3, 4; \lambda_1, \lambda_2)$ is clearly equivalent to a partition of the edges of $G(n, 3; \lambda_1, \lambda_2)$ into sets of size 6, each of which induces a copy of $K_4$; for each $i \in \mathbb{Z}_3$, $\mathbb{Z}_n \times \{i\}$ is a group. As such, $G(n, 3; \lambda_1, \lambda_2)$ is said to be the associated graph of the $GDD$. These two notions will be used interchangeably throughout this paper.

To construct these designs we must first define the nesting of a $GDD(V, B) = GDD(n, 3, 3; \lambda_1, \lambda_2)$ as follows. A nesting of

$GDD(V, B)$ with associated graph $G(n, 3; \lambda_1, \lambda_2)$ is defined to be a function of $f : B \to V$ such that $\{\{x, f(b)\} \mid x \in b \in B\} = E(G(n, 3; \lambda_1, \lambda_2))$. More informally, a $GDD$ with block size 3 is said to be nested if a fourth point can be added to each block such that the edges gained from the nesting cover precisely the same edges as the original $GDD$. So each pair $\{u, v\}$ of vertices occurs together in twice as many blocks of size 4 in the nested design as the number of triples containing $\{u, v\}$ in the original $GDD$. We will use the following theorem provided by Jin Hua Wang [14].

**Theorem 2.2.** *There exists a nesting of a* $GDD(t, n, 3; \lambda_1 = 0, \lambda_2 = \lambda)$ *if and only if* $\lambda t(n-1) \equiv 0 \pmod 6$ *and* $n \geq 4$.

# 3 Generalizing Clatworthy Design $S1$

We first find a small $S1$-design.

**Lemma 3.1.** *There exists a* $GDD(2, 3, 4; 2, 1)$.

**Proof.** To produce a $GDD(2, 3, 4; 2, 1)$, let $V = \mathbb{Z}_2 \times \mathbb{Z}_3$ and $B = \{\{(0, a), (1, a), (0, a+1), (1, a+1)\} \mid a \in \mathbb{Z}_3\}$; for each $l \in \mathbb{Z}_3$, $\mathbb{Z}_2 \times \{l\}$ is a group.

**Theorem 3.2.** *There exists a* $GDD(n, 3, 4; 2, 1)$ *if and only if* $n \equiv 2 \pmod 6$.

**Proof.** We start by proving the necessity, so suppose there exists a

$GDD(n, 3, 4; 2, 1)$. Since each block contains six edges, the number of blocks in any such design is

$$b = \frac{|E(G(n, 3; 2, 1))|}{6} = \frac{3(\frac{2n(n-1)}{2} + 3n^2)}{6} = n^2 - \frac{n}{2}.$$

Clearly the number of blocks is an integer, so $n$ must be even.

For each block, each vertex contains 3 of its incident edges. So the number of blocks containing each vertex $v$ is

$$d_{G(n,3;2,1)}(v) = \frac{2(n-1)+2n}{3} = \frac{4n-2}{3},$$

which must be an integer. Thus $n \equiv 2 \pmod 3$. Since $n$ must also be even, $n \equiv 2 \pmod 6$ is a necessary condition.

Next we prove the sufficiency, so suppose that $n \equiv 2 \pmod 6$. We will show there exists a $GDD(n,3,4;2,1)$, $(\mathbb{Z}_n \times \mathbb{Z}_3, B)$ with groups $\mathbb{Z}_n \times \{l\}$ for each $l \in \mathbb{Z}_3$. Since Lemma 3.1 produces a $GDD(2,3,4;2,1)$, we can assume that $n \geq 8$. The design will be described as a graph decomposition of the graph $G(n,3;2,1)$.

For each $i \in \mathbb{Z}_{n/2}$, let $B(i)$ be a copy of $S2$ on the vertices in $C(i) = \{2i, 2i+1\} \times \mathbb{Z}_3$, where for each $l \in \mathbb{Z}_3$, $\{2i, 2i+1\} \times \{l\}$ is a group. By Theorem 2.2, there exists a $(3,1)$-$GDD$, $(Z_n, \{\{2i, 2i+1\} \mid i \in \mathbb{Z}_{n/2}\}, B_1)$, that has nesting $f$ of type $2^{n/2}$. Let $B_1(l) = \{\{(x,l),(y,l),(z,l),(f(b),l+1)\}, \{(x,l+1),(y,l+1),(z,l+1),(f(b),l)\} \mid l \in Z_3, \{x,y,z\} \in B_1\}$, reducing the sums in the second coordinate of each vertex modulo 3. Then define the blocks in the design as follows:

$$B \;=\; (\textstyle\bigcup_{i\in\mathbb{Z}_{n/2}} B(i)) \cup (\bigcup_{l\in\mathbb{Z}_3} B_1(l))$$

We first count the number of blocks we get in the construction to see if it equals $b = n^2 - \frac{n}{2}$ (calculated above when proving the necessity).

$$
\begin{aligned}
|B| &= |(\bigcup_{i\in\mathbb{Z}_{n/2}} B(i))| + |(\bigcup_{l\in\mathbb{Z}_3} B_1(l))| \\
&= (3)(n/2) + (2)(3)((\tfrac{n(n-1)-n}{2})/3) \\
&= \tfrac{3n}{2} + \tfrac{2(n(n-1)-n)}{2} \\
&= \tfrac{3n+2n^2-2n-2n}{2} \\
&= \tfrac{2n^2-n}{2} \\
&= n^2 - \tfrac{n}{2}
\end{aligned}
$$

Since $|B| = b$ it suffices to check that each edge occurs in at least the correct number (that is, $\lambda_1$ or $\lambda_2$) of blocks in $B$. We consider each edge, $e = \{(x,a),(y,b)\}$, in turn.

1. Suppose $e$ joins two vertices in $C(i)$ for some $i \in \mathbb{Z}_{n/2}$. Then clearly $e$ occurs in $\lambda_1 = 2$ blocks in $B(i)$ if $e$ joins two vertices in the same group and $e$ occurs in $\lambda_2 = 1$ block in $B(i)$ if $e$ joins two vertices in different groups, as required.

2. Next suppose that $e = \{(x,a),(y,a)\}$ for some $a \in \mathbb{Z}_3$ and $0 \le x, y < n$ where for each $i \in \mathbb{Z}_{n/2}$, $e$ does not join two vertices in $C(i)$. Let $\{x, y, z_1\}$ be the triple in $B_1(a)$ that contains $\{x,y\}$, and suppose $f(\{x,y,z_1\}) = z_2$ is the vertex added to the triple by the nesting. Then the $\lambda_1 = 2$ blocks containing the edge $\{(x,a),(y,a)\}$ are as follows: $\{(x,a),(y,a),(z_1,a),(z_2,a+1)\}$, $\{(x,a),(y,a),(z_1,a),$ $(z_2,a+2)\}$.

3. Finally suppose $e = \{(x,a),(y,b)\}$ where $a, b \in \mathbb{Z}_3$, $a \ne b$ and where for each $i \in \mathbb{Z}_{n/2}$, $e$ does not join two vertices in $C(i)$. We can assume that $b \equiv a+1 \pmod 3$. Since $\{x,y\} \in G(n,3;2,1)$, exactly one of the following occurs: either there exists a triple $t_1 = \{x, z_3, z_4\} \in B_1(a)$ such that $f(t_1) = y$ or there exists a triple $t_2 = \{y, z_3, z_4\} \in B_1(a)$ such that $f(t_2) = x$. Therefore $e = \{(x,a),(y,b = a+1)\}$ occurs in $\lambda_2 = 1$ of the following blocks: $\{(x,a),(z_3,a),(z_4,a),(y,a+1)\}$ or $\{(x,a),(y,a+1),(z_3,a+1),(z_4,a+1)\}$.

Thus, every edge is covered the correct number of times by the blocks, so $n \equiv 2 \pmod 6$ is a sufficient condition for a $GDD(n, 3, 4; 2, 1)$ to exist. ■

# 4 A Corollary - Generalizing $S5$

It turns out that the necessary conditions for the existence of a $GDD(n, 3, 4; 10, 5)$, generalizing the Clatworthy design $S5$, are the same as for the existence of a

$GDD(n, 3, 4; 2, 1)$ (i.e. $S1$-design). So we immediately obtain the following corollary.

**Corollary 4.1.** *There exists a* $GDD(n, 3, 4; 10, 5)$ *if and only if* $n \equiv 2 \pmod 6$.

**Proof.** The necessity follows since the degree of each vertex, namely $(20n - 10)/3$ must be an integer. The sufficiency follows by taking five copies of the $S1$-design constructed in Theorem 3.2.

# 5 Generalizing Clatworthy Design $S3$

To complete this design we will use the nesting described in the Preliminaries section, as well as Theorem 2.2 provided by Jin Hua Wang [14].

We first find a small $S3$-design.

**Lemma 5.1.** *There exists a* $GDD(2, 3, 4; 6, 3)$*; and a* $GDD(4, 3, 4; 6, 3)$.

**Proof.** To produce a $GDD(2, 3, 4; 6, 3)$, let $V = \mathbb{Z}_2 \times \mathbb{Z}_3$ and each $l \in \mathbb{Z}_3$, $\mathbb{Z}_2 \times \{l\}$ is a group, and take three copies of each block produced in the small $S1$-design described in the proof of Lemma 3.1.

When $n = 4$, four base blocks are provided that can be rotated "vertically and horizontally" producing 42 blocks as required. Formally, a $GDD(4,3,4;6,3)$ is produced by $(\mathbb{Z}_4 \times \mathbb{Z}_3, B)$, where

$B = \{\{(i,a),(i+1,a),(i+2,a),(i,a+1)\},\{(i,a),(i+1,a),(i+1,a+1),(i+2,a+1)\},\{(i,a),(i+2,a),(i,a+1),(i+3,a+2)\} \mid i \in \mathbb{Z}_4, a \in \mathbb{Z}_3\} \cup \{\{(i,a),(i+1,a),(i+2,a),(i+3,a)\} \mid i \in \mathbb{Z}_2, a \in \mathbb{Z}_3\}$, and where for each $l \in \mathbb{Z}_3$, $\mathbb{Z}_4 \times \{l\}$ is a group.

**Theorem 5.2.** *There exists a $GDD(n,3,4;6,3)$ if and only if $n$ is even, except possibly if $n = 6$.*

**Proof.** We start by proving the necessity, so suppose there exists a

$GDD(n,3,4;6,3)$. Since each block contains six edges, the number of blocks in any such design is

$$b = \frac{|E(G(n,3;6,3))|}{6} = \frac{3(\frac{6n(n-1)}{2}) + 3(3n^2)}{6} = 3n^2 - \frac{3n}{2}.$$

Clearly the number of blocks is an integer, so $n$ must be even.

For each block, each vertex contains 3 of its incident edges. So the number of blocks containing each vertex $v$ is

$$d_{G(n,3;6,3)}(v) = \frac{6(n-1) + 2(3n)}{3} = 4n - 2,$$

which makes no restrictions on $n$. Thus $n$ is even is a necessary condition.

To prove the sufficiency we assume that $n$ is even, $n \neq 6$, and show there exists a $GDD(n,3,4;6,3)$. We will consider three cases in turn: $n \equiv 2 \pmod 6$, $n \equiv 4 \pmod 6$, and $n \equiv 0 \pmod 6$.

First suppose $n \equiv 2 \pmod 6$. Since $n \equiv 2 \pmod 6$ is the necessary condition for the existence of a $GDD(n, 3, 4; 2, 1)$ (i.e. $S1$-design), we immediately obtain the following corollary.

**Corollary 5.3.** *There exists a $GDD(n, 3, 4; 6, 3)$ if and only if $n \equiv 2 \ (mod\ 6)$*

**Proof.** The necessity follows since the number of blocks, namely $3n^2 - \frac{3n}{2}$ must be an integer. The sufficiency follows by taking three copies of the $S1$-design constructed in Theorem 3.2.

Now suppose $n \equiv 4 \pmod 6$. Since Lemma 5.1 produces a $GDD(4, 3, 4; 6, 3)$, we can assume that $n \geq 10$. The design will be described as a graph decomposition of the graph $G(n, 3; 6, 3)$.

For each $i \in \mathbb{Z}_{n/2}$, let $B(i)$ be a copy of $S3$ on the vertices in $C(i) = \{2i, 2i+1\} \times \mathbb{Z}_3$, where for each $l \in \mathbb{Z}_3$, $\{2i, 2i+1\} \times \{l\}$ is a group. By Theorem 2.2, there exists a $(3,3)$-$GDD$, $(Z_n, \{\{2i, 2i+1\} \mid i \in \mathbb{Z}_{n/2}\}, B_1)$, that has nesting $f$ of type $2^{n/2}$. Let $B_1(l) = \{\{(x,l), (y,l), (z,l), (f(b), l+1)\}, \{(x, l+1), (y, l+1), (z, l+1), (f(b), l)\} \mid l \in Z_3, \{x, y, z\} \in B_1\}$, reducing the sums in the second coordinate of each vertex modulo 3. Then define the blocks in the design as follows:

$$B \;=\; (\textstyle\bigcup_{i \in \mathbb{Z}_{n/2}} B(i)) \cup (\bigcup_{l \in \mathbb{Z}_3} B_1(l))$$

We first count the number of blocks we get in the construction to see if it equals $b = 3n^2 - \frac{3n}{2}$ (calculated above when proving the necessity).

$$
\begin{aligned}
|B| &= |(\bigcup_{i\in\mathbb{Z}_{n/2}} B(i))| + |(\bigcup_{l\in\mathbb{Z}_3} B_1(l))| \\
&= (9)(n/2) + (3)(2)(3)((\tfrac{n(n-1)-n}{2})/3) \\
&= \tfrac{9n}{2} + \tfrac{6(n(n-1)-n)}{2} \\
&= \tfrac{9n+6n^2-6n-6n}{2} \\
&= \tfrac{6n^2-3n}{2} \\
&= 3n^2 - \tfrac{3n}{2}
\end{aligned}
$$

Since $|B| = b$ it suffices to check that each edge occurs in at least the correct number (that is, $\lambda_1$ or $\lambda_2$) of blocks in $B$. We consider each edge, $e = \{(x,a),(y,b)\}$, in turn.

1. Suppose $e$ joins two vertices in $C(i)$ for some $i \in \mathbb{Z}_{n/2}$. Then clearly $e$ occurs in $\lambda_1 = 6$ blocks in $B(i)$ if $e$ joins two vertices in the same group and $e$ occurs in $\lambda_2 = 3$ block in $B(i)$ if $e$ joins two vertices in different groups, as required.

2. Next suppose that $e = \{(x,a),(y,a)\}$ for some $a \in \mathbb{Z}_3$ and $0 \le x, y < n$ where for each $i \in \mathbb{Z}_{n/2}$, $e$ does not join two vertices in $C(i)$. Let $\{x,y,z_1\}, \{x,y,z_3\}$, and $\{x,y,z_5\}$ be the triples in $B_1(a)$ that contain $\{x,y\}$, and suppose $f(\{x,y,z_1\}) = z_2$, $f(\{x,y,z_3\}) = z_4$, and $f(\{x,y,z_5\}) = z_6$ are the vertices added to the triples by the nesting. Then the $\lambda_1 = 6$ blocks containing the edge $\{(x,a),(y,a)\}$ are as follows: $\{(x,a),(y,a),(z_1,a),(z_2,a+1)\}$, $\{(x,a),(y,a),(z_1,a),(z_2,a+2)\}$, $\{(x,a),(y,a),(z_3,a),(z_4,a+1)\}$, $\{(x,a),(y,a),(z_3,a),(z_4,a+2)\}$, $\{(x,a),(y,a),(z_5,a),(z_6,a+1)$, and $\{(x,a),(y,a),(z_5,a),(z_6,a+2)$.

3. Finally suppose $e = \{(x,a),(y,b)\}$ where $a,b \in \mathbb{Z}_3$, $a \ne b$ and where for each $i \in \mathbb{Z}_{n/2}$, $e$ does not join two vertices in $C(i)$. We can assume that $b \equiv a+1 \pmod 3$. Since $\{x,y\} \in G(n,3;6,3)$, exactly one of the following occurs for each of the $\lambda_2 = 3$ $\{x,y\}$ edges: either there exists a triple $t_1 = \{x,z_7,z_8\} \in B_1(a)$ such that $f(t_1) = y$ or

there exists a triple $t_2 = \{y, z_7, z_8\} \in B_1(a)$ such that $f(t_2) = x$. The same argument can be made for the other two $\{x, y\}$ edges using $\{z_9, z_{10}, z_{11}, z_{12}\}$. Therefore $e = \{(x, a), (y, b = a+1)\}$ occurs in $\lambda_2 = 3$ of the following blocks:
$\{(x, a), (z_7, a), (z_8, a), (y, a+1)\}$ or $\{(x, a), (y, a+1), (z_7, a+1), (z_8, a+1)\}$, $\{(x, a), (z_9, a), (z_{10}, a), (y, a+1)\}$ or $\{(x, a), (y, a+1), (z_9, a+1), (z_{10}, a+1)\}$, and $\{(x, a), (z_{11}, a), (z_{12}, a), (y, a+1)\}$ or $\{(x, a), (y, a+1), (z_{11}, a+1), (z_{12}, a+1)\}$.

Thus, $n \equiv 4 \pmod 6$ is a sufficient condition for a $GDD(n, 3, 4; 6, 3)$ to exist.

Now suppose $n \equiv 0 \pmod 6$. Since $n = 6$ is the possible exception, we can assume that $n \geq 12$. Similarly to the $n \equiv 4 \pmod 6$ case, there exists a $(3,3)$-$GDD$, $(Z_n, \{\{2i, 2i+1\} \mid i \in \mathbb{Z}_{n/2}\}, B_1)$, that has nesting $f$ of type $2^{n/2}$ by Theorem 2.2. Therefore, the arguments for the $n \equiv 0 \pmod 6$ case are essentially the same for the $n \equiv 4 \pmod 6$ case.

Thus, $n \equiv 0 \pmod 6$ is a sufficient condition for a $GDD(n, 3, 4; 6, 3)$ to exist. ■

# 6 Generalizing Clatworthy Design $R104$

With two possible exceptions, the existence of a generalized $R104$-design is settled in this section. We use a similar construction that is used in the previous section. As before, we first find some small $R104$-designs.

**Lemma 6.1.** *There exists a* $GDD(3, 3, 4; 3, 1)$, *and a* $GDD(12, 3, 4; 3, 1)$.

**Proof.** To produce a $GDD(3, 3, 4; 3, 1)$, let $V = \mathbb{Z}_3 \times \mathbb{Z}_3$ and $B = \{\{(i, a), (i+1, a), (i+2, a), (i, a+1)\} \mid i \in \mathbb{Z}_3, a \in \mathbb{Z}_3\}$; for each $l \in \mathbb{Z}_3$, $\mathbb{Z}_3 \times \{l\}$ is a group.

When $n = 12$, 4 base blocks are provided that can be rotated producing 144 blocks towards the required number of blocks. Let the 36 vertices be labeled $0, 1, 2, ..., 35$ and partitioned into three groups such that vertices with labels that are 0 (mod 3) are one group, 1 (mod 3) are the second group, and 2 (mod 3) are the third group. Then consider the following four blocks: $\{0,2,6,9\}, \{0,9,10,22\}, \{0,6,18,21\}$, and $\{0,8,20,25\}$. When these four blocks are rotated they cover the edges of difference $3, 6, 9, 15, 18$ twice, difference 12 three times, and differences $1, 2, 4, 5, 7, 8, 10, 11, 13, 14,$
$16, 17$ once. In other words, we have covered the mixed edges the required one time and the pure edges two of the required three times, with the exception of the pure edges of difference 12 which are completely covered. Finally, to cover the pure edges that are left we use Theorem 2.1; this exists since $n = 12$ and $u = n/3 \geq 4$, to put a $(4,1) - GDD$ of type $3^{n/3}$ on each level with group $G_i$. The edges of difference 12 make 4 triangles so the vertices in each triangle form a group in the $(4,1) - GDD$ of type $3^4$. On each level, the $(4,1) - GDD$ of type $3^{n/3}$ produces 9 blocks for a total of 27 blocks. Thus, the $144 + 27 = 171$ which is the required number of blocks (see (∗) below).

**Theorem 6.2.** *There exists a $GDD(n, 3, 4; 3, 1)$ if and only if $n \equiv 0, 3$ (mod 12), except possibly if $n = 24, 36$.*

**Proof.** We start by proving the necessity, so suppose there exists a

$GDD(n, 3, 4; 3, 1)$. Since each block contains six edges, the number of blocks in any such design is

$$b = \frac{|E(G(n,3;3,1))|}{6} = \frac{3(\frac{3n(n-1)}{2}) + 3(n^2)}{6} = \frac{5n^2 - 3n}{4}. \qquad (*)$$

Clearly the number of blocks is an integer, so $n \equiv 0, 3$ (mod 12).

For each block, each vertex contains 3 of its incident edges. So the number of blocks containing each vertex $v$ is

$$d_{G(n,3;3,1)}(v) = \frac{3(n-1)+2(n)}{3} = \frac{5}{3}n - 3,$$

which means $n \equiv 0 \pmod 3$. Thus $n \equiv 0, 3 \pmod{12}$ is a necessary condition.

To prove the sufficiency we assume that $n \equiv 0, 3 \pmod{12}$, $n \neq 24, 36$, and show there exists a $GDD(n, 3, 4; 3, 1)$. We will consider two cases in turn: $n \equiv 3 \pmod{12}$ and $n \equiv 0 \pmod{12}$, $n \geq 48$.

First suppose $n \equiv 3 \pmod{12}$. Since Lemma 6.1 produces a $GDD(3, 3, 4; 3, 1)$, we can assume that $n \geq 15$. The design will be described as a graph decomposition of the graph $G(n, 3; 3, 1)$.

For each $i \in \mathbb{Z}_{n/3}$, let $B(i)$ be a copy of $R104$ on the vertices in $C(i) = \{3i, 3i+1, 3i+2\} \times \mathbb{Z}_3$, where for each $l \in \mathbb{Z}_3$, $\{3i, 3i+1, 3i+2\} \times \{l\}$ is a group. By Theorem 2.2, there exists a $(3,1)$-$GDD$, $(Z_n, \{\{3i, 3i+1, 3i+2\} \mid i \in \mathbb{Z}_{n/3}\}, B_1)$, that has nesting $f$ of type $3^{n/3}$. Let $B_1(l) = \{\{(x,l), (y,l), (z,l), (f(b), l+1)\}, \{(x, l+1), (y, l+1), (z, l+1), (f(b), l)\} \mid l \in Z_3, \{x, y, z\} \in B_1\}$, reducing the sums in the second coordinate of each vertex modulo 3. By Theorem 2.1 (since $n \geq 15$ and $u = n/3 \geq 4$), for each $l \in Z_3$ let $B'(l)$ be a copy of a $(4,1)-GDD$ of type $3^{n/3}$ on the vertex set $Z_n \times \{l\}$ with groups in $\{\{3i, 3i+1, 3i+2\} \times \{l\} \mid i \in Z_{n/3}\}$. Then define the blocks in the design as follows:

$$B = (\textstyle\bigcup_{i \in \mathbf{Z}_3} B(i)) \cup (\bigcup_{l \in \mathbf{Z}_3} B_1(l)) \cup (\bigcup_{l \in \mathbf{Z}_3} B'(l))$$

We first count the number of blocks we get in the construction to see if it equals $b = \frac{5n^2-3n}{4}$ (calculated above when proving the necessity).

$$
\begin{aligned}
|B| &= |(\bigcup_{i\in\mathbb{Z}_3} B(i))| + |(\bigcup_{l\in\mathbb{Z}_3} B_1(l))| + |(\bigcup_{l\in\mathbb{Z}_3} B'(l))| \\
&= (9)(n/3) + (2)(3)((\tfrac{n(n-1)-2n}{2})/3) + 3(\tfrac{(\binom{n}{2}-n)}{6}) \\
&= \tfrac{9n}{3} + n^2 - n - 2n + \tfrac{\frac{n(n-1)}{2}-n}{2} \\
&= 3n + n^2 - 3n + \tfrac{n^2-n}{4} - \tfrac{n}{2} \\
&= \tfrac{4n^2+n^2-n-2n}{4} \\
&= \tfrac{5n^2-3n}{4}
\end{aligned}
$$

Since $|B| = b$ it suffices to check that each edge occurs in at least the correct number (that is, $\lambda_3$ or $\lambda_1$) of blocks in $B$. We consider each edge, $e = \{(x,a),(y,b)\}$, in turn.

1. Suppose $e$ joins two vertices in $C(i)$ for some $i \in \mathbb{Z}_{n/3}$. Then clearly $e$ occurs in $\lambda_1 = 3$ blocks in $B(i)$ if $e$ joins two vertices in the same group and $e$ occurs in $\lambda_2 = 1$ block in $B(i)$ if $e$ joins two vertices in different groups, as required.

2. Next suppose that $e = \{(x,a),(y,a)\}$ for some $a \in \mathbb{Z}_3$ and $0 \leq x, y < n$ where for each $i \in \mathbb{Z}_{n/2}$, $e$ does not join two vertices in $C(i)$. Let $\{x, y, z_1\}$ be the triple in $B_1(a)$ that contains $\{x,y\}$, and suppose $f(\{x,y,z_1\}) = z_2$ is the vertex added to the triple by the nesting. Then the $\lambda_1 = 3$ blocks containing the edge $\{(x,a),(y,a)\}$ are as follows: $\{(x,a),(y,a),(z_1,a),(z_2,a+1)\}$, $\{(x,a),(y,a),(z_1,a),(z_2, a+2)\}$, and in one block in $B'(a)$.

3. Finally suppose $e = \{(x,a),(y,b)\}$ where $a, b \in \mathbb{Z}_3$, $a \neq b$ and where for each $i \in \mathbb{Z}_{n/2}$, $e$ does not join two vertices in $C(i)$. We can assume that $b \equiv a+1 \pmod 3$. Since $\{x,y\} \in G(n,3;3,1)$, exactly one of the following occurs: either there exists a triple $t_1 = \{x, z_3, z_4\} \in B_1(a)$ such that $f(t_1) = y$ or there exists a triple $t_2 = \{y, z_3, z_4\} \in B_1(a)$ such that $f(t_2) = x$. Therefore $e = \{(x,a),(y,b = a+1)\}$ occurs in $\lambda_2 = 1$ of the following blocks: $\{(x,a),(z_3,a),(z_4,a),(y,a+1)\}$ or $\{(x,a),(y,a+1),(z_3,a+1),(z_4,a+1)\}$.

Thus, $n \equiv 3 \pmod{12}$ is a sufficient condition for a $GDD(n, 3, 4; 3, 1)$ to exist.

Now suppose $n \equiv 0 \pmod{12}$, $n \geq 48$.

For each $i \in \mathbb{Z}_{n/12}$, let $B(i)$ be a copy of the $GDD(12, 3, 4; 3, 1)$ created in Lemma 6.1 on the vertices in $C(i) = \{12i, 12i+1, 12i+2, 12i+3, 12i+4, 12i+5, 12i+6, 12i+7, 12i+8, 12i+9, 12i+10, 12i+11\} \times \mathbb{Z}_3$, where for each $l \in \mathbb{Z}_3$, $\{12i, 12i+1, 12i+2, 12i+3, 12i+4, 12i+5, 12i+6, 12i+7, 12i+8, 12i+9, 12i+10, 12i+11\} \times \{l\}$ is a group. By Theorem 2.2, there exists a $(3,1)-GDD$, $(\mathbb{Z}_n, \{\{12i, 12i+1, 12i+2, 12i+3, 12i+4, 12i+5, 12i+6, 12i+7, 12i+8, 12i+9, 12i+10, 12i+11\} \mid i \in \mathbb{Z}_{n/12}\}, B_1)$, that has nesting $f$ of type $12^{n/12}$. Let $B_1(l) = \{\{(x,l), (y,l), (z,l), (f(b), l+1)\}, \{(x, l+1), (y, l+1), (z, l+1), (f(b), l)\} \mid l \in Z_3, \{x,y,z\} \in B_1\}$, reducing the sums in the second coordinate of each vertex modulo 3. By Theorem 2.1 (since $n \geq 48$ and $u = n/3 \geq 4$), for each $l \in Z_3$ let $B'(l)$ be a copy of a $(4,1)-GDD$ of type $12^{n/12}$ on the vertex set $Z_n \times \{l\}$ with groups in $\{\{12i, 12i+1, 12i+2, 12i+3, 12i+4, 12i+5, 12i+6, 12i+7, 12i+8, 12i+9, 12i+10, 12i+11\} \times \{l\} \mid i \in Z_{n/3}\}$. Then define the blocks in the design as follows:

$$B = \left(\bigcup_{i \in \mathbb{Z}_{12}} B(i)\right) \cup \left(\bigcup_{l \in \mathbb{Z}_3} B_1(l)\right) \cup \left(\bigcup_{l \in \mathbb{Z}_3} B'(l)\right)$$

We first count the number of blocks we get in the construction to see if it equals $b = \frac{5n^2-3n}{4}$ (calculated above when proving the necessity).

$$\begin{aligned}
|B| &= \left|\left(\bigcup_{i \in \mathbb{Z}_{12}} B(i)\right)\right| + \left|\left(\bigcup_{l \in \mathbb{Z}_3} B_1(l)\right)\right| + \left|\left(\bigcup_{l \in \mathbb{Z}_3} B'(l)\right)\right| \\
&= (171)\left(\tfrac{n}{12}\right) + 6\left(\left(\tfrac{n(n-1)-11n}{2}\right)/3\right) + 3\left(\frac{\binom{n}{2} - \left(\frac{(12)(11)}{2}\right)\left(\frac{n}{12}\right)}{6}\right) \\
&= \frac{171n}{12} + n^2 - n - 11n + \frac{\frac{n(n-1)}{2} - \frac{11n}{2}}{2} \\
&= \frac{57n}{4} + n^2 - 12n + \frac{n^2-n}{4} - \frac{11n}{4} \\
&= \frac{57n + 4n^2 - 48n + n^2 - n - 11n}{4} \\
&= \frac{5n^2-3n}{4}
\end{aligned}$$

Since $|B| = b$ it suffices to check that each edge occurs in at least the correct number (that is, $\lambda_3$ or $\lambda_1$) of blocks in $B$. We consider each edge, $e = \{(x, a), (y, b)\}$, in turn.

1. Suppose $e$ joins two vertices in $C(i)$ for some $i \in \mathbb{Z}_{n/3}$. Then clearly $e$ occurs in $\lambda_1 = 3$ blocks in $B(i)$ if $e$ joins two vertices in the same group and $e$ occurs in $\lambda_2 = 1$ block in $B(i)$ if $e$ joins two vertices in different groups, as required.

2. Next suppose that $e = \{(x, a), (y, a)\}$ for some $a \in \mathbb{Z}_3$ and $0 \leq x, y < n$ where for each $i \in \mathbb{Z}_{n/2}$, $e$ does not join two vertices in $C(i)$. Let $\{x, y, z_1\}$ be the triple in $B_1(a)$ that contains $\{x, y\}$, and suppose $f(\{x, y, z_1\}) = z_2$ is the vertex added to the triple by the nesting. Then the $\lambda_1 = 3$ blocks containing the edge $\{(x, a), (y, a)\}$ are as follows: $\{(x, a), (y, a), (z_1, a), (z_2, a+1)\}$, $\{(x, a), (y, a), (z_1, a), (z_2, a+2)\}$, and in one block in $B'(a)$.

3. Finally suppose $e = \{(x, a), (y, b)\}$ where $a, b \in \mathbb{Z}_3$, $a \neq b$ and where for each $i \in \mathbb{Z}_{n/2}$, $e$ does not join two vertices in $C(i)$. We can assume that $b \equiv a + 1 \pmod 3$. Since $\{x, y\} \in G(n, 3; 3, 1)$, exactly one of the following occurs: either there exists a triple $t_1 = \{x, z_3, z_4\} \in B_1(a)$ such that $f(t_1) = y$ or there exists a triple $t_2 = \{y, z_3, z_4\} \in B_1(a)$ such that $f(t_2) = x$. Therefore $e = \{(x, a), (y, b = a + 1)\}$ occurs in $\lambda_2 = 1$ of the following blocks: $\{(x, a), (z_3, a), (z_4, a), (y, a+1)\}$ or $\{(x, a), (y, a+1), (z_3, a+1), (z_4, a+1)\}$.

Thus, $n \equiv 0 \pmod{12}$, $n \geq 48$, is a sufficient condition for a $GDD(n, 3, 4; 3, 1)$ to exist. ■

# 7 Generalizing Clatworthy Design $R105$

With two possible exceptions, the existence of a generalized $R105$-design is settled in this section.

**Theorem 7.1.** *There exists a $GDD(n,3,4;6,2)$ if and only if $n \equiv 0 \pmod 3$, except possibly if $n = 6, 9$.*

**Proof.** We start by proving the necessity, so suppose there exists a

$GDD(n,3,4;6,2)$. Since each block contains six edges, the number of blocks in any such design is

$$b = \frac{|E(G(n,3;6,2))|}{6} = \frac{3(\frac{6n(n-1)}{2}) + 3(2n^2)}{6} = \frac{n(5n-3)}{2}. \quad (*)$$

Clearly the number of blocks is an integer, so there are no restrictions on $n$ because either $n$ or $5n-3$ is even.

For each block, each vertex contains 3 of its incident edges. So the number of blocks containing each vertex $v$ is

$$d_{G(n,3;6,2)}(v) = \frac{6(n-1)+2(2n)}{3} = \frac{10}{3}n - 2,$$

which means $n \equiv 0 \pmod 3$ is a necessary condition.

To prove the sufficiency we assume that $n \equiv 0 \pmod 3$, $n \neq 6, 9$, and show there exists a $GDD(n,3,4;6,2)$.

For each $i \in \mathbb{Z}_{n/3}$, let $B(i)$ be a copy of $R105$ on the vertices in $C(i) = \{3i, 3i+1, 3i+2\} \times \mathbb{Z}_3$, where for each $l \in \mathbb{Z}_3$, $\{3i, 3i+1, 3i+2\} \times \{l\}$ is a group. By Theorem 2.2, there exists a $(3,2)$-$GDD$, $(Z_n, \{\{3i, 3i+1, 3i+2\} \mid i \in \mathbb{Z}_{n/3}\}, B_1)$, that has nesting $f$ of type $3^{n/3}$. Let $B_1(l) = \{\{(x,l),(y,l),(z,l),(f(b), l+$

$1)\}, \{(x, l+1), (y, l+1), (z, l+1), (f(b), l)\} \mid l \in Z_3, \{x, y, z\} \in B_1\}$, reducing the sums in the second coordinate of each vertex modulo 3. By Theorem 2.1 (since $n \geq 12$ and $u = n/3 \geq 4$), for each $l \in Z_3$ let $B'(l)$ be a copy of a $(4,1)-GDD$ of type $3^{n/3}$ on the vertex set $Z_n \times \{l\}$ with groups in $\{\{3i, 3i+1, 3i+2\} \times \{l\} \mid i \in Z_{n/3}\}$. Then define the blocks in the design as follows:

$$B = (\bigcup_{i \in \mathbf{Z}_3} B(i)) \cup (\bigcup_{l \in \mathbf{Z}_3} B_1(l)) \cup (\bigcup_{l \in \mathbf{Z}_3} B'(l))$$

We first count the number of blocks we get in the construction to see if it equals $b = \frac{5n^2-3n}{2}$ (calculated above when proving the necessity).

$$\begin{aligned}
|B| &= |(\bigcup_{i \in \mathbf{Z}_3} B(i))| + |(\bigcup_{l \in \mathbf{Z}_3} B_1(l))| + |(\bigcup_{l \in \mathbf{Z}_3} B'(l))| \\
&= (18)(n/3) + (2)(2)(3)((\tfrac{n(n-1)-2n}{2})/3) + 6(\tfrac{(\binom{n}{2})-n)}{6}) \\
&= \tfrac{18n}{3} + 2n^2 - 2n - 4n + \tfrac{n(n-1)-2n}{2} \\
&= 6n + 2n^2 - 6n + \tfrac{n^2-3n}{2} \\
&= \tfrac{4n^2+n^2-3n}{2} \\
&= \tfrac{5n^2-3n}{2}
\end{aligned}$$

Since $|B| = b$ it suffices to check that each edge occurs in at least the correct number (that is, $\lambda_3$ or $\lambda_1$) of blocks in $B$. We consider each edge, $e = \{(x, a), (y, b)\}$, in turn.

1. Suppose $e$ joins two vertices in $C(i)$ for some $i \in \mathbb{Z}_{n/3}$. Then clearly $e$ occurs in $\lambda_1 = 6$ blocks in $B(i)$ if $e$ joins two vertices in the same group and $e$ occurs in $\lambda_2 = 2$ block in $B(i)$ if $e$ joins two vertices in different groups, as required.

2. Next suppose that $e = \{(x, a), (y, a)\}$ for some $a \in \mathbb{Z}_3$ and $0 \leq x, y < n$ where for each $i \in \mathbb{Z}_{n/2}$, $e$ does not join two vertices in $C(i)$. Let $\{x, y, z_1\}$ and $\{x, y, z_3\}$ be the triples in $B_1(a)$ that contain $\{x, y\}$, and suppose $f(\{x, y, z_1\}) = z_2$ and $f(\{x, y, z_3\}) = z_4$ are the vertices added to the triples by the nesting. Then the $\lambda_1 = 6$ blocks containing

the edge $\{(x,a),(y,a)\}$ are as follows:
$\{(x,a),(y,a),(z_1,a),(z_2,a+1)\}$, $\{(x,a),(y,a),(z_1,a),(z_2, a+2)\}$, $\{(x,a),(y,a),(z_3,a),(z_4,a+1)\}$, $\{(x,a),(y,a), (z_3,a),(z_4,a+2)\}$ and in two blocks in $B'(a)$.

3. Finally suppose $e = \{(x,a),(y,b)\}$ where $a,b \in \mathbb{Z}_3$, $a \neq b$ and where for each $i \in \mathbb{Z}_{n/2}$, $e$ does not join two vertices in $C(i)$. We can assume that $b \equiv a+1 \pmod 3$. Since $\{x,y\} \in G(n,3;6,2)$, exactly one of the following occurs: either there exists a triple $t_1 = \{x,z_5,z_6\} \in B_1(a)$ such that $f(t_1) = y$ or there exists a triple $t_2 = \{y,z_5,z_6\} \in B_1(a)$ such that $f(t_2) = x$. The same argument can be made for the other $\{x,y\}$ edge using $\{z_7,z_8\}$. Therefore $e = \{(x,a),(y,b=a+1)\}$ occurs in $\lambda_2 = 2$ of the following blocks:
$\{(x,a),(z_5,a),(z_6,a),(y,a+1)\}$ or $\{(x,a),(y,a+1),(z_5,a+ 1),(z_6,a+1)\}$, and $\{(x,a),(z_7,a),(z_8,a),(y,a+1)\}$ or $\{(x, a),(y,a+1),(z_7,a+1),(z_8,a+1)\}$.

Thus, $n \equiv 6,9 \pmod{12}$, $n \neq 6,9$, is a sufficient condition for a $GDD(n,3,4;6,2)$ to exist. ■

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