Bounds on the Upper k-Domination Number and the Upper k-Star-Forming Number of a Graph

Odile Favaron
LRI, UMR 8623, Université Paris-Sud and CNRS, 91405 Orsay, France

Abstract

A subset A of vertices of a graph G is a k-dominating set if every vertex not in A has at least k neighbors in A and a k-star-forming set if every vertex not in A forms with k vertices of A a not necessarily induced star $K_{1,k}$. The maximum cardinalities of a minimal k-dominating set and of a minimal k-star-forming set of G are respectively denoted by $\Gamma_k(G)$ and $\text{SF}_k(G)$. We determine upper bounds on $\Gamma_k(G)$ and $\text{SF}_k(G)$ and describe the structure of the extremal graphs attaining them.
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1 Introduction

We consider simple undirected graphs $G = (V(G), E(G))$ of order $n = |V(G)|$, minimum degree $\delta(G)$ and maximum degree $\Delta(G)$. We often use the abbreviations V, δ, Δ for $V(G), \delta(G), \Delta(G)$. The subgraph induced by a subset A of V is denoted $G[A]$. Its number of edges, minimum degree and maximum degree are denoted $e(A), \delta(A), \Delta(A)$. The number of edges between two subsets A and B is $e(A, B)$. The degree $d(x)$ of a vertex x of V is equal to its number of neighbors in V and $d_A(x)$ is its number of neighbors in the subset A.

For a positive integer k, a subset A is a *k-dominating set*, briefly k-DS, if $d_A(x) \geq k$ for every vertex $x \in V \setminus A$ and a *k-independent set* if $\Delta(A) \leq k-1$. The minimum and maximum cardinalities of a minimal k-dominating set of G are respectively denoted $\gamma_k(G)$ and $\Gamma_k(G)$. The minimum and maximum cardinalities of a maximal k-independent set of G are respectively denoted $i_k(G)$ and $\beta_k(G)$. By the definitions, $\gamma_k(G) \leq \Gamma_k(G)$ and $i_k(G) \leq \beta_k(G)$ for every k and G. These notions, introduced by Fink and Jacobson in [4], generalize those of domination and independence which correspond to the case $k = 1$. Hence $\gamma_1(G) = \gamma(G)$, $\Gamma_1(G) = \Gamma(G)$, $i_1(G) = i(G)$ and $\beta_1(G) = \beta(G)$.

An independent set is dominating if and only if it is maximal and in this case, it is a minimal dominating set. This implies that every graph satisfies

$$\gamma(G) \leq i(G) \leq \beta(G) \leq \Gamma(G). \tag{1}$$

This inequality chain can be only partially generalized by using the k-domination and the k-independence because for $k \geq 2$, a maximal k-independent set is not necessarily k-dominating. Hence we cannot compare its cardinality to $\gamma_k(G)$ and $\Gamma_k(G)$ and there exist graphs such that $\gamma_k(G) > i_k(G)$ or $\Gamma_k(G) < \beta_k(G)$ as observed in [3]. To overcome this inconvenience, a new definition was proposed by Chellali and Favaron in [1]. A subset A of vertices of G is a *k-star-forming-set*, briefly k-SFS, if every vertex of $V \setminus A$ forms with k vertices of A a not necessarily induced star $K_{1,k}$. In other words, for every vertex x in $V \setminus A$, either $d_A(x) \geq k$ or x has a neighbor y in A such that $d_A(y) \geq k-1$ (or both). The whole set V itself is a k-SFS. The minimum and maximum cardinalities of a minimal (under inclusion) k-SFS are respectively denoted $\mathrm{sf}_k(G)$ and $\mathrm{SF}_k(G)$. A k-independent set is maximal if and only if it is a k-SFS and in this case, A is a minimal k-SFS. Therefore the chain (1) is completely generalized and for every graph G and every positive integer k we have

$$\mathrm{sf}_k(G) \leq i_k(G) \leq \beta_k(G) \leq \mathrm{SF}_k(G). \tag{2}$$

In this paper, we are interested in the upper parameters SF_k and Γ_k and we call Γ_k-set (SF_k-set resp.) a minimal k-DS (k-SFS) of maximum cardinality. For $k = 1$, the 1-SFS are the dominating sets, and thus $\mathrm{SF}_1(G) = \Gamma_1(G) = \Gamma(G)$. It is well-known that $\Gamma(G) \leq n - \delta$ for every graph and the graphs satisfying $\Gamma(G) = n - \delta$ have been descibed by Cockayne and Mynhardt in [2]. For $k \geq \Delta + 1$, V is a minimal k-DS and a minimal k-SFS, and thus $\mathrm{SF}_{\Delta+1}(G) = \Gamma_{\Delta+1}(G) = n$. For the other values of k, a k-DS is a k-SFS which implies $\mathrm{sf}_k(G) \leq \gamma_k(G)$ for every graph. But a minimal k-DS is not necessarily a minimal k-SFS and $\mathrm{SF}_k(G)$ can be larger or smaller than $\Gamma_k(G)$ as shown by examples in [1]. Our purpose is to determine for $k \leq \Delta$ upper bounds on $\mathrm{SF}_k(G)$ and $\Gamma_k(G)$ generalizing

$\Gamma(G) \leq n - \delta$ and to describe the structures of the extremal graphs attaining the bounds. The result depends on the position of k with respect to the minimum degree δ of G.

2 Upper bounds on Γ_k and SF_k for $\delta \leq k \leq \Delta$

Theorem 1 *Let G be a graph of order n, minimum degree δ and maximum degree Δ and let k be a positive integer such that $\delta \leq k \leq \Delta$. Then*

$$\mathrm{SF}_k(G) \leq n-1 \quad \text{and} \quad \Gamma_k(G) \leq n-1.$$

Proof. The whole set V of vertices of G, which is a k-dominating set and a k-star-forming set, is not a minimal k-dominating set nor a minimal k-star-forming set since for every vertex v of degree Δ, the set $V \setminus \{v\}$ is k-dominating and thus k-star-forming. Therefore $\Gamma_k(G) \leq n-1$ and $\mathrm{SF}_k(G) \leq n-1$. ∎

Definition 2 *Family $\mathcal{F}_k$*

Let k be a positive integer. A graph $G = (V, E)$ belongs to the family $\mathcal{F}_k$ if it contains a vertex v such that $\Delta(V \setminus \{v\}) \leq k-1$ and at least one vertex of $N[v]$ has degree at least k in G.

Theorem 3 *If $G \in \mathcal{F}_k$, then $\delta(G) \leq k \leq \Delta(G)$ and $\mathrm{SF}_k(G) = n-1$.*

Proof. The inequalities $\delta(G) \leq k$ and $k \leq \Delta(G)$ are respectively implied by the first and the second condition in the definition of $\mathcal{F}_k$. We note that if $k < \Delta$, then v is the unique vertex of G of degree Δ. Since $d(v) \geq k$ or $d_{V\setminus\{v\}}(x) = k-1$ for some neighbor x of v, the set $V\setminus\{v\}$ is k-star-forming. Moreover for all $x \in V\setminus\{v\}$ and all $y \in N_{V\setminus\{v\}}(x)$, $d_{V\setminus\{v,x\}}(x) \leq k-1$ and $d_{V\setminus\{v,x\}}(y) \leq k-2$. Therefore $V\setminus\{v,x\}$ is not k-star-forming and $V\setminus\{v\}$ is a minimal k-star-forming set of order $n-1$. Hence $\mathrm{SF}_k(G) \geq n-1$. By Theorem 1, $\mathrm{SF}_k(G) = n-1$. ∎

Theorem 4 *A graph G of order n, minimum degree δ and maximum degree Δ satisfies $\mathrm{SF}_k(G) = n-1$ for some positive integer k with $\delta \leq k \leq \Delta$ if and only if $G \in \mathcal{F}_k$.*

Proof. The part "if" is proved in Theorem 3. To prove Part "only if", we consider a graph G with $\delta \leq k \leq \Delta$ and $\mathrm{SF}_k(G) = n-1$. Let $S = V \setminus \{v\}$ be a $\mathrm{SF}_k(G)$-set.

If $\Delta(S) \geq k$, let $x \in S$ with $d_S(x) \geq k$. When possible, we choose $x \in N(v)$. If v is adjacent to x, let w be any vertex in $N_S(x)$. Since $d_{S\setminus\{w\}}(x) \geq k-1$, the set $S\setminus\{w\}$ is k-star-forming, a contradiction with the minimality of the k-SFS S. Hence all the neighbors of v have degree at most $k-1$ and v is not adjacent to x. If $d(v) \geq k$, then $d_{S\setminus\{x\}}(v) = d_S(v) \geq k$ and $S \setminus \{x\}$ is a k-DS and thus a k-SFS, a contradiction. Therefore v is not adjacent to x and $d(v) \leq k-1$. Since S is k-star-forming and $S \setminus \{x\}$ is not, v has a neighbor u in S such that $d_S(u) \geq k-1$ and $d_{S\setminus\{x\}}(u) < k-1$. Hence $u \in N_S(x)$ and $d_S(u) = k-1$. Therefore there exists a vertex t in $N_S(x) \setminus N_S(u)$. Both vertices v and t have a neighbor of degree at least $k-1$ in $S \setminus \{t\}$. Therefore $S \setminus \{t\}$ is a k-star-forming set, a contradiction.

Hence $\Delta(S) \leq k-1$ and since $\Delta \geq k$, v or one of its neighbors has degree at least k. Thus G belongs to $\mathcal{F}_k$. ∎

Definition 5 *Family $\mathcal{G}_k$*

Let k be a positive integer. A graph $G = (V, E)$ belongs to the family $\mathcal{G}_k$ if it contains a vertex v such that either $d(v) > k$ and $\Delta(V \setminus \{v\}) \leq k-1$, or $d(v) = k$ and $d(x) \leq k-1$ for all $x \in V \setminus N[v]$.

Theorem 6 *If $G \in \mathcal{G}_k$, then $\delta \leq k \leq \Delta$ and $\Gamma_k(G) = n-1$.*

Proof. The inequalities $\delta(G) \leq k \leq \Delta(G)$ are obvious from the definition of $\mathcal{G}_k$. Since $d(v) \geq k$, the set $S = V \setminus \{v\}$ is k-dominating. Let $x \in S$ and $S' = S \setminus \{x\}$. If $d_S(x) \leq k-1$, then x is not k-dominated by S'. If $d_S(x) \geq k$, then $d(v) = k$ and $x \in N(v)$. In this case, v is not k-dominated by S'. Therefore S is a minimal k-dominating set of G of order $n-1$ and $\Gamma_k(G) \geq n-1$. By Theorem 1, $\Gamma_k(G) = n-1$. ∎

Theorem 7 *A graph G of order n, minimum degree δ and maximum degree Δ satisfies $\Gamma_k(G) = n-1$ for some integer k with $\delta \leq k \leq \Delta$ if and only if $G \in \mathcal{G}_k$.*

Proof. The part "if" is proved in Theorem 6. To prove Part "only if", we consider a graph $G = (V, E)$ with $\delta \leq k \leq \Delta$ and $\Gamma_k(G) = n-1$. Let $S = V \setminus \{v\}$ be a $\Gamma_k(G)$-set. Then $d(v) \geq k$. Let $x \in S \setminus N(v)$. Since $S \setminus \{x\}$ is not k-dominating, $d(x) = d_S(x) \leq k-1$. Let $y \in N(v)$. Since $S \setminus \{y\}$ is not k-dominating, either $d(v) = k$ or $d_S(y) \leq k-1$ (or both). Therefore $G \in \mathcal{G}_k$. ∎

When $k = 1$, we know that $\Gamma_1(G) = \mathrm{SF}_1(G) = \Gamma(G)$. The graphs of $\mathcal{F}_1 = \mathcal{G}_1$ consist of one star when $\delta = 1$ or of the disjoint union of one star and isolated vertices when $\delta = 0$. We show that for $k \geq 2$, the two families $\mathcal{F}_k$ and $\mathcal{G}_k$ not disjoint and from $k \geq 3$, not included in each other.

Definition 8 *Family $\mathcal{I}_k$*

Let k be a positive integer. A graph $G = (V, E)$ belongs to the family $\mathcal{I}_k$ if it contains a vertex v such that $d(v) \geq k$ and $\Delta(V \setminus \{v\}) \leq k - 1$.

Theorem 9 *1. $\mathcal{F}_k \cap \mathcal{G}_k = \mathcal{I}_k$.*

2. For $k \geq 2$, $\mathcal{G}_k \setminus \mathcal{F}_k \neq \emptyset$ and if $G \in \mathcal{G}_k \setminus \mathcal{F}_k$, then $\Delta(G) > k$.

3. For $k \geq 3$, $\mathcal{F}_k \setminus \mathcal{G}_k \neq \emptyset$ and if $G \in \mathcal{F}_k \setminus \mathcal{G}_k$, then $\Delta(G) = k$.

Proof. 1. Let $G \in \mathcal{I}_k$ and let v be as in the definition of $\mathcal{I}_k$. The couple G, v satifies the conditions of the definitions 2 and 5. Hence $\mathrm{SF}_k(G) = \Gamma_k(G) = n - 1$ and $G \in \mathcal{F}_k \cap \mathcal{G}_k$.

Conversely let $G = (V, E) \in \mathcal{F}_k \cap \mathcal{G}_k$ and following the definition of $\mathcal{F}_k$, let $x \in V$ such that $\Delta(V \setminus \{x\}) \leq k - 1$ and at least one vertex of $N[x]$ at degree at least k in G. If $d(x) \geq k$, then $G \in \mathcal{I}_k$ by taking $v = x$ in the definition of $\mathcal{I}_k$. If $d(x) < k$, let $S = V \setminus \{y\}$ be a minimal k-DS of G of order $n - 1$ (S exists since $G \in \mathcal{G}_k$). Then $d(y) \geq k$, y is necessarily a neighbor of x of degree exactly k and since $\Delta(V \setminus \{x\}) \leq k - 1$, $\Delta(V \setminus \{y\}) \leq k$. Assume $\Delta(V \setminus \{y\}) = k$ and let $z \in V \setminus \{y\}$ such that $d_{V \setminus \{y\}}(z) = k$. Then $z \in N(x)$, $d(z) = k$ and the vertices y and z are not adjacent. Hence $V \setminus \{y, z\}$ is a k-DS, a contradiction with he minimality of the k-DS $V \setminus \{y\}$. Therefore $\Delta(V \setminus \{y\}) \leq k - 1$ and thus $G \in \mathcal{I}_k$ by taking $v = y$ in the definition of $\mathcal{I}_k$.

2. For $k \geq 2$, an example of an arbitrarily large graph in $\mathcal{G}_k \setminus \mathcal{F}_k$ is obtained by joining a vertex v to the centers $c_1, \ldots, c_k$ of k stars $K_{1,p}$ with $p \geq k$ leaves. For these graphs of minimum degree $\delta = 1$ and maximum degree $\Delta = p + 1 > k$, $V \setminus \{v\}$ is a minimal k-DS but not a minimal k-SFS of G. One can check that $V \setminus \{c_1, c_2, \cdots, c_k\}$ is a largest minimal k-SFS of G. Hence $\mathrm{SF}_k(G) = n - k < \Gamma_k(G)$ and $G \notin \mathcal{F}_k$.

More generally, if $G \in \mathcal{G}_k$ and $\Delta(G) = k$, then the vertex v in the definition of $\mathcal{G}_k$ has degree $d(v) = k$ and $\Delta(V \setminus \{v\}) \leq k - 1$, implying $G \in \mathcal{F}_k$. Hence if $G \in \mathcal{G}_k \setminus \mathcal{F}_k$, then $\Delta(G) > k$.

3. For $k = 2$, $\mathcal{F}_2 = \mathcal{I}_2$ is the family of graphs obtained by joining a vertex v to at least two vertices of the disjoint union of K_1's and K_2's. Hence $\mathcal{F}_2 \subseteq \mathcal{G}_2$.

For $k \geq 3$, an example of an arbitrarily large graph in $\mathcal{F}_k \setminus \mathcal{G}_k$ is obtained by joining two vertices v and w to the centers $c_1, c_2, \cdots, c_{k-1}$ of $k-1$ subdivided stars (each ray is subdivided by an arbitrary number of vertices) with $k-2$ leaves. For these graphs of minimum degree $\delta = 1$ and maximum degree $\Delta = k$, the set $V \setminus \{w\}$ is a minimal k-SFS of G but is not k-dominating. One can check that $V \setminus \{c_1, c_2, \cdots, c_{k-1}\}$ is a largest minimal k-DS of G. Hence $\mathrm{SF}_k(G) = n-1$, $\Gamma_k(G) = n-k+1$ and $G \in \mathcal{F}_k \setminus \mathcal{G}_k$.

More generally, if $G \in \mathcal{F}_k$ and $\Delta(G) > k$, then G has a unique vertex v of degree $\Delta(G)$ and $V \setminus \{v\}$ is the unique $SF_k(G)$-set. Then $\Delta(V \setminus \{v\}) \leq k-1$, implying $G \in \mathcal{G}_k$. Hence if $G \in \mathcal{F}_k \setminus \mathcal{G}_k$, then $\Delta(G) = k$. ∎

3 Upper bounds on Γ_k and SF_k for $1 \leq k \leq \delta$

Theorem 10 *Let G be a graph of order n and minimum degree δ and let k be an integer such that $1 \leq k \leq \delta$. Then $SF_k(G) \leq n - \delta + k - 1$ and $\Gamma_k(G) \leq n - \delta + k - 1$.*

Proof. Let S be a set of vertices of G of order $|S| \geq n - \delta + k$ and let $x \in S$. Then $|V \setminus S| \leq \delta - k$, $d_S(x) \geq \delta - |V \setminus S| \geq k$ and $d_S(v)) \geq \delta - |V \setminus (S \cup \{v\})| \geq k+1$ for all $v \in V \setminus S$. Hence $S \setminus \{x\}$ is a k-dominating set and thus also a k-star-forming set. Therefore no k-dominating set nor k-star-forming set of order more than $n - \delta + k - 1$ can be minimal, which implies the result. ∎

Definition 11 *Family $\mathcal{H}_k$*

Let k be a positive integer. A graph $G = (V, E)$ belongs to the family $\mathcal{H}_k$ if V is the disjoint union of two non-empty sets S and T, $|S| \geq k$, $G[S]$ is $(k-1)$-regular, $\delta(T) \geq |T| - |S| + k - 1$ and all the edges between S and T exist.

Theorem 12 *Let G be a graph of order n in $\mathcal{H}_k$.*

1. *G has minimum degree $\delta \geq k$ and $SF_k(G) = \Gamma_k(G) = n - \delta + k - 1$.*

2. *If k is even, then $n - \delta$ is odd.*

Proof. 1. For any vertex x, $d(x) = k - 1 + |T|$ if $x \in S$ and $d(x) \geq |S| + \delta(T) \geq k - 1 + |T|$ if $x \in T$. Hence

$$\delta(G) = k - 1 + |T| \geq k. \tag{3}$$

The set S is k-dominating, and thus k-star-forming, since every vertex of T is k-dominated by S. If x is any vertex of S, then $d_{S\setminus\{x\}}(x) = k-1$ and each neighbor of x in S has degree $k-2$ in $S \setminus \{x\}$. Therefore S is a minimal k-dominating set and a minimal k-star-forming set of order $n - |T| = n - \delta + k - 1$ by (3). The result follows from Theorem 10.

2. If k is even, then the order $n - \delta + k - 1$ of the $(k-1)$-regular set S must be even. Therefore $n - \delta$ is odd. ■

Theorem 13 *Let $G = (V, E)$ be a graph of order n and minimum degree δ. If k is an integer such that $2 \le k \le \delta$, then $SF_k(G) = n - \delta + k - 1$ if and only if $G \in \mathcal{H}_k$.*

Proof. The part "if" is proved in Theorem 12. To prove Part "only if", we consider a graph $G = (V, E)$ with $\delta \ge k \ge 2$ and $\mathrm{SF}_k(G) = n - \delta + k - 1$. Let S be a $SF_k(G)$-set. Since $|V \setminus S| = \delta - k + 1$, every vertex of S (respectively $V \setminus S$) has at least $k-1$ (respectively k) neighbors in S. Let $A = \{x \in S \mid d_S(x) = k-1\}$, $B = \{x \in S \mid d_S(x) \ge k\}$ and $P = \{x \in V \setminus S \mid d_S(x) = k\}$. Note that for all $v \in A \cup P$, $d(v) \ge \delta$ implies that v is adjacent to all the vertices of $V \setminus (S \cup \{v\})$. In particular, all the edges between A and $V \setminus S$ exist.

Suppose first that $\Delta(S) \ge k$, i.e. $B \ne \emptyset$, and let $x \in B$. Since $S \setminus \{x\}$ is not a k-star-forming set and x is k-dominated by $S \setminus \{x\}$, x has at least one neighbor y in P such that every vertex of $N_S(y) \setminus \{x\}$ has less than $k-1$ neighbors in $S \setminus \{x\}$. Therefore the $k-1$ vertices of $N_S(y) \setminus \{x\}$ are in A and are adjacent to x. Since all the edges between A and P exist, $|A| = k-1 \ge 1$ since $k \ge 2$. This implies that every vertex of $A \cup (V \setminus S)$ has at least one neighbor in B. Hence for all $z \in A$, $S \setminus \{z\}$ is a k-star-forming set, in contradiction to the minimality of S.

Therefore $\Delta(S) = k-1$ and $G[S]$ is $(k-1)$-regular. Each vertex of $S = A$ is adjacent to all the vertices of $V \setminus S$. Moreover, since each vertex of $V \setminus S$ has degree at least $\delta = |V \setminus S| + k - 1$ in G, $\delta(V \setminus S) \ge |V \setminus S| + k - 1 - |S|$. Thus $G \in \mathcal{H}_k$. ■

The following corollary is a consequence of $\Gamma_1(G) = \mathrm{SF}_1(G)$ when $k = 1$ and of Theorems 12 and 13 when $k \ge 2$.

Corollary 14 *Let G be a graph of order n and minimum degree δ. If $SF_k(G) = n - \delta + k - 1$ for some positive integer $k \le \delta$, then $\Gamma_k(G) = \mathrm{SF}_k(G)$.*

Now we study the family of graphs such that $\Gamma_k(G) = n - \delta + k - 1$.

Definition 15 *Family $\mathcal{L}_k$*

Let k be a positive integer. The graph $G = (V, E)$ belongs to the family $\mathcal{L}_k$ if V is partitioned into $A \cup B \cup P \cup Q$ with $B, P \neq \emptyset$ and the following properties are satisfied:

(Pa) $d_{A\cup B}(x) = k - 1$ *for all* $x \in A$

(Pb) $d_{A\cup B}(x) \geq k$ *for all* $x \in B$

(Pc) *All the edges between A and $P \cup Q$ exist*

(Pd) $d_B(x) = k - |A|$ *for all* $x \in P$

(Pe) $d_B(x) \geq k - |A| + 1$ *for all* $x \in Q$

(Pf) *$G[P]$ is complete*

(Pg) *All the edges between P and Q exist*

(Ph) $d_P(x) \geq 1$ *for all* $x \in B$

(Pi) $d(x) \geq |P| + |Q| + k - 1$ *for all* $x \in B \cup Q$.

For a graph $G \in \mathcal{L}_k$, we denote $|A|$, $|B|$, $|P|$ and $|Q|$ respectively by a, b, p and q. The construction of a graph satisfying Properties (Pa) to (Pi) is not possible for every quadruplet a, b, p, q. For instance, (Pb) implies

$$a + b \geq k + 1 \tag{4}$$

since $b \neq 0$, and since $p \neq 0$, (Pd) and (Ph) imply

$$k - a \geq 1 \quad \text{and} \quad p(k - a) \geq b. \tag{5}$$

Therefore $p \geq 2$ and $b \geq 2$. Similarly, considering the number $e(A, B)$ of edges between A and B gives

$$\sum_{x\in B} d_{A\cup B}(x) - 2e(B) = \sum_{x\in B} d_A(x) = e(A, B) = \sum_{x\in A} d_B(x) = \sum_{x\in A} d_{A\cup B}(x) - 2e(A)$$

By (Pa) and (Pb), $\sum_{x\in B} d_{A\cup B}(x) \geq kb$ and $\sum_{x\in A} d_{A\cup B}(x) = a(k - 1)$. Since $2e(B) \leq b(b - 1)$ and $2e(A) \geq 0$, we get

$$b(k - (b - 1)) \leq a(k - 1). \tag{6}$$

Inequalities (4), (5), (6) are not yet sufficient for a quadruplet to correspond to a graph in $\mathcal{L}_k$. The determination of a complete system of necessary and sufficient conditions on a, b, p, q to correspond to a graph is rather complicated. We merely give examples of graphs in $\mathcal{L}_k$, which shows that the conditions (Pa) to (Pi) are not incompatible.

Proposition 16 *Let n and ℓ be two integers with $2 \leq \ell \leq n/2$ and let $G_{n,\ell}$ be obtained from a clique K_n by deleting the ℓ edges of a matching $\{x_1y_1, x_2y_2, \cdots, x_\ell y_\ell\}$. Then $G \in \mathcal{L}_k$ for every k such that $\ell - 1 \leq k \leq 2\ell - 3$.*

Proof. For $\ell - 1 \leq k \leq 2\ell - 3$, the integer $p = 2\ell - k - 1$ is such that $2 \leq p \leq \ell$. The partition $B = \{x_1, \cdots, x_p\}$, $P = \{y_1, \cdots, y_p\}$, $A = \{x_{p+1}, x_{p+2}, \cdots, x_\ell, y_{p+1}, y_{p+2}, \cdots, y_\ell\}$, $Q = V \setminus (A \cup B \cup P)$ of V satisfies Conditions (Pa) to (Pi) of Definition 15. Hence $G \in \mathcal{L}_k$. ■

Theorem 17 *Let G be a graph of order n and minimum degree δ. If $G \in \mathcal{L}_k$ for some positive integer k, then $k < \delta \leq n-2$ and $\Gamma_k(G) = n-\delta+k-1$.*

Proof. Every vertex x in $A \cup P$ has degree $p+q+k-1$ by (Pa) and (Pc) if $x \in A$ and by (Pc), (Pd), (Pf) and (Pg) if $x \in P$. Every vertex x in $B \cup Q$ has degree at least $p+q+k-1$ by (Pi). Hence

$$\delta = p + q + k - 1 = n - (a + b) + k - 1. \tag{7}$$

By (4) and since $p \geq 2$, $k < \delta \leq n - 2$.

By (Pc), (Pd) and (Pe), $d_{A\cup B}(x) = k$ if $x \in P$ and $d_{A\cup B}(x) \geq k + 1$ if $x \in Q$. Hence $A \cup B$ is a k-dominating set of G. Moreover for each $x \in A \cup B$, $(A \cup B) \setminus \{x\}$ is no more a k-DS of G, by (Pa) if $x \in A$ and by (Ph), (Pc) and (Pd) if $x \in B$. Therefore $A \cup B$ is a minimal k-dominating set of order $a + b = n - (p + q) = n - \delta + k - 1$ by (7). The result follows from Theorem 10. ■

Theorem 18 *Let $G = (V, E)$ be a graph of order n and minimum degree δ. If k is an integer such that $1 \leq k \leq \delta$ then $\Gamma_k(G) = n - \delta + k - 1$ if and only if $G \in \mathcal{H}_k \cup \mathcal{L}_k$.*

Proof. The part "if" is proved in Theorems 12 and 17. To prove Part "only if", we consider a graph $G = (V, E)$ with $\delta \geq k \geq 1$ and $\Gamma_k(G) = n-\delta+k-1$. Let S be a $\Gamma_k(G)$-set. Then $\delta = n - |S| + k - 1$. For every $x \in S$, $\delta \leq d(x) \leq d_S(x) + |V \setminus S| = d_S(x) + \delta - k + 1$. Therefore $\delta(S) \geq k - 1$.

If $\Delta(S) = k-1$, then $G[S]$ is $(k-1)$-regular. Let $V \setminus S = T$. Since $\delta = n - |S| + k - 1$, all the edges between S and T exist. Moreover $\delta(T) \geq \delta - |S| \geq |T| - |S| + k - 1$. Therefore $G \in \mathcal{H}_k$.

If $\Delta(S) \geq k$, let $A = \{x \in S \mid d_S(x) = k-1\}$, $B = \{x \in S \mid d_S(x) \geq k\} \neq \emptyset$, $P = \{x \in V \setminus S \mid d_S(x) = k\}$ and $Q = \{x \in V \setminus S \mid d_S(x) > k\}$. Since the set S is k-dominating, the four sets A, B, P, Q partition V. Properties (Pa) and (Pb) come from the definition of A and B. By the minimality of S, $S \setminus \{x\}$ is not k-dominating for any $x \in B$, implying that every vertex of B has a neighbor in P (Property (Ph)). Hence $P \neq \emptyset$. From (Pa) and $\delta = n - |S| + k - 1$, we get $d_{P \cup Q}(x) \geq n - |S| = |P| + |Q|$ for all $x \in A$ (Property (Pc)). Properties (Pd) and (Pe) are consequences of the definition of P and Q together with (Pc). Properties (Pf) and (Pg) are consequences of the definition of P together with $d(x) \geq \delta = |P| + |Q| + k - 1$ for all $x \in P$. Finally (Pi) comes from $\delta = |P| + |Q| + k - 1$. Therefore the partition $A \cup B \cup P \cup Q$ of V satisfies Properties (Pa) to (Pi) and $G \in \mathcal{L}_k$.
■

The extremal cases $k = 1$ and $k = \delta$ are of particular interest. As they are considered in both Sections 2 and 3, they can be used to verify the compatibility of the results of the two sections.

Particular case k=1

For every graph G of minimum degree $\delta \geq 1$, $SF_1(G) = \Gamma_1(G) = \Gamma(G) \leq n - \delta$ and the extremal graphs are those of $\mathcal{H}_1 \cup \mathcal{L}_1$ by Theorem 18.

From Definition 11, the graphs G of $\mathcal{H}_1$ are as follows: V is partitioned into two sets S and T, $G[S]$ is independent and all the edges between S and T exist. The edges of $G[T]$ are optional subject to $\delta(T) + |S| \geq \delta$.

Let (A, B, P, Q) be a partition of a graph G of $\mathcal{L}_1$ as in Definition 15. Necessarily, $a = 0$ and $p \geq b$ since $a \leq k - 1$ and $p(k - a) \geq b$. By (7), the minimum degree of G is $\delta = p + q$. Hence, for all $x \in B$ we have

$$p + q \leq d(x) = d_B(x) + d_P(x) + d_Q(x) \leq d_B(x) + d_P(x) + q. \qquad (8)$$

Therefore $\sum_{x \in B}(d_P(x) + d_B(x)) \geq pb$, i.e., $e(B,P) + 2e(B) \geq pb$. By Property (Pd), $e(B,P) = p$. Hence $2e(B) \geq p(b-1)$ and since $2e(B) \leq b(b-1)$, we get $b \geq p$. Therefore $b = p$ and $2e(B) = b(b-1)$, i. e. $G[B]$ is complete. By (Pd) and (Ph), the equality $b = p$ implies that $d_P(x) = 1$, and thus the edges between B and P form a perfect matching. Then $d_B(x) + d_P(x) = p$ for all $x \in B$ and the first inequality in (8) implies that $d_Q(x) = q$ for all $x \in B$. The graphs G of $\mathcal{L}_1$ are thus as follows: V is partitioned into three sets B, P, Q with $|B| \geq 2$ and $|P| \geq 2$, $G[B]$ and

$G[P]$ are complete, the edges between B and P form a perfect matching, and all the edges between $B \cup P$ and Q exist. The graph is completed by edges in $G[Q]$ subject to (Pi).

Note that the two families $\mathcal{H}_1$ and $\mathcal{L}_1$ were already determined by Cockayne and Mynhardt in [2] as those of graphs satisfying $\Gamma(G) = n - \delta(G)$.

When $\delta = 1$, the case $k = 1$ is also considered in Section 2. The reader can verify that the graphs of minimum degree 1 of $\mathcal{F}_1 = \mathcal{G}_1$ obtained in Section 2 and the graphs of minimum degree 1 of $\mathcal{H}_1$ (since those of of $\mathcal{L}_1$ have minimum degree $\delta \geq 2$) obtained in Section 3 are the same, namely the stars.

Particular case $k = \delta \geq 2$

The case $k = \delta$ is considered in both sections 2 and 3. The family of graphs of minimum degree $\delta \geq 2$ satisfying $\Gamma_\delta(G) = n - 1$ is equal to $\mathcal{G}_\delta$ by Theorem 7 and to $\mathcal{H}_\delta$ by Theorem 18 (since by Theorem 17, the graphs in $\mathcal{L}_k$ are such that $\delta > k$). Similarly the family of graphs of minimum degree $\delta \geq 2$ satisfying $SF_\delta(G) = n - 1$ is equal to $\mathcal{F}_\delta$ by Theorem 4 and to $\mathcal{H}_\delta$ by Theorem 13. It can be checked that the graphs of minimum degree $\delta = k$ of $\mathcal{F}_k$, $\mathcal{G}_k$ and $\mathcal{H}_k$ form the same family of graphs obtained by joining a vertex v to every vertex of a $(\delta - 1)$-regular graph.

We finish with some examples of graphs in $\mathcal{L}_k \cap \mathcal{H}_k$, $\mathcal{L}_k \setminus \mathcal{H}_k$ or $\mathcal{H}_k \setminus \mathcal{L}_k$.

- By Proposition 16, the graph $G_{n,\ell}$ of minimum degree $\delta = n - 2$ is in $\mathcal{L}_{\ell-1}$. When ℓ is even, the set $S = \{x_1, x_2, \cdots, x_{\frac{\ell}{2}}, y_1, y_2, \cdots, y_{\frac{\ell}{2}}\}$ has order $\ell = n - \delta + (\ell - 1) - 1$ and $G[S]$ is $(\ell - 2)$-regular. The partition $S, V \setminus S$ satisfies the conditions of Definition 11 for $k = \ell - 1$. Therefore $G_{n,\ell} \in \mathcal{H}_{\ell-1} \cap \mathcal{L}_{\ell-1}$.

- By the second part of Theorem 12, the graph $G_{n,\ell}$, for which $n - \delta = 2$ is even, cannot belong to $\mathcal{H}_k$ for k even. Hence by Proposition 16, $G_{n,\ell} \in \mathcal{L}_k \setminus \mathcal{H}_k$ for every even k between $\ell - 1$ and $2\ell - 3$.

- For $k \geq 3$ of any parity and $n \geq k + 4$, consider a graph L with vertex set $\{a_1, b_1, b_2, \cdots, b_{k+1}, p_1, p_2, q_1, q_2, \cdots, q_{n-k-4}\}$. All the edges exist in L except the six edges a_1b_1, a_1b_3, p_1b_1, p_1b_2, p_2b_3 and p_2b_4. Then $\delta = n - 3$. The sets $A = \{a_1\}$, $B = \{b_1, b_2, \cdots, b_{k+1}\}$, $P = \{p_1, p_2\}$, $Q = \{q_1, q_2, \cdots, q_{n-k-4}\}$ satisfy Conditions (Pa) to (Pi) of the definition of $\mathcal{L}_k$. Hence $\Gamma_k(L) = n - \delta + k - 1 = k + 2$ and the set $A \cup B$ is a $\Gamma_k(L)$-set of order $k + 2$. If $L \in \mathcal{H}_k$ then it would contain a $(k-1)$-regular subgraph S of order $k + 2$. In such a subgraph, $k + 2$ edges are missing. This is clearly impossible if $k+2>6$. This is also impossible when $k=3$ or 4 because of the disposition of the six missing edges of L. Therefore $L \in \mathcal{L}_k \setminus \mathcal{H}_k$.

- For every $k \geq 1$ and $n \geq k+1$, the complete graph K_n belongs to $\mathcal{H}_k$ (consider a partition S, T where S is any set of $n-\delta+k-1=k$ vertices). But $K_n \notin \mathcal{L}_k$ since every graph of $\mathcal{L}_k$ has minimum degree at most $n-2$ by Theorem 17. Therefore $K_n \in \mathcal{H}_k \setminus \mathcal{L}_k$.

4 Open question

The examples of graphs in $\mathcal{G}_k \setminus \mathcal{F}_k$ or in $\mathcal{F}_k \setminus \mathcal{G}_k$ given in Theorem 9, show that when $k \geq \delta$, $\Gamma_k(G)$ can be larger or smaller than $\mathrm{SF}_k(G)$.

When $k<\delta$, the graphs in $\mathcal{L}_k \setminus \mathcal{H}_k$ satisfy $\mathrm{SF}_k(G) < \Gamma_k(G) = n-\delta+k-1$, but if $\mathrm{SF}_k(G) = n-\delta+k-1$ then $\Gamma_k(G)$ is also equal to $n-\delta+k-1$ by Corollary 14. Hence the following question can be considered:

Does every graph G of minimum degree δ satisfy $\mathrm{SF}_k(G) \leq \Gamma_k(G)$ for every positive integer $k<\delta$?

References

[1] M. Chellali and O. Favaron, On k-star-forming sets in graphs, *J. Combin. Math. Combin. Comput.* 68 (2009) 205-214.

[2] E. J. Cockayne and C. M. Mynhardt, Domination and irredundance in cubic graphs, *Discrete Math.* 167/168 (1997) 205-214.

[3] O. Favaron, k-domination and k-independence in graphs, *Ars Combin.* 25C (1988) 159–167.

[4] J.F. Fink and M.S. Jacobson, n-domination in graphs. *Graph Theory with Applications to Algorithms and Computer Science.* John Wiley and Sons. New York (1985) 283–300.