C_4-Frames of $M(b, n)$

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Abstract

Let $M(b, n)$ be the complete multipartite graph with b parts $B_0, ..., B_{b-1}$ of size n. A 4-cycle system of $M(b, n)$ is said to be a *frame* if the 4-cycles can be partitioned into sets $S_1, ..., S_z$ such that for $1 \leq j \leq z$, S_j induces a 2-factor of $M(b, n) \setminus B_i$ for some $i \in \mathbb{Z}_b$. The existence of a C_4-frame of $M(b, n)$ has been settled when $n = 4$ [6]. In this paper, we completely settle the existence question of a C_4-frame of $M(b, n)$ for all $b \neq 2$ and n.

1 Introduction

Let $M(b, n)$ be the complete simple multipartite graph with b parts $B_0, ..., B_{b-1}$ of size n. The vertex set, $V(M(b, n))$, is always chosen to be $\mathbb{Z}_b \times \mathbb{Z}_n$, with parts $\{j\} \times \mathbb{Z}_n$ for each $j \in \mathbb{Z}_b$. The edge set, $E(M(b, n))$, is $\{\{(i, s), (j, t)\} \mid i, j \in \mathbb{Z}_b, i < j$, and $s, t \in \mathbb{Z}_n\}$. Let C_z denote a cycle of length z.

An H-decomposition of a graph G is a partition of $E(G)$, each element of which induces a copy of H. A *z-cycle system* of

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a graph G is a set of z-cycles that partition the edges of G. A z-cycle system is a C_z-decompostion of G. There has been considerable interest in 4-cycle systems of bipartite and multipartite graphs. Sotteau has shown in [7] that a complete bipartite graph can be decomposed into cycles of even length under certain conditions. This result has been extended to multipartite graphs in [1] by Billington and Cavenagh. Billington and Hoffman produced a *gregarious* 4-cycle-system of multipartite graphs in [2] (a gregarious 4-cycle has each vertex in a different part).

A *2-factor* of a graph G is a spanning *2-regular* subgraph of G. A *2-factorization* of G is a set of edge-disjoint 2-factors, the edges of which partition $E(G)$. A C_z-factorization is a 2-factorization such that each component of each 2-factor is a cycle of length z; each 2-factor of a C_z-factorization is known as a C_z-factor. C_z-factorizations are also known as *resolvable* C_z*-decompositions.*

A *frame* of the multipartite graph $M(b, n)$ is a collection of sets of edges, $S_1, \ldots, S_z$, that partition $E(M(b, n))$ such that for $1 \le j \le z$, S_j induces a 2-factor of $M(b, n) \setminus B_i$ for some $i \in \mathbb{Z}_b$. A C_4-frame is a frame such that each component of each 2-factor is a 4-cycle.

The existence of a C_4-frame of $M(b, 4)$ was needed for a construction of C_4-factorizations in [6]. The existence of said frame was completely settled in that paper. In this paper, we completely settle the problem for all $M(b, n)$.

2 Preliminary Results

We begin by finding some necessary conditions for the existence of a C_4-frame of $M(b, n)$.

Lemma 1 *If there exists a C_4-frame of $M(b, n)$, then*

1. $b \neq 2$,

2. *$|E(M(b,n))| \equiv 0 \ (mod\ 4)$,*

3. *$(b-1)n \equiv 0 \ (mod\ 4)$, and*

4. *at least one of b and n is even.*

Proof If $b = 0$ or $b = 1$, then there are no edges to partition in $M(b,n)$. There are no edges joining vertices in the same part in $M(b,n)$. So in order to produce 2-factors of $M(b,n) \setminus (\{d\} \times \mathbb{Z}_n)$, it must contain more than one part. If $b = 2$, then $M(b,n) \setminus (\{d\} \times \mathbb{Z}_n)$ contains only one part. So $b \neq 2$.

Since a C_4-frame of $M(b,n)$ is a 4-cycle-system, the number of edges in $M(b,n)$, $\binom{b}{2}n^2$, must be divisible by four. Also, in order to produce C_4-factors of $M(b,n) \setminus (\{d\} \times \mathbb{Z}_n)$ for $d \in \mathbb{Z}_b$, the number of vertices of $M(b,n) \setminus (\{d\} \times \mathbb{Z}_n)$ for $d \in \mathbb{Z}_b$ must be divisible by four. So $(b-1)n \equiv 0 \pmod 4$.

Each of the C_4-factors consists of $(b-1)n$ edges. In the multipartite graph $M(b,n)$, there are $\binom{b}{2}n^2$ edges. So the number of C_4-factors in a C_4-frame of $M(b,n)$ is

$$\frac{\binom{b}{2}n^2}{(b-1)n} = \frac{1}{2}bn,$$

which implies that at least one of b and n is even. ■

Lemma 2 *[3] Suppose $a \equiv 1(mod\ 4)$. Then near C_4-factorizations of λK_a exist for all even λ.*

3 C_4-Frames of $M(b,4m)$

In this section, we produce C_4-frames of the multipartite graph $M(b,n)$ with each part size a multiple of four. We use three constructions for producing a C_4-frame of $M(b,4m)$ based on the parity of b and m. When b is odd, the parity of m is irrelevant. When b is even, we have a construction for m is even and m is odd. It can be easily seen that $M(b, 4m)$ satisfies the necessary conditions in Lemma 1 as long as $b \neq 2$.

Theorem 1 *Let b be odd. There exists a C_4-frame of $M(b, 4m)$ for all $b \neq 2$.*

Proof Let F' be a near 1-factorization on the vertex set $\mathbb{Z}_b$, and for each $d \in \mathbb{Z}_b$ let F'_d be the near 1-factor in F' with deficiency d; so each vertex in $\mathbb{Z}_b \setminus \{d\}$ occurs in exactly one edge in F'_d.

Let F be a 1-factorization on the vertex set $\mathbb{Z}_m \times \mathbb{Z}_m$, and for each $t \in \mathbb{Z}_m$, let F_t be a 1-factor in F. Let $K(B_x, B_y)$ be the complete simple bipartite graph on the parts $B_x = \{x\} \times \mathbb{Z}_{4m}$ and $B_y = \{y\} \times \mathbb{Z}_{4m}$, $0 \leq x < y \leq b-1$. Let $K(B_{x,k}, B_{y,l})$ be the complete simple bipartite graph on parts $B_{x,k} = \{x\} \times \{4k, 4k+1, 4k+2, 4k+3\}$ and $B_{y,l} = \{y\} \times \{4l, 4l+1, 4l+2, 4l+3\}$, $0 \leq x < y \leq b-1$, $k, l \in \mathbb{Z}_m$.

Notice that

$$K(B_x, B_y) = \bigcup_{\substack{\{k,l\} \in E(F_t) \\ t \in \mathbb{Z}_m}} K(B_{x,k}, B_{y,l}).$$

For each $\{k, l\} \in E(F_t)$, define a C_4-factorization of $K(B_{x,k}, B_{y,l})$, consisting of two C_4-factors:

$$\pi_{xk,yl}(0) = \{((x, 4k), (y, 4l), (x, 4k+2), (y, 4l+2)),$$

$$((x, 4k+1), (y, 4l+1), (x, 4k+3), (y, 4l+3))\}$$

$$\pi_{xk,yl}(1) = \{((x, 4k), (y, 4l+1), (x, 4k+2), (y, 4l+3)),$$

$$((x, 4k+1), (y, 4l+2), (x, 4k+3), (y, 4l)\}$$

For each $d \in \mathbb{Z}_b$, let

$$M_d = \bigcup_{\{x,y\} \in E(F'_d)} K(B_x, B_y),$$

which has a C_4-factorization, P_d, consisting of the $2m$ C_4-factors:

$$M_d(j,t) = \bigcup_{\substack{\{x,y\} \in E(F'_d) \\ \{k,l\} \in E(F_t)}} \pi_{xk,yl}(j) \text{ for each } j \in \mathbb{Z}_2,\ t \in \mathbb{Z}_m.$$

Notice that

$$M(b, 4m) = \bigcup_{d \in \mathbb{Z}_b} M_d,$$

each edge of which therefore occurs in exactly one cycle in

$$\bigcup_{\substack{d \in \mathbb{Z}_b \\ j \in \mathbb{Z}_2 \\ t \in \mathbb{Z}_m}} M_d(j,t).$$

Notice also that each $M_d(j,t)$ is a C_4-factor of $M(b,4m) \setminus (\{d\} \times \mathbb{Z}_{4m})$ so the 4-cycles in

$$P(b) = \bigcup_{d \in \mathbb{Z}_b} P_d,$$

form a C_4-frame of $M(b, 4m)$.

■

Theorem 2 *Suppose b and m are even. There exists a C_4-frame of $M(b,4m)$ for all $b \neq 2$.*

Proof Partition the vertices $\mathbb{Z}_b \times \mathbb{Z}_{4m}$ into blocks $B = \{B_{i,j} \mid i \in \mathbb{Z}_b, j \in \mathbb{Z}_m\}$, each of size 4, where $B_{i,j} = \{\{i\} \times \{4j, 4j+1, 4j+$

$2, 4j+3\}\}$. Let $K(B_{i,j}, B_{k,l})$ be the complete simple bipartite graph on parts $B_{i,j}$ and $B_{k,l}$. Notice that

$$M(b,4m) = \bigcup_{\substack{i,k\in\mathbb{Z}_b\\ i<k\\ j,l\in\mathbb{Z}_m}} K(B_{i,j}, B_{k,l}).$$

For $i,k \in \mathbb{Z}_b$ and $j,l \in \mathbb{Z}_m$, let

$$S(b,4m) = \bigcup_{\substack{i,k\in\mathbb{Z}_b\\ i<k\\ j\in\mathbb{Z}_m}} K(B_{i,j}, B_{k,j})$$

and let

$$T(b,4m) = \bigcup_{\substack{i,k\in\mathbb{Z}_b\\ i<k\\ j,l\in\mathbb{Z}_m\\ j\neq l}} K(B_{i,j}, B_{k,l}).$$

Notice that $M(b,4m) = S(b,4m) \cup T(b,4m)$ so that in order to produce a C_4-frame of $M(b,4m)$, we may combine C_4-frames of $S(b,4m)$ and $T(b,4m)$. We begin by producing a C_4-frame of $T(b,4m)$.

Let F be a 1-factorization of K_m with vertex set $\mathbb{Z}_m$, and for each $x \in \mathbb{Z}_{m-1}$, let F_x be a 1-factor in F. For each $\{f,g\} \in F_x$, let $K(\{f\},\{g\})$ be the complete simple bipartite graph with parts $\{f\} \times \mathbb{Z}_b$ and $\{g\} \times \mathbb{Z}_b$.

Let P be a proper-edge coloring of $K(\{f\},\{g\})$ such that the edges in $\{\{(f,d),(g,d)\} \mid d \in \mathbb{Z}_b\}$ receive different colors. All such $K(\{f\},\{g\})$ produced by the 1-factorization must have the same proper-edge coloring P, else a frame is not guaranteed. For each $d \in \mathbb{Z}_b$, let P_d be the set of edges colored d in the coloring P, and let $P_d^- = P_d \setminus \{(f,d),(g,d)\}$ be a subset of P_d.

Remark A proper-edge coloring of $K(\{f\},\{g\})$ such that the edges in $\{\{(f,d),(g,d)\} \mid d \in \mathbb{Z}_b\}$ receive different colors is equivalent to an idempotent latin square. The edge (i,j) is colored k in the coloring if the (i,j)-entry of the idempotent latin square contains symbol k. Note for all $n \neq 2$, there exists an idempotent latin square of order n [5].

For each edge $e = \{(f,k),(g,l)\} \in P_d^-$, define a C_4-factorization of $K(B_{f,k}, B_{g,l})$ consisting of two C_4-factors:

$$\pi_{fk,gl}(0) = \{((4f,k),(4g,l),(4f+2,k),(4g+2,l)),$$
$$((4f+1,k),(4g+1,l),(4f+3,k),(4g+3,l))\}$$
$$\pi_{fk,gl}(1) = \{((4f,k),(4g+1,l),(4f+2,k),(4g+3,l)),$$
$$((4f+1,k),(4g+2,l),(4f+3,k),(4g,4))\}.$$

For each $d \in Z_b$ and $x \in \mathbb{Z}_{m-1}$, let

$$M_{d,x} = \bigcup_{\substack{e \in P_d^- \\ \{f,g\} \in E(F_x)}} K(B_{f,k}, B_{g,l}),$$

which has a C_4-factorization, $P_{d,x}$, consisting of the 2 C_4-factors

$$M_{d,x}(j) = \bigcup_{\substack{e \in P_d^- \\ \{f,g\} \in (F_x)}} \pi_{fk,gl}(j) \text{ for } j \in \mathbb{Z}_2.$$

For each $d \in Z_b$, let

$$M_d = \bigcup_{x \in \mathbb{Z}_{m-1}} M_{d,x}$$

be the graph with vertex set $(\mathbb{Z}_b \times \mathbb{Z}_{4m}) \setminus (\{d\} \times \mathbb{Z}_{4m})$, which has a C_4-factorization, P_d, consisting of the $2(m-1)$ C_4-factors

$$M_d(j) = \bigcup_{x \in \mathbb{Z}_{m-1}} M_{d,x}(j) \text{ for } j \in \mathbb{Z}_2.$$

Notice that

$$T(b,4m) = \bigcup_{d \in \mathbb{Z}_b} M_d,$$

each edge of which therefore occurs in exactly one cycle in

$$\bigcup_{\substack{d \in \mathbb{Z}_b \\ x \in \mathbb{Z}_{m-1} \\ j \in \mathbb{Z}_2}} M_{d,x}(j).$$

Notice also that each $M_{d,x}(j)$ is a C_4-factor of $M(b,4m) \setminus (\{d\} \times \mathbb{Z}_{4m})$ so the 4-cycles in

$$P(b) = \bigcup_{d \in \mathbb{Z}_b} P_d,$$

form a C_4-frame of $T(b,4m)$.

There are two constructions for a C_4-frame of $S(b,4m)$ based on the value of b. To begin, we need an almost resolvable $b-1$-cycle decomposition of $2K_b$, which is given in [4].

Define $C = \{(c_0(i), c_1(i), \ldots, c_{b-2}(i)) \mid i \in \mathbb{Z}_b, c_{b-2}(i) = \infty, c_j(i) = i + (-1)^{j+1} \lceil j/2 \rceil \text{ for } 0 \leq j \leq b-3\} \cup \{(0,1,\ldots,b-2)\}$ to be a $(b-1)$-cycle system of $2K_b$ on the vertex set $V = \mathbb{Z}_{b-1} \cup \{\infty\}$. Let $c' = (0,1,\ldots,b-2)$. For each $d \in V$, let C_d be the cycle with deficiency d.

Case 1: $b \equiv 0 \pmod 4$

For each $c = (c_0, c_1, \ldots, c_{b-2}) \in C \setminus \{c'\}$, say $c = C_d$, $t \in \mathbb{Z}_m$, and $j \in \mathbb{Z}_2$, define a C_4-factor, $P(c,t,j)$, of $(V \times \mathbb{Z}_{4m}) \setminus (\{d\} \times \mathbb{Z}_{4m})$. Let $\gamma = 2j + (1 + (-1)^{i+1})$. The first subscripts are reduced modulo $(b-1)$, and if the second subscript is x, $(x-4t)$ is reduced modulo 4):

$P(c,t,j) = \{((c_i, \gamma+4t)), (c_{i+1}, \gamma+4t)), (c_i, 1+\gamma+4t)), (c_{i+1}, 1+\gamma+4t))) \mid -1 \le i \le \frac{b}{2}-2\} \cup \{((c_i, 2j+2+4t), (c_{i+1}, 2j+4t), (c_i, 2j+3+4t), (c_{i+1}, 2j+1+4t)) \mid \frac{b}{2}-1 \le i \le b-3\}$.

Also, for each $t \in \mathbb{Z}_m$ and $j \in \mathbb{Z}_2$, define a C_4-factor, $P(c,t,j)$, of $(V \times \mathbb{Z}_{4m}) \setminus (\{\infty\} \times \mathbb{Z}_{4m})$ as follows (with subscripts similarly reduced):

$P(c',t,j) = \{((i, 2j+4t), (i+1, 2j+2+4t), (i, 2j+1+4t), (i+1, 2j+3+4t)) \mid i \in \mathbb{Z}_{b-1}\}$.

Notice that for each $c \in C$, $t \in \mathbb{Z}_m$, and $j \in \mathbb{Z}_2$, $P(c,t,j)$ is a C_4-factor of $M(b,4m) \setminus (\{d\} \times \mathbb{Z}_{4m})$ where $c = C_d$ so the 4-cycles in

$$\bigcup_{\substack{c \in C \\ t \in \mathbb{Z}_t \\ j \in \mathbb{Z}_2}} P(c,t,j)$$

form a C_4-frame of $S(b,4m)$.

Case 2: $b \equiv 2 \pmod 4$

For each $c = (c_0, c_1, \ldots, c_{b-2}) \in C \setminus \{c'\}$, say $c = C_d$ and $t \in \mathbb{Z}_m$, define $2m$ C_4-factors, $P_0(c,t)$ and $P_1(c,t)$, of $(V \times \mathbb{Z}_{4m}) \setminus (\{d\} \times \mathbb{Z}_{4m})$ as follows (with the first subscripts reduced modulo $(b-1)$ and if the second subscript is x, $(x-4t)$ is reduced modulo 4):

1. $P_0(c,t) = \{((c_i, 1+(-1)^{i+1}+4t), (c_{i+1}, 1+(-1)^{i+1}+4t), (c_i, 2+(-1)^{i+1}+4t), (c_{i+1}, 2+(-1)^{i+1}+4t)) \mid -2 \le i \le \frac{b}{2}-2\} \cup \{((c_i, 0), (c_{i+1}, 2), (c_i, 1), (c_{i+1}, 3)) \mid \frac{b}{2}-1 \le i \le b-3\}$, and

2. $P_1(c,t) = \{((c_i, 3+(-1)^{i+1}+4t), (c_{i+1}, 3+(-1)^{i+1}+4t), (c_i, 4+(-1)^{i+1}+4t), (c_{i+1}, 4+(-1)^{i+1}+4t)) \mid 0 \le i \le \frac{b}{2}-$

$2\} \cup \{((c_i,2),(c_{i+1},0),(c_i,3),(c_{i+1},1)) \mid \frac{b}{2}-1 \le i \le b-2\}$

Also, for each $t \in \mathbb{Z}_m$ and $j \in \mathbb{Z}_2$, define a C_4-factor, $P(c,t,j)$, of $(\mathbb{Z}_b \times \mathbb{Z}_{4m}) \setminus (\{\infty\} \times \mathbb{Z}_{4m})$ as follows (with subscripts similarly reduced):

$P(c',t,j) = \{((i,2j+4t),(i+1,2j+2+4t),(i,2j+1+4t),(i+1,2j+3+4t)) \mid i \in \mathbb{Z}_{b-1}\}$.

Notice that for each $c \in C$, $t \in \mathbb{Z}_m$ and each $j \in \mathbb{Z}_2$, $P_j(c,t)$ and $P(c,t,j)$ are C_4-factors of $M(b,4m) \setminus (\{d\} \times \mathbb{Z}_{4m})$ where $c = C_d$ so the 4-cycles in

$$\bigcup_{\substack{c\in C\\ t\in\mathbb{Z}_t\\ j\in\mathbb{Z}_2}} P_j(c,t) \cup \bigcup_{\substack{c\in C\\ t\in\mathbb{Z}_t\\ j\in\mathbb{Z}_2}} P(c,t,j)$$

form a C_4-frame of $S(b,4m)$. ∎

Theorem 3 *Let b be even, and let m be odd. There exists a C_4-frame of $M(b,4m)$ for all $b \neq 2$.*

Proof Let $B, S(b,4m)$, and $T(b,4m)$ be defined as in Theorem 2. Then a C_4-frame of $M(b,4m)$ can again be produced by combining C_4-frames of $S(b,4m)$ and $T(b,4m)$. We use the construction in Theorem 2 to produce a C_4-frame of $S(b,4m)$. So it remains to produce a C_4-frame of $T(b,4m)$.

Partition the vertices $\mathbb{Z}_b \times \mathbb{Z}_{4m}$ into blocks $D = \{D_{i,j} \mid i \in \mathbb{Z}_b, j \in \mathbb{Z}_{2m}\}$, each of size 2, where $D_{i,j} = \{\{i\} \times \{2j, 2j+1\}\}$. Let $K(D_{i,j}, D_{k,l})$ be the complete simple bipartite graph on parts $D_{i,j}$ and $D_{k,l}$. Notice that

$$T(b,4m) = \bigcup_{\substack{i,k\in\mathbb{Z}_b\\ i<k\\ j,l\in\mathbb{Z}_{2m}\\ j\neq l}} K(D_{i,j}, D_{k,l}).$$

Let K_{2m} be the complete graph with vertex set $\mathbb{Z}_{2m}$. Let F be a 1-factorization of K_{2m} such that $F_{2m-1} = \{\{0,1\},\{2,3\},\ldots \{2m-2, 2m-1\}\}$ is a 1-factor in F. Let $F' = F \setminus F_{2m-1}$ and for each $x \in \mathbb{Z}_{2m-2}$, let F'_x be a 1-factor in F'. For each $\{f, g\} \in F'_x$, let $K_{f,g}$ be the complete simple bipartite graph on parts $\{f\} \times \mathbb{Z}_b$ and $\{g\} \times \mathbb{Z}_b$.

Let P be a proper-edge coloring of $K(\{f\},\{g\})$ such that the edges in $\{\{(f,d),(g,d)\} \mid d \in \mathbb{Z}_b\}$ receive different colors. All such $K(\{f\},\{g\})$ produced by the 1-factorization must have the same proper-edge coloring P, else a frame is not guaranteed. For each $d \in \mathbb{Z}_b$, let P_d be the set of edges colored d in the coloring P, and let $P_d^- = P_d \setminus \{(f,d),(g,d)\}$ be a subset of P_d.

For each edge $e = \{(f,k),(g,l)\} \in P_d^-$, define a C_4-factorization of $K(D_{f,k}, D_{g,l})$ consisting of the following C_4-factor:

$$\pi_{fk,gl} = \{((2f,k),(2g,l),(2f+1,k),(2g+1,l))\}.$$

For each $d \in \mathbb{Z}_b$ and $x \in \mathbb{Z}_{2m-2}$, let

$$M_{d,x} = \bigcup_{\substack{e \in P_d^- \\ \{f,g\} \in (F'_x)}} K(D_{f,k}, D_{g,l}),$$

which has a C_4-factorization, $P_{d,x}$, consisting of the C_4-factor

$$M(d,x) = \bigcup_{\substack{e \in P_d^- \\ \{f,g\} \in (F'_x)}} \pi_{fk,gl}.$$

For each $d \in \mathbb{Z}_b$, let

$$M_d = \bigcup_{x \in \mathbb{Z}_{2m-2}} M_{d,x}$$

be the graph with vertex set $(\mathbb{Z}_b \times \mathbb{Z}_{4m}) \setminus (\{d\} \times \mathbb{Z}_{4m})$, which has a C_4-factorization, P_d, consisting of the $2m-2$ C_4-factors

$$M(d) = \bigcup_{x \in \mathbb{Z}_{2m-2}} M(d,x).$$

Notice that

$$T(b, 4m) = \bigcup_{d \in \mathbb{Z}_b} M_d,$$

each edge of which therefore occurs in exactly one cycle in

$$\bigcup_{\substack{d \in \mathbb{Z}_b \\ x \in \mathbb{Z}_{2m-2}}} M(d, x).$$

Notice also that each $M(d, x)$ is a C_4-factor of $M(b, 4m) \setminus (\{d\} \times \mathbb{Z}_{4m})$ so the 4-cycles in

$$P(b) = \bigcup_{d \in \mathbb{Z}_b} P_d$$

form a C_4-frame of $T(b, 4m)$. ■

4 C_4-Frames of $M(b, n)$

In this section, we produce C_4-frames of $M(b, n)$ when each part size is not a multiple of four.

Theorem 4 *Let $b \neq 2$. There exists a C_4-frame of $M(b, n)$ if and only if:*

1. $|E(M(b, n))| \equiv 0 \ (mod\ 4)$,
2. $(b-1)n \equiv 0 \ (mod\ 4)$, *and*
3. *at least on of b and n is even.*

Proof It has been shown that conditions (1-3) are necessary. So assume that conditions (1-3) are satisfied. Also assume that $n \neq 4m$.

Since $(b-1)n \equiv 0 \pmod 4$ and $n \not\equiv 0 \pmod 4$, either $(b-1) \equiv 0 \pmod 4$ or $(b-1), n \equiv 2 \pmod 4$. If $(b-1) \equiv 0 \pmod 4$, then b is odd; by condition 3), n must be even. So $n \equiv 2 \pmod 4$. So we must produce a C_4-frame of $M(b,n)$ when $(b-1) \equiv 0 \pmod 4$ and $n \equiv 2 \pmod 4$ and when $(b-1), n \equiv 2 \pmod 4$. In either case, we can write $n = 2m$, and the construction for both cases is the same.

Let F' be a near 1-factorization on the vertex set $\mathbb{Z}_b$, and for each $d \in \mathbb{Z}_b$ let F'_d be the near 1-factor in F' with deficiency d; so each vertex in $\mathbb{Z}_b \setminus \{d\}$ occurs in exactly one edge in F'_d. Let F be a 1-factorization on the vertex set $\mathbb{Z}_m \times \mathbb{Z}_m$, and for each $t \in \mathbb{Z}_m$, let F_t be a 1-factor in F.

Let $K(D_x, D_y)$ be the complete simple bipartite graph on the parts $D_x = \{x\} \times \mathbb{Z}_{2m}$ and $D_y = \{y\} \times \mathbb{Z}_{2m}$, $0 \le x < y \le b-1$. Let $K(D_{x,k}, D_{y,l})$ be the complete simple bipartite graph on parts $D_{x,k} = \{x\} \times \{2k, 2k+1\}$ and $D_{y,l} = \{y\} \times \{2l, 2l+1\}$, $0 \le x < y \le b-1$, $k, l \in \mathbb{Z}_m$.

Notice that for each $F_t \in F$

$$K(D_x, D_y) = \bigcup_{\substack{\{k,l\} \in F_t \\ t \in \mathbb{Z}_m}} K(D_{x,k}, D_{y,l}).$$

For each $\{k,l\} \in E(F_t)$, define a C_4-factorization of $K(D_{i,j}, D_{k,l})$, consisting of the C_4-factor:

$$\pi_{xk,yl} = \{((x,2k),(y,2l),(x,2k+1),(y,2l+1))\}.$$

For each $d \in \mathbb{Z}_b$, let

$$M_d = \bigcup_{\{x,y\} \in E(F'_d)} K(D_x, D_y),$$

which has a C_4-factorization, P_d, consisting of the m C_4-factors:

$$M_d(t) = \bigcup_{\substack{\{x,y\} \in E(F'_d) \\ \{k,l\} \in E(F_t)}} \pi_{xk,yl}, \text{ for each } t \in \mathbb{Z}_m.$$

Notice that

$$M(b, 2m) = \bigcup_{d \in \mathbb{Z}_b} M_d,$$

each edge of which therefore occurs in exactly one cycle in

$$\bigcup_{\substack{d \in \mathbb{Z}_b \\ t \in \mathbb{Z}_m}} M_d(t).$$

Notice also that each $M_d(t)$ is a 2-factor of $M(b, 2m) \setminus (\{d\} \times \mathbb{Z}_{2m})$ so the 4-cycles in

$$P(b) = \bigcup_{d \in \mathbb{Z}_b} P_d,$$

form a C_4-frame of $M(b, 2m)$. ■

5 C_4-Frames of $\lambda M(b, n)$

We now investigate for which values of λ does there exist a C_4-frame of $\lambda M(b, n)$. The graph $\lambda M(b, n)$ is formed by replacing each edge in $M(b, n)$ with λ edges.

The necessary conditions for the existence of a C_4-frame of $\lambda M(b, n)$ are those found in Lemma 1 with a slight difference. The number of parts must still be not equal to 2; the number of edges must still be divisible by four; and $(b-1)n$ must still be congruent to zero modulo 4. The condition that at least one of b and n is even is a consequence of the fact that there are $\frac{1}{2}bn$ C_4-factors in a C_4-frame of $M(b, n)$. In a C_4-frame of $\lambda M(b, n)$, there must be $\frac{1}{2}\lambda bn$ C_4-factors. If λ is even, then there is no parity restriction on b or n, but if λ is odd, the parity restriction stands.

Theorem 5 *Let $b \neq 2$. There exists a C_4-frame of $\lambda M(b, n)$ if and only if:*

1. $|E(\lambda M(b,n))| \equiv 0 \ (mod\ 4)$,

2. $(b-1)n \equiv 0 \ (mod\ 4)$, *and*

3. *if* λ *is odd, then at least one of* b *and* n *is even.*

Proof It has been shown that conditions (1-3) are necessary. So assume that conditions (1-3) are satisfied. Notice that if there exists a C_4-frame of $M(b,n)$, then there exists a C_4-frame of $\lambda M(b,n)$. Notice also that if λ is odd, then a C_4-frame of $\lambda M(b,n)$ exists if and only if there exists a C_4-frame of $M(b,n)$. So we assume that there does not exist a C_4-frame of $M(b,n)$ and λ is even.

If a C_4-frame of $M(b,n)$ does not exist, then one of the necessary conditions for its existence must be violated. However, that violation must not negate the existence of a C_4-frame of $\lambda M(b,n)$ when λ is even. The violation is that both b and n are odd. Therefore, we must produce a C_4-frame of $\lambda M(b,n)$ when λ is even and both b and n are odd. Since $(b-1)n \equiv 0 \pmod 4$ and n is odd, $b \equiv 1 \pmod 4$.

If we can produce a C_4-frame of $2M(b,n)$, then we may repeat the construction $\frac{\lambda}{2}$ times to produce a C_4-frame of $\lambda M(b,n)$. So all we need to show is that there exists a C_4-frame of $2M(b,n)$ when n is odd and $b \equiv 1 \pmod 4$.

Let N' be a near C_4-factorization of $2K_b$ on the vertex set $\mathbb{Z}_b$ [3], and for each $d \in \mathbb{Z}_b$, let N'_d be the near C_4-factor in N' with deficiency d; so each vertex in $\mathbb{Z}_b \setminus \{d\}$ occurs in exactly one 4-cycle in N'_d. Each N'_d contains $\frac{b-1}{4}$ 4-cycles, (w,x,y,z), with $w < x,y,z$. For $s \in \mathbb{Z}_{\frac{b-1}{4}}$, let $c_d(s)$ be the s^{th} 4-cycle in N'_d. So $N'_d = \{c_d(s) = (w,x,y,z) \mid w,x,y,z \in \mathbb{Z}_b, w < x,y,z, \text{ and } s \in \mathbb{Z}_{\frac{b-1}{4}}\}$.

Let F' be a near 1-factorization on the vertex set $\mathbb{Z}_n$, and for each $t \in \mathbb{Z}_n$, let F'_t be the near 1-factor in F' with deficiency t; so each vertex in $\mathbb{Z}_n \setminus \{t\}$ occurs in exactly one edge in F'_t.

Given $c_d(s) \in N'_d$ and $\{f,g\} \in E(F'_t)$, define two 4-cycles on the vertices $\{w,x,y,z\} \times \{f\}$ and $\{w,x,y,z\} \times \{g\}$ as follows:

$$\pi_d(s,f,g) = \{((w,f),(x,g),(y,f),(z,g)),$$

$$((w,g),(x,f),(y,g),(z,f)) \mid (w,x,y,z) \in N'_d, f,g \in \mathbb{Z}_n\}.$$

Also define a 4-cycle on the vertices $\{w,x,y,z\} \times \{t\}$ as follows:

$$\pi_d(s,t) = \{((w,t),(x,t),(y,t),(z,t)) \mid (w,x,y,z) \in N'_d, t \in \mathbb{Z}_n\}.$$

Let $d \in \mathbb{Z}_b$, $s \in \mathbb{Z}_{\frac{b-1}{4}}$, and $t \in \mathbb{Z}_n$. Notice that

$$P_d(s,t) = \pi_d(s,t) \cup \bigcup_{\{f,g\} \in E(F'_t)} \pi_d(s,f,g)$$

is a C_4-factor on the vertices $\{w,x,y,z\} \times \mathbb{Z}_n$ for $w,x,y,z \in \mathbb{Z}_b$. Notice also that for each $c_d(s) \in N'_d$,

$$P_d(t) = \bigcup_{c_d(s) \in N'_d} P_d(s,t)$$

is a C_4-factor of $2M(b,n) \setminus (\{d\} \times \mathbb{Z}_n)$.

For each $d \in \mathbb{Z}_b$, $c_d(s) = (w,x,y,z) \in N'_d$ and $t \in \mathbb{Z}_n$, let $M_d(s,t)$ be the multipartite graph on vertices $\{w,x,y,z\} \times \mathbb{Z}_n$ for $w,x,y,z \in \mathbb{Z}_b$ induced by the edges of the C_4-factor $P_d(s,t)$.

Let

$$M_d(t) = \bigcup_{s \in \mathbb{Z}_{\frac{b-1}{4}}} M_d(s,t)$$

be the multipartite graph on vertices $(\mathbb{Z}_b \times \mathbb{Z}_n) \setminus (\{d\} \times \mathbb{Z}_n)$ induced by the edges of the C_4-factor $P_d(t)$.

Let

$$M_d = \bigcup_{t \in \mathbb{Z}_n} M_d(t),$$

which has a C_4-factorization,

$$P_d = \bigcup_{t \in \mathbb{Z}_n} P_d(t),$$

consisting of t C_4-factors.

Notice that

$$2M(b,n) = \bigcup_{d \in \mathbb{Z}_b} M_d,$$

each edge of which occurs in exactly one cycle in

$$\bigcup_{\substack{d \in \mathbb{Z}_b \\ s \in \mathbb{Z}_{\frac{b-1}{4}} \\ t \in \mathbb{Z}_n}} P_d(s,t).$$

Notice also that each $P_d(s,t)$ is a C_4-factor of $2M(b,n) \setminus (\{d\} \times \mathbb{Z}_n)$ so the 4-cycles in

$$P(b) = \bigcup_{d \in \mathbb{Z}_b} P_d$$

form a C_4-frame of $2M(b,n)$. ■

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