Bounds for Independent Roman Domination in Graphs

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In memory of Ralph Stanton – a mentor and a friend.

Abstract

A Roman dominating function on a graph G is a function $f : V(G) \to \{0, 1, 2\}$ such that every vertex u with $f(u) = 0$ is adjacent to a vertex v with $f(v) = 2$. The weight of a Roman dominating function f is the value $f(V(G)) = \sum_{u \in V(G)} f(u)$. A Roman dominating function f is an independent Roman dominating function if the set of vertices for which f assigns positive values is independent. The independent Roman domination number $i_R(G)$ of G is the minimum weight of an independent Roman dominating function of G.

We show that if T is a tree of order n, then $i_R(T) \leq 4n/5$, and characterize the class of trees for which equality holds. We present bounds for $i_R(G)$ in terms of the order, maximum and minimum degree, diameter and girth of G. We also present Nordhaus-Gaddum inequalities for independent Roman domination numbers.

Keywords: Domination; Roman domination; Independent Roman domination.

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1 Introduction

The mathematical concept of Roman domination is the oldest known type of domination in which mobile guards are used, and has its historical roots in the time of the ancient Roman Empire. Some of the earliest references to this protection strategy can be found in [2, 9, 10, 11]. Also see [7] for a brief summary of this background.

Cockayne, Dreyer, Hedetniemi and Hedetniemi [5] were the first authors to study Roman domination in graphs. Let $G = (V(G), E(G))$ denote a simple graph of order n. For a function $f : V(G) \to \{0, 1, 2\}$, let $(V_0; V_1; V_2)$ be the ordered partition of $V(G)$ induced by f, where $V_i = \{v \in V(G) : f(v) = i\}$ and $|V_i| = n_i$ for $i = 0, 1, 2$. There is a $1-1$ correspondence between the functions $f : V(G) \to \{0, 1, 2\}$ and the ordered partitions $(V_0; V_1; V_2)$ of $V(G)$, hence we write $f = (V_0; V_1; V_2)$.

A function $f = (V_0; V_1; V_2)$ is a *Roman dominating function*, or just RDF, if every vertex $u \in V_0$ is adjacent to at least one vertex $v \in V_2$. The weight of an RDF is the value $f(V(G)) = \sum_{u \in V} f(u)$. The *Roman domination number* of a graph G, denoted by $\gamma_R(G)$, is the minimum weight of an RDF on G. A function $f = (V_0; V_1; V_2)$ is called a *γ_R-function* if it is an RDF and $f(V(G)) = \gamma_R(G)$.

An RDF $f = (V_0, V_1, V_2)$ in G is *an independent RDF*, or just IRDF, if $V_1 \cup V_2$ is independent [1]. The *independent Roman domination number* $i_R(G)$ of G is the minimum weight of an IRDF of G. We refer to an IRDF with minimum weight as an *i_R-function.*

Cockayne et al. [5] mentioned the study of independent Roman dominating functions in graphs as an open problem, which was subsequently studied by Adabi, Targhi, Jafari Rad and Moradi in [1]. We continue the study of independent Roman dominating functions. In Section 3.1 we obtain an upper bound for the independent Roman domination number of a tree T in terms of its order. We then prove in Section 3.2 that in general there is no non-trivial upper bound for the independent Roman domination number of a graph in terms of its order. In Section 4 we obtain upper bounds for this parameter in terms of order, maximum and minimum degree, diameter and girth, and in Section 5 we give some Nordhaus-Gaddum inequalities for the independent Roman domination number of a graph and its complement. We conclude with a few open problems in Section 6.

2 Definitions and previous results

For notation and graph theory terminology in general we follow [4, 6]. Specifically, we denote the *open neighbourhood* of a vertex v of G by $N_G(v)$, or just $N(v)$, and its *closed neighbourhood* by $N[v]$. For a set $S \subseteq V(G)$, $N(S) = \cup_{v\in S} N(v)$ and $N[S] = \cup_{v\in S} N[v]$. The *degree* $\deg x$ of a vertex x denotes the number of neighbours of x in G. A set $S \subseteq V(G)$ is a *dominating set* if $N[S] = V(G)$. The *domination number* $\gamma(G)$ of G is the minimum cardinality of a dominating set of G. A set $S \subseteq V(G)$ is an *independent dominating set* if S is independent and dominating; the minimum cardinality of such a set is the *independent domination number* $i(G)$. It is obvious that $i_R(G) \le 2i(G)$ for any graph G. For $S \subseteq V(G)$ we denote the subgraph of G induced by S by $G[S]$.

We make use of the following results. The second part of the statement of Lemma 1, $V_2 \subseteq V_2'$, is implicit in the proof given in [1].

Lemma 1 [1] *Let $f = (V_0; V_1; V_2)$ be an RDF of a graph G. If V_2 is independent then there exists an IRDF $g = (V_0'; V_1'; V_2')$ of G such that $w(g) \le w(f)$ and $V_2 \subseteq V_2'$.*

The next statement follows by taking $f = (\varnothing; V(G); \varnothing)$ and applying Lemma 1.

Observation 2 [1] *For any graph G of order n, $i_R(G) \le n$.*

It is also helpful to notice that if X and $Y = V(G) - N[X]$ are independent sets, then $f = (V_0; V_1; V_2)$ with $V_1 = Y$ and $V_2 = X$ is an IRDF of G and thus $i_R(G) \le 2|X| + |Y|$. This observation helps us to determine i_R for paths, cycles and complete bipartite graphs. Note that the inequality in Proposition 3(i) is strict if and only if $n \equiv 1 \pmod 3$.

Proposition 3 (i) [1] *If $G_n = P_n$, $n \ge 2$, or $G_n = C_n$, $n \ge 3$, then*

$$i_R(G_n) = \gamma_R(G_n) = \lfloor \tfrac{2(n+1)}{3} \rfloor \le 2\gamma(G_n) = 2i(C_n) = 2\lceil \tfrac{n}{3} \rceil.$$

(ii) *For any $n \ge m \ge 1$, $i_R(K_{m,n}) = m + 1 = i(G) + 1$.*

Theorem 4 [3] *If T is a tree of order $n \ge 3$, then $\gamma_R(T) \le \frac{4n}{5}$. Equality holds if and only if there exists a partition $V(T) = X_1 \cup \cdots \cup X_k$ of $V(T)$ such that $G[X_i] \cong P_5$ for each i, and the subgraph induced by the central vertices of these paths is connected.*

3 Upper bounds in terms of order

In this section we first present an upper bound for the independent Roman domination number of a tree in terms of its order. We then prove that the bound of Observation 2 is the best possible for general graphs.

3.1 Trees

A *stem* in a tree T is a vertex that is adjacent to a leaf. Let $L(T)$ and $S(T)$ be the set of all leaves and the set of all stems of T respectively. We begin with a simple lemma.

Lemma 5 *Let T be a tree of order at least three. Then*

(i) T has an i_R-function $f = (V_0; V_1; V_2)$ such that $L(T) \cap V_2 = \varnothing$, and
(ii) for any IRDF $f = (V_0; V_1; V_2)$ of T, $V_1 \cap S(T) = \varnothing$.

Proof. *(i)* Let $f = (V_0; V_1; V_2)$ be an i_R-function such that $|L(T) \cap V_2|$ is minimum and suppose $u \in L(T) \cap V_2$. Let v be the stem adjacent to u. Then $f(v) = 0$. Now if $f(w) = 2$ for some $w \in N(v) - \{u\}$, then $g = (V_0; V_1 \cup \{u\}; V_2 - \{u\})$ is an IRDF such that $w(g) < w(f)$, which is impossible. Hence $N[v] \cap V_2 = \{u\}$. Let $U = V_1 \cap N[v]$ and consider the function $f' = (V_0'; V_1'; V_2')$ defined by $f'(a) = f(a)$ if $a \in V(T) - N[v]$, $f'(v) = 2$ and $f'(a) = 0$ if $a \in N(v)$. Then f' is an IRDF of T and $w(f') = w(f) - |U|$. Thus, by the minimality of f, $U = \varnothing$. But then f' is an i_R-function such that $|L(T) \cap V_2'| < |L(T) \cap V_2|$, contradicting the choice of f.

(ii) If $v \in V_1 \cap S(T)$ and u is a leaf adjacent to v, then $f(u) = 0$ since $V_1 \cup V_2$ is independent. But then V_2 does not dominate V_0, a contradiction.

■

We next consider trees with diameter at most four.

Lemma 6 *If T has order $n \geq 3$ and* $\operatorname{diam} T \leq 4$, *then $i_R(T) \leq \frac{4n}{5}$, and equality holds if and only if $T = P_5$.*

Proof. If $\operatorname{diam} T = 2$, then T has a dominating vertex and $i_R(T) = 2 < \frac{4n}{5}$. If $\operatorname{diam} T = 3$, then T has exactly two stems u and v, and all the other vertices are leaves. Thus $i_R(T) = 2+\min\{\deg u - 1, \deg v - 1\} \le 1+\frac{n}{2} < \frac{4n}{5}$ since $n \ge 4$.

Now assume that $\operatorname{diam} T = 4$. Let $P = x, w, v, u, y$ be a diametrical path in T. Let $N(v)$ consist of k_1 leaves and k_2 stems, and let $|V(G)\backslash N[v]| = m$. Then $m \ge k_2$ and $n = 1 + m + k_1 + k_2$. Each vertex in $V(G)\backslash N[v]$ is a leaf of T. Let $f = (V_0; V_1; V_2)$ be an i_R-function of T such that $L(T) \cap V_2 = \varnothing$. (Such a function exists by Lemma 5 (*i*).) By Lemma 5 (*ii*), $f(u) = 0$ or $f(u) = 2$.

- If $f(u) = 2$, then $f(v) = 0$ and thus $f(u') = 2$ for each stem $u' \in N(v)$, $f(\ell) = 1$ for each leaf $\ell \in N(v)$, and $f(y') = 0$ for each leaf $y' \in V(G)\backslash N[v]$. Thus $w(f) = k_1 + 2k_2$.
- If $f(u) = 0$, then $f(z) = 2$ for some $z \in N(u)$, and by the choice of f, $f(v) = 2$. Then $f(u') = 0$ for each $u' \in N(v)$, so $f(y') = 1$ for each leaf $y' \in V(G)\backslash N[v]$. Thus $w(f) = m + 2$.

Therefore $i_R(T) = \min\{k_1 + 2k_2, m + 2\}$. If $k_1 + 2k_2 \le m + 2$, then $m \ge k_1 + 2k_2 - 2$ and thus $n \ge 2k_1 + 3k_2 - 1$. Hence

$$i_R(T) \le \frac{k_1 + 2k_2}{2k_1 + 3k_2 - 1} n \le \frac{2k_2}{3k_2 - 1} n \le \frac{4n}{5} \quad \text{since } k_2 \ge 2.$$

Moreover, equality holds if and only if $k_1 = 0$, $k_2 = 2$ and $n = 3k_2 + 2k_1 - 1 = 5$, that is, $T = P_5$. If $m + 2 \le k_1 + 2k_2$, then

$$i_R(T) \le \frac{m+2}{1 + m + k_1 + k_2} n \le \frac{k_1 + 2k_2}{2k_1 + 3k_2 - 1} n \le \frac{4n}{5},$$

and equality holds if and only if $k_1 = 0$, $k_2 = 2$ and $m + 2 = k_1 + 2k_2 = 4$, that is, $m = 2$ and thus $T = P_5$. ■

We now state and prove our main result.

Theorem 7 *For any tree T on $n \ge 3$ vertices, $i_R(T) \le \frac{4n}{5}$, and equality holds if and only if*

P1 *there exists a partition $V(T) = X_1 \cup \cdots \cup X_k$ of $V(T)$ such that $G[X_i] \cong P_5$ for each i, and the subgraph induced by the central vertices of these paths is connected.*

Proof. Suppose the statement is not true and let T be a tree of smallest order n for which it does not hold. By Lemma 6, $\operatorname{diam} T \ge 5$. Let x, y be two leaves with $d(x, y) = \operatorname{diam} T$ and let P be the diametrical $x-y$ path. Let w_1, w, v, u be distinct vertices such that w_1, w, v, u, y is a subpath of P; note that $w_1 \neq x$. Let T_i, $i = 1, 2$, be the components of $T - vw$, where $w \in V(T_1)$. Say $n_i = |V(T_i)|$, $i = 1, 2$. By the choice of T, $i_R(T_i) \leq \frac{4n_i}{5}$. Moreover, $\operatorname{diam} T_2 \leq 4$ and thus by Lemma 6, $i_R(T_2) = \frac{4n_2}{5}$ if and only if $T_2 = P_5$ (in which case v is the central vertex of T_2), while $i_R(T_1) = \frac{4n_1}{5}$ if and only if **P1** holds.

Let f_i be an i_R-function for T_i such that $f_i(\ell) < 2$ for each $\ell \in L(T_i)$; such functions f_i exist by Lemma 5 (i). In particular, if **P1** holds for T_i, let $f_i(z) = 2$ if $z \in S(T_i)$ and $f(z) = 0$ otherwise.

First suppose that $\min\{f_1(w), f_2(v)\} < 2$. Then $f_1 \cup f_2$ is an RDF of T such that V_2 is independent, in which case Lemma 1 implies that $i_R(T) \leq w(f_1 \cup f_2) = \frac{4n}{5}$. If $i_R(T) < \frac{4n}{5}$, then $\gamma_R < \frac{4n}{5}$ and, by Theorem 4, T does not satisfy **P1**. Suppose $i_R(T) = \frac{4n}{5}$. Then $i_R(T_i) = \frac{4n_i}{5}$, $i = 1, 2$, **P1** holds for T_1, and $T_2 = P_5$. Let $H = v_1, ..., v_5$ be the copy of P_5 in T_1 that contains w. By the choice of f_i, $f_1(v_1) = f_1(v_3) = f_1(v_5) = f_2(v) = 0$ (since v is the central vertex of T_2), $f_1(v_2) = f_1(v_4) = 2$, and $f(a) = 2$ if a is a stem of T_2.

- If $f_1(w) = 0$ and $w = v_3$, then **P1** holds for T. If $f_1(w) = 0$ and $w \neq v_3$, then we may assume without loss of generality that $w = v_1$. Define the function g on T by $g(a) = f_1(a)$ if $a \in V(T_1) - V(H)$, $g(v_5) = 1 = g(a)$ if $a \in L(T_2)$, $g(v_3) = g(v) = 2$ and $g(a) = 0$ otherwise. Then g is an IRDF of T and $w(g) < w(f_1 \cup f_2) = i_R(T)$, a contradiction.
- If $f_1(w) = 2$, we may assume that $w = v_2$. Define the function g' on T by $g'(a) = f_1(a)$ if $a \in V(T_1) - \{w, v_1\}$, $g(v_1) = 1 = g'(a)$ if $a \in L(T_2)$, $g'(v) = 2$ and $g'(a) = 0$ otherwise. Then g' is an IRDF of T and $w(g') < w(f_1 \cup f_2) = i_R(T)$, again a contradiction.

Thus if $\min\{f_1(w), f_2(v)\} < 2$, then $i_R(T) \leq \frac{4n}{5}$ and equality holds if and only if **P1** holds, contrary to our assumption that the statement does not hold for T. Therefore $f_1(w) = f_2(v) = 2$. By the choice of f_i, $w \notin L(T_1)$, $v \notin L(T_2)$ and $T_2 \neq P_5$. Thus $\deg_T(w), \deg_T(v) \geq 3$. We also note that now **P1** does not hold for T_2 (otherwise we would have $f_2(v) = 0$) and so, by Lemma 6, $w(f_2) < \frac{4n_2}{5}$. We now prove four facts.

Fact 1. If $\deg_T v = 3$, then $i_R(T) < \frac{4n}{5}$.

Proof of Fact 1. Define g_2 on $V(T_2)$ by $g_2(v) = g_2(a) = 0$ if a is a leaf of T_2 at distance two from v, $g_2(a) = 2$ if a is a stem of T_2 adjacent to v, and

$g_2(a) = 1$ if a is a leaf in T_2 adjacent to v. Since $\deg_{T_2} v = 2$, g_2 is an IRDF for T_2 with $w(g_2) = 3$ if v is adjacent to a leaf and $w(g_2) = 4$ otherwise. In the latter case T_2 has at least six vertices, and in either case $w(g_2) < \frac{4n_2}{5}$. Thus $f \cup g_2$ is an IRDF of T of weight less than $\frac{4n}{5}$ and so $i_R(T) < \frac{4n}{5}$. ◇

We therefore assume that $\deg_T v \geq 4$ and proceed with Fact 2.

Fact 2. If $N_{T_2}(v)$ contains a vertex c of degree at least three, then $i_R(T) < \frac{4n}{5}$.

Proof of Fact 2. Without loss of generality assume that $c = u$. Thus there are at least two leaves adjacent to u, one of which is y; say y' is another leaf adjacent to u. If each vertex in $N_{T_2}(v)$ has degree at least three, then T_2 has at least $2\deg_{T_2} v$ leaves at distance two from v. Since $f_2(v) = 2$, it follows that $f_2(a) = 0$ if $a \in N_{T_2}(v)$ and thus $f_2(a) = 1$ if $a \in L(T_2)$. Hence $w(f_2) \geq 2 + 2\deg_{T_2} v$. But then the function g_2, defined on $V(T_2)$ by $g_2(a) = 2$ if $a \in N_{T_2}(v)$ and $g_2(a) = 0$ otherwise, has weight $w(g_2) = 2\deg_{T_2} v < w(f_2) = i_R(T_2)$, which is impossible.

Therefore there is vertex $b \in N_{T_2}(v)$ such that $\deg b \leq 2$. Delete b, y' and (if it exists) the leaf b' adjacent to b from T to obtain the tree T_3. By the choice of T, $i_R(T_3) \leq \frac{4(n-2)}{5}$. Let f_3 be an i_R-function of T_3 such that $f_3(\ell) \neq 2$ for each $\ell \in L(T_3)$; f_3 exists by Lemma 5 (i). Since u is adjacent, in T_3, to the leaf y, Lemma 5 (ii) implies that $f_3(u) \neq 1$. Hence $f_3(u) = 2$ or $f_3(u) = 0$.

- If $f_3(u) = 2$, then $f_3(v) = 0$. The function g_3 defined by $g_3(a) = f_3(a)$ if $a \in V(T_3)$, $g_3(y') = 0$ and $g_3(b) = \deg b$ is an IRDF of T with weight at most $\frac{4(n-2)}{5} + \deg b < \frac{4n}{5}$.
- If $f_3(u) = 0$, then $f(z) = 2$ for some $z \in N(u)$. By the choice of f_3, $f_3(v) = 2$ while $f_3(z) = 1$ for each leaf z adjacent to u. Then, similar to g_3, the function h_3 defined by $h_3(a) = f_3(a)$ if $a \in V(T_3)$, $h_3(b) = 0$, $h_3(b') = 1$ if b' exists, and $h_3(y') = 1$ is an IRDF of T with weight less than $\frac{4n}{5}$.

Therefore $i_R(T) < \frac{4n}{5}$. ◇

Since the statement does not hold for T, Fact 2 implies that every vertex in $N_{T_2}(v)$ has degree at most two. In particular, $\deg u = 2$. Consider arbitrary $v' \in N_T(w)\backslash\{w_1\}$ and any leaf ℓ of T at maximum distance from w such that v' lies on the $w-\ell$ path in T. Since P is a diametrical path of T, $d(w,\ell) \leq d(w,y) = 3$. Moreover, if $d(w,\ell) = 3$, then we may interchange the roles of v and v', and find that v' has exactly the same properties as those deduced above for v. We now (weakly) partition $N(w)\backslash\{v, w_1\}$ into the three subsets $A_1 = N(w) \cap L(T)$, $A_2 = \{a \in N(w)\backslash\{v, w_1\} : N(a)\backslash\{w\} \subseteq L(T)\}$ and $A_3 = N(w) - \{v, w_1\} - A_1 - A_2$, where some of these sets may be empty. Thus each vertex in A_3 plays the same role as v and thus has at least three neighbours other than w. Let $\alpha_i = |A_i|$, $i = 1, 2, 3$. We proceed with Fact 3.

Fact 3. If $T - w$ has no component $F \in \{K_1, K_2\}$, then $i_R(T) < \frac{4n}{5}$.

Proof of Fact 3. Now $A_1 = \varnothing$ and $\deg_T a \geq 3$ whenever $a \in A_2$. Hence $\alpha_2 + \alpha_3 = \deg w - 2 \geq 1$. Let T_{11} and T_{12} be the two subtrees of $T_1 - ww_1$, where $w \in V(T_{11})$. By the choice of T, $i_R(T_{11}) \leq \frac{4|V(T_{11})|}{5}$ and $i_R(T_{12}) \leq \frac{4|V(T_{12})|}{5}$. Let f_{12} be an i_R-function for T_{12}. Define g_1 on $V(T_1)$ by $g_1(a) = f_{12}(a)$ if $a \in V(T_{12})$, $g_1(a) = 2$ if $a \in A_2 \cup A_3$, $g_1(a) = 0$ if a is a leaf or stem adjacent to some vertex in $A_2 \cup A_3$, $g_1(a) = 1$ if a is a leaf at distance two from some vertex in A_3. Furthermore, $g_1(w) = 0$. It follows that g_1 is an IRDF for T_1 with weight at most $\frac{4n_1}{5}$. Therefore $g_1 \cup f_2$ is an IRDF for T with $w(g_1 \cup f_2) = w(g_1) + w(f_2) < \frac{4n_1}{5} + \frac{4n_2}{5} = \frac{4n}{5}$ (since $w(f_2) < \frac{4n_2}{5}$), as desired. ◇

We conclude that $T - w$ has a component $F \in \{K_1, K_2\}$. Say r is the vertex of F adjacent to w, let $\rho = |V(F)|$ and define $T_4 = T - F - \{u, y\}$. Then $i_R(T_4) \leq \frac{4(n-\rho-2)}{5}$.

Fact 4. If f_4' is an i_R-function of T_4 with $\max\{f'(v), f'(w)\} = 2$, then $i_R(T) < \frac{4n}{5}$.

Proof of Fact 4. If $f_4'(v) = 2$, then we define g_4 on $V(T)$ by $g_4(a) = f_4'(a)$ if $a \in V(T_4)$, $g_4(u) = 0$, $g_4(y) = 1$ and $g_4(r) = \rho$. This implies that $i_R(T) \leq i_R(T_4) + \frac{\rho+1}{\rho+2} < \frac{4n}{5}$ since $\rho \in \{1, 2\}$. Similarly, if $f_4'(w) = 2$ we obtain $i_R(T) < \frac{4n}{5}$. ◇

Therefore no i_R-function of T_4 assigns the value 2 to v or w. Let f_4 be an i_R-function of T_4 such that $f_4(\ell) < 2$ for each leaf of T_4. Then $f_4(w) \neq 2$ and $f_4(v) \neq 2$. If $f_4(v) = 1$, then v is not a stem (Lemma 5 *(ii)*). Then each $a \in N_{T_4}(v) - \{w\}$ has degree exactly two (as concluded after the proof of Fact 2), and $f_4(a) = 0$, by independence. But then $f_4(\ell) = 2$, where ℓ is the leaf adjacent to a, contrary to the fact that f_4 satisfies the condition in Lemma 5 *(i)*. Therefore $f_4(v) = 0$. Since $f_4(w) < 2$, $f_4(u') = 2$ for some $u' \in N(v) - \{w, u\}$; by the choice of f_4, $u' \notin L(T)$ and thus u' is a stem.

Let T_{4i} be the two components of $T_4 - vw$, where $v \in V(T_{42})$, and let f_{4i} be the restriction of f_4 to T_{4i}. Since $f_4(w), f_4(v) < 2$, f_{4i} is an IRDF for T_{4i}, $i = 1, 2$. Since $\deg_T v \geq 4$, $\deg_{T_{42}} v \geq 2$. The arguments in the preceding paragraph show that $f_{42}(a) = 2$ if and only if a is a stem of T_{42} adjacent to v, $f_{42}(a) = 1$ if and only if a is a leaf of T_{42} adjacent to v, and $f_{42}(a) = 0$ otherwise. Thus if v is adjacent in T_{42} to k_1 leaves and k_2 stems, then $w(f_{42}) = k_1 + 2k_2$. Define the function h on T_{42} by $h(v) = 2$, $h(a) = 1$ if a is a leaf at distance two from v, and $h(a) = 0$ otherwise. Then $w(h) = 2 + k_2 \leq k_1 + 2k_2 = w(f_{42})$ since $k_1 + k_2 = \deg_{T_{42}} v \geq 2$. Moreover, since $f_{41}(w) < 2$, $f_{41} \cup h$ is an RDF of T_4 such that $\{a \in V(T_4) : (f_{41} \cup h)(a) = 2\}$ is independent. By Lemma 1 there exists an IRDF h' of T_4 such that $w(h') \leq w(f_{41} \cup h) = w(f_{41}) + w(h) \leq w(f_{41}) + w(f_{42}) = w(f_4)$ and $h'(v) = h(v) = 2$. Since f_4 is an i_R-function of T_4, equality holds throughout, hence $w(h') = w(f_4)$. But now h' is an i_R-function of T_4 such that $h'(v) = 2$, and this contradicts Fact 4.

We have therefore shown that the bound holds for all trees of order at least three. We have also shown that **P1** holds whenever $i_R(T) = \frac{4n}{5}$, while the fact that **P1** does not hold if $i_R(T) < \frac{4n}{5}$ follows from Theorem 4. The proof of the theorem is therefore complete. ■

3.2 General graphs

We next show that in general there is no upper bound better than the bound of Observation 2 for the independent Roman domination number in a graph G.

Proposition 8 *For any $t \in (0,1)$ there are an integer n and a connected graph G on n vertices such that $i_R(G) > tn$.*

Proof. Let $t \in (0,1)$. Since

$$\lim_{n \longrightarrow +\infty} \frac{n}{n+1} = 1,$$

there is a rational number $\frac{p}{p+1}$ such that $\frac{p}{p+1} > t$. Let $G_1 \cong K_m$, where $m > 3p$, and let G be the graph obtained by adding m leaves to each vertex in G_1. It is obvious that $i_R(G) = 2 + (m-1)m$. Let $n = m^2 + m$. Since $m^2 > 3pm \geq 2pm + m$, we obtain $m^2 + p(m^2) \geq 2pm + m + p(m^2)$ and so $m^2 + p(m^2) - m - pm \geq pm + p(m^2)$. Now $(p+1)(m-1)m \geq p(m + m^2)$ and thus $(m-1)m \geq \frac{p}{p+1}(m^2 + m)$. Therefore $i_R(G) > tn$. ■

If x is a vertex of maximum degree in a graph G, then $(N(x); V(G) - N[x]; \{x\})$ is an RDF for G and by Lemma 1 we obtain the following bound, in which equality holds for complete bipartite graphs, for example (see Proposition 3).

Proposition 9 *For any graph G of order n, $i_R(G) \leq n - \Delta(G) + 1$.*

Since the chromatic number $\chi(G) \leq 1+\Delta(G)$ for any graph G, Proposition 9 links the independent Roman domination number and the chromatic number as follows.

Corollary 10 *For any connected graph G of order $n > 1$, $i_R(G) \leq n - \chi(G) + 2$, with equality if and only if $G = K_n$.*

Corollary 11 *If k is an integer such that $k \geq \frac{n}{\Delta(G)-1}$, then $i_R(G) \leq \frac{k-1}{k}n$.*

Proof. Since $n \leq k(\Delta(G)-1)$, $kn \leq k(\Delta(G)-1)+(k-1)n$ and Proposition 9 gives $i_R(G) \leq n - \Delta(G) + 1 \leq \frac{k-1}{k}n$. ■

The special case $k = 5$ in Corollary 11 implies that if $n \leq 5(\Delta(G) - 1)$ and $\Delta(G) > 1$, then $i_R(G) \leq \frac{4n}{5}$.

Lemma 12 *Let G be a graph with $\Delta(G) \geq 3$. For any edge $e \in E(G)$, $i_R(G) \leq i_R(G - e) + \Delta(G) - 3$.*

Proof. Let $e = xy \in E(G)$ and f be an IRDF for $G - e$. If $\{f(x), f(y)\} \neq \{2\}$, then $i_R(G) \leq i_R(G - e)$ by Lemma 1. So we may assume that $f(x) = f(y) = 2$. We define g on $V(G)$ by $g(v) = f(v)$ if $v \notin (N[x] \setminus \{y\})$, $g(x) = 0$, and $g(v) = 1$ if $v \in (N(x) \setminus \{y\}$. Then g is an RDF for G, and by Lemma 1 the result follows. ■

Since any connected graph contains a spanning tree, the following result is a direct consequence of Theorem 7 and Lemma 12.

Corollary 13 *If G is a cubic graph of order n, then $i_R(G) \leq \frac{4n}{5}$.*

The bound in Corollary 13 also follows from the bound $i(G) \leq \frac{2n}{5}$ for a cubic graph $G \neq K_{3,3}$ of order n, which was proved in [8]. Also see Problem 4 in Section 6.

4 Upper bounds in terms of degree, diameter and girth

In this section we present simple upper bounds for $i_R(G)$ in terms of maximum and minimum degree, diameter and girth. We begin by improving the bound given in Proposition 9 for some graphs.

Proposition 14 *If v is a vertex of a graph G such that $k = \Delta(G - N[v]) \geq 2$, then $i_R(G) \leq n - \deg v - k + 2$.*

Proof. Let u be a vertex of $G - N[v]$ of degree k. The function

$$f = (N(u) \cup N(v); V(G) - (N[u] \cup N[v]); \{u, v\})$$

is an RDF of G such that $\{u, v\}$ is independent and $w(f) = 4 + (n - \deg v - 2 - k) = n - \deg - k + 2$. The result follows from Lemma 1. ■

Corollary 15 (*i*) *Let v be a vertex of a graph G such that $\deg v = \Delta(G)$, and let $k = \Delta(G - N[v])$. If $k \geq 2$, then $i_R(G) \leq n - \Delta(G) - k + 2$.*

(*ii*) *If G is a graph of order n and $\operatorname{diam} G \geq 3$, then $i_R(G) \leq n - 2\delta(G) + 2$.*

(*iii*) *If G is an r-regular graph of order n with $1 < r < \frac{n-1}{3}$, then $i_R(G) \leq n - r$.*

(*iv*) *If G is a cubic graph of order $n \geq 10$, then $i_R(G) \leq n - 4$.*

Proof. (*i*) Immediate from Proposition 14.

(*ii*) Let u and v be two vertices with $d(u, v) \geq 3$ and apply Proposition 14.

(*iii*) Let v be any vertex of G. There are at most $r(r-1)$ edges between $N(v)$ and $G - N[v]$. Let $G_1 = G - N[v]$. Then

$$|E(G_1)| \geq r\frac{(n - r - 1)}{2} - r(r - 1).$$

The restrictions on r imply that $(r - 1)(n - 1 - 3r) > 0$. Therefore $(n - r - 1)(r - 1) - 2r(r - 1) > 0$ and so

$$\frac{(n - r - 1)r}{2} - r(r - 1) > \frac{n - r - 1}{2}.$$

We deduce that

$$|E(G_1)| > \frac{n - r - 1}{2} = \frac{|V(G_1)|}{2},$$

hence $k = \Delta(G_1) \geq 2$.

(*iv*) Let v be any vertex of G and note that $|N[N[v]]| \leq 10$. If $V(G) - N[N[v]] \neq \varnothing$, then $k = \Delta(G - N[v]) = 3$ and the result follows from Proposition 14. If $V(G) - N[N[v]] = \varnothing$, then $n = 10$ and $G - N[v]$ is a 2-regular graph of order six, i.e., $G - N[v] \in \{C_6, 2K_3\}$, hence $i_R(G - N[v]) = 4$ and the result follows. ■

We next give some bounds in terms of diameter and girth.

Proposition 16 (*i*) *For any graph G of order n, $i_R(G) \leq n - \lceil \frac{\text{diam}\, G\ -1}{3} \rceil$.*

(*ii*) *For any graph G of order n and girth $g(G)$, $\lfloor \frac{2(g(G)+1)}{3} \rfloor \leq i_R(G) \leq n - \left\lceil \frac{g(G)-2}{3} \right\rceil$.*

Proof. (*i*) Let P be a diametrical path in G and let f be an i_R-function of P. By Proposition 3, $w(f) = \lfloor \frac{2(2+\text{diam}\, G)}{3} \rfloor$. We define h on $V(G)$ by $h(x) = f(x)$ if $x \in V(P)$, and $h(x) = 1$ otherwise. Then h is an RDF for G. By Lemma 1, $i_R(G) \leq w(h) = n - \lceil \frac{-1+\text{diam}\, G}{3} \rceil$. The proof of (*ii*) is similar. ■

5 Nordhaus-Gaddum type bounds

We now present some Nordhaus-Gaddum inequalities. Let $\delta, \overline{\delta}, \Delta, \overline{\Delta}$ denote the minimum and maximum degrees of G and $\overline{G}$, respectively.

Proposition 17 *For any graph G of order $n \geq 3$,*

$$5 \leq i_R(G) + i_R(\overline{G}) \leq n + 3.$$

Equality holds in the lower bound if and only if G or $\overline{G}$ is K_3, or (δ, Δ) or $(\overline{\delta}, \overline{\Delta}) = (1, n-1)$, and in the upper bound if and only if G or $\overline{G}$ is C_5 or $\frac{n}{2}K_2$.

Proof. Since $n \geq 3$, $i_R(G), i_R(\overline{G}) \geq 2$. If $i_R(G) = 2$, then there is a vertex $x \in V(G)$ such that $N[x] = V(G)$. But x is an isolated vertex in $\overline{G}$, hence $i_R(\overline{G}) \geq 3$. Similarly if $i_R(\overline{G}) = 2$, then $i_R(G) \geq 3$. We deduce that $i_R(G) + i_R(\overline{G}) \geq 5$. Suppose equality holds and assume $i_R(G) < i_R(\overline{G})$. Then $i_R(G) = 2$ and $i_R(\overline{G}) = 3$, and as above G has a vertex x of degree $n-1$. If $G = K_3$, then $i_R(G) + i_R(\overline{G}) = 5$, so suppose $n \geq 4$. If

$\delta(G-x) \geq 1$, then $\overline{G-x}$ has order at least three and no universal vertex, hence $i_R(\overline{G-x}) \geq 3$ and $i_R(\overline{G}) \geq 4$, which is not the case. Hence $G-x$ has an isolated vertex and $\delta(G)=1$.

For the upper bound, by Proposition 9,

$$\begin{aligned} i_R(G)+i_R(\overline{G}) &\leq (n-\Delta(G)+1)+(n-\Delta(\overline{G})+1) \\ &= n-\Delta(G)+\delta(G)+3 \leq n+3. \end{aligned}$$

If $i_R(G)+i_R(\overline{G}) = n+3$, then equality holds throughout the calculation, and $\delta(G) = \Delta(G)$. Hence G is k-regular for some k. We may assume that $k \leq \frac{n-1}{2}$, since the argument is symmetric in G and $\overline{G}$. Also, since $i_R(K_n)+i_R(\overline{K_n}) = n+2$, we may assume that $k \geq 1$. From equality we obtain $i_R(G) = n-k+1$ and $i_R(\overline{G}) = k+2$. Let $v \in V(G)$. If some vertex $u \in V(G)\backslash N[v]$ has at least two neighbours in $V(G)\backslash N[v]$, then by Corollary 15(iii), $i_R(G) \leq n-k$, a contradiction. Hence every vertex not in $N[v]$ has at least $k-1$ neighbours in $N(v)$. The same argument applies to $\overline{G}$, hence, in G, each vertex in $N(v)$ has at most one neighbour outside $N[v]$.

Counting the edges joining $N(v)$ and $V(G)-N[v]$ from both sides yields $(k-1)(n-k-1) \leq k$, which simplifies to $n \leq \frac{k}{k-1}+k+1$ for $k > 1$. Since $n \geq 2k+1$, we have $2k+1 \leq \frac{k}{k-1}+k+1$, which requires $k \leq 2$. If $k=2$, then $n \leq 5$ and $n \geq 2k+1 = 5$, which implies that $G = C_5$. The only 1-regular graph of order n is $G \cong \frac{n}{2}K_2$, and $i_R(G)+i_R(\overline{G}) = n+3$. ■

Proposition 18 *If G is a connected graph of order n with* $\operatorname{diam} G \geq 3$, *then*

$$6 \leq i_R(G)+i_R(\overline{G}) \leq n-\delta(G)+4.$$

These bounds are sharp.

Proof. Since $\operatorname{diam} G \geq 3$, we obtain $i_R(G) \geq 3$. Also $i_R(\overline{G}) \geq 3$ since G is connected. The lower bound follows. For the upper bound, Proposition 9 and Corollary 15(ii) imply

$$\begin{aligned} i_R(G)+i_R(\overline{G}) &\leq n-2\delta(G)+2+n-\Delta(\overline{G})+1 \\ &= n-2\delta(G)+2+\delta(G)+2 \\ &= n-\delta(G)+4. \end{aligned}$$

To see the sharpness of both bounds consider the cycle C_6. ■

6 Open problems

As mentioned in Section 2, $i_R(G) \leq 2i(G)$ for any graph G. Paths and cycles of order $n \not\equiv 1 \pmod{n}$ are examples of graphs for which equality holds, but equality holds for other graphs as well.

Problem 1 *Characterize the class of graphs G (or the class of trees, or the class of cubic graphs, etc.) such that $i_R(G) = 2i(G)$.*

It is easy to see that the only graphs G such that $i_R(G) = i(G)$ are the edgeless graphs. Complete bipartite graphs are examples of graphs for which $i_R(G) = i(G) + 1$; in fact, if G has an i-set X such that some $v \in X$ is adjacent to all vertices of $G - X$, then $i_R(G) = i(G) + 1$. Conversely, if $i_R(G) = i(G) + 1$ and $f = (V_0, V_1, V_2)$ is an i_R-function of G, then $|V_1| + |V_2| = |V_1 \cup V_2| \geq i(G) = i_R(G) - 1 = |V_1| + 2|V_2| - 1$, so $|V_2| \leq 1$. But $V_2 = \varnothing$ if and only if G is edgeless. Hence $|V_2| = 1$ and thus all vertices in V_0 are adjacent to the single vertex in V_2. Therefore $i_R(G) = i(G) + 1$ if and only if G has a vertex v such that $\deg v \geq \frac{n}{2}$ and $V(G) - N[v]$ is independent or empty.

Problem 2 *Characterize the class of graphs G (or the class of trees, or the class of cubic graphs, etc.) such that $i_R(G) = i(G)+k$ for (fixed) $2 \leq k \leq i$.*

Problem 3 *Characterize the class of graphs G (or trees) with $i_R(G) = \gamma_R(G)$.*

Problems 1 and 2 were addressed, with some success, in [1], but the results there are in terms of the existence of independent sets with various properties and do not give a description of the classes of graphs in terms of easily determined properties.

It was also proved in [1] that if $\Delta(G) \leq 3$, then $i_R(G) = \gamma_R(G)$, so these parameters are equal for cubic graphs. As stated in Corollary 13, $i_R(G) \leq \frac{4n}{5}$ for a cubic graph G of order n. It is also known [8] that $i(G) \leq \frac{2n}{5}$ for a cubic graph $G \neq K_{3,3}$ of order n, and equality is only known to hold for the Cartesian product $K_5 \square K_2$. However, $i_R(K_{3,3}) = 4 < \frac{4 \cdot 6}{5}$ and $i_R(K_5 \square K_2) = 6 < \frac{4 \cdot 10}{5}$, so it is unlikely that there exists a cubic graph G with $i_R(G) = \frac{4n}{5}$. Is it possible that $i_R(G) \leq \frac{2n}{3}$ or even that $i_R(G) \leq \frac{6n}{10}$ except for a finite number of small graphs?

Problem 4 *Find a sharp bound for i_R and thus γ_R for cubic graphs.*

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