

The Algorithmic Complexity of Domination Digraphs

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ABSTRACT. Let $G = (V, E)$ be an undirected graph and let $\pi = \{V_1, V_2, \dots, V_k\}$ be a partition of the vertices V of G into k blocks V_i . From this partition one can construct the following digraph $D(\pi) = (\pi, E(\pi))$, the vertices of which correspond one-to-one with the k blocks V_i of π , and there is an arc from V_i to V_j if every vertex in V_j is adjacent to at least one vertex in V_i , that is, V_i dominates V_j . We call the digraph $D(\pi)$ the *domination digraph* of π . A triad is one of the 16 digraphs on three vertices having no loops or multiple arcs. In this paper we study the algorithmic complexity of deciding if an arbitrary graph G has a given digraph as one of its domination digraphs, and in particular, deciding if a given triad is one of its domination digraphs. This generalizes results for the domatic number.

Keywords: digraph, domatic partition, complexity
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1 Introduction

A domatic partition of a graph is a partition of the vertex set into blocks such that each block dominates the whole graph, or in other words, each block dominates every other block. It is well-known that any graph without isolated vertices has a domatic partition into two blocks, but Garey and Johnson [2] showed that it is \mathcal{NP} -complete to determine if a graph has a domatic partition into three blocks. In this paper we generalize these ideas, relaxing the requirement that every block dominates every other block.

We will use the following notation. A graph $G = (V, E)$ has order $n = |V|$. For a vertex $v \in V$, the *open neighborhood* of v is the set $N(v) = \{u \in V \mid uv \in E\}$, $\deg(v) = |N(v)|$ and the *closed neighborhood* is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, the *open neighborhood* of S is

the set $N(S) = \bigcup_{v \in S} N(v)$, and the *closed neighborhood* of S is the set $N[S] = N(S) \cup S$. A set S is *independent* if no two vertices in S are adjacent, and a *dominating set* if $N[S] = V$. Given two disjoint sets S and S' , we say that S *dominates* S' , denoted $S \Rightarrow S'$, if $S' \subseteq N(S)$; that is, every vertex in S' is dominated by, or is adjacent to, at least one vertex in S . We use the notation $S \not\Rightarrow S'$ to mean that S does not dominate S' .

Given two vertices $u, v \in V$, $d(u, v)$ denotes the *distance* between u and v ; $ecc(u)$ denotes the *eccentricity* of vertex u , which equals the maximum value of $d(u, v)$ over all vertices $v \in V$; and $diam(G)$ denotes the diameter of the graph.

Let $D = (V, A)$ be a directed graph (or *digraph*) with a set of vertices V and a set $A \subseteq V \times V$ of directed edges, called *arcs*. If $(u, v) \in A$, we write $u \rightarrow v$ and say u *dominates* v . A digraph $D = (V, A)$ is *complete* if for every $u, v \in V$, either $u \rightarrow v$, or $v \rightarrow u$, or both. A complete digraph D is a *tournament* if for every $u, v \in V$, either $u \rightarrow v$, or $v \rightarrow u$, but not both. A complete digraph D is *transitive* if its vertices can be ordered u_1, u_2, \dots, u_n , such that $u_i \rightarrow u_j \in A$ if and only if $i < j$. A *diad* or *triad* is a directed graph having two or three vertices, respectively, with no loops ($u \rightarrow u$) or multiple edges (two or more arcs $u \rightarrow v$ and $u \rightarrow v$). It is known (and easily shown) that there are exactly three diads and 16 triads.

Let $\pi = \{V_1, V_2, \dots, V_k\}$ be a partition of the vertices V of a graph G . From this partition we can construct a digraph $D(\pi) = (\pi, E(\pi))$, the vertices of which correspond one-to-one with the k blocks V_i of π , and there is an arc from V_i to V_j , denoted $V_i \rightarrow V_j$, if $V_i \Rightarrow V_j$. We call the digraph $D(\pi)$ the *domination digraph* of π , and we say that π is a $D(\pi)$ -*partition*. Let $\mathcal{D}(G)$ denote the family of all domination digraphs obtained from arbitrary partitions π of the vertices of a graph G . This concept was introduced in [4].

In this paper we consider the general question: what can you say about the graphs $D(\pi) \in \mathcal{D}(G)$? In particular, given an arbitrary triad, say T , is $T \in \mathcal{D}(G)$? We show that the problem is polynomial-time solvable for some T , and \mathcal{NP} -complete for some T .

2 Diads

There are just three diads: (i) the diad denoted $\overline{K_2}$ having no arc, (ii) the diad denoted $\text{ARC}(2)$ having one arc, and (iii) the diad denoted K_2 consisting of two vertices u and v and the two arcs $u \rightarrow v$ and $v \rightarrow u$. For any graph G it is easy to determine if any one of the three diads is in $\mathcal{D}(G)$.

We will need the following, oft-cited, theorem due to Ore [5].

Theorem 1 (Ore) *The complement $V - S$ of any minimal dominating set S in a graph G without isolated vertices contains a dominating set.*

Theorem 2 *For any graph G ,*

- (a) $\overline{K_2} \in \mathcal{D}(G)$ *if and only if* $\text{diam}(G) \geq 3$ *or* G *is disconnected;*
- (b) $\text{ARC}(2) \in \mathcal{D}(G)$ *if and only if* G *is neither complete nor empty;*
- (c) $K_2 \in \mathcal{D}(G)$ *if and only if* G *has no isolated vertex.*

Proof. (a) If π is a $\overline{K_2}$ -partition, then there are vertices $u_1 \in V_1$ and $u_2 \in V_2$ such that $V_2 \not\ni u_1$ and $V_1 \not\ni u_2$. That is, $N[u_1] \subseteq V_1$ and $N[u_2] \subseteq V_2$. Thus, u_1 and u_2 are at distance at least 3. Conversely, if u_1 and u_2 are at distance at least 3, then $\{N[u_1], V - N[u_1]\}$ is a $\overline{K_2}$ -partition.

(b) If G is neither complete nor empty, then there exists vertices x and y such that x is not adjacent to y but y is not isolated. In this case, $\{V - \{y\}, \{y\}\}$ is an $\text{ARC}(2)$ -partition.

(c) It follows from Ore's Theorem, that if S is a minimal dominating set in a graph G without isolated vertices, then $\{S, V - S\}$ is a K_2 -partition. \square

3 Triads

We consider the algorithmic complexity of the problem of deciding if an arbitrary graph G has a given triad, say T , as one of its domination digraphs, that is, if $T \in \mathcal{D}(G)$. That is, we will investigate the 16 decision problems corresponding to the 16 triads. Let us denote by TRIAD one of these 16 digraphs. The general decision problem can be stated in the following form:

TRIAD

INSTANCE: Graph $G = (V, E)$.

QUESTION: Does G have a TRIAD-partition $\pi = \{A, B, C\}$, that is, such that $D(\pi)$ is isomorphic to TRIAD?

Thus, for example, when we refer to one of the 16 triads, say SINK EDGE, we mean both the triad itself, and the corresponding decision problem. The reference should be clear by context.

Figure 1 below illustrates the 16 triads, each with an assigned name, and the status of the associated decision problem.

In the next section we give the proofs for the triads where polynomial-time algorithms are known, and in Section 5 we give the proofs for the triads which are known to be \mathcal{NP} -hard.

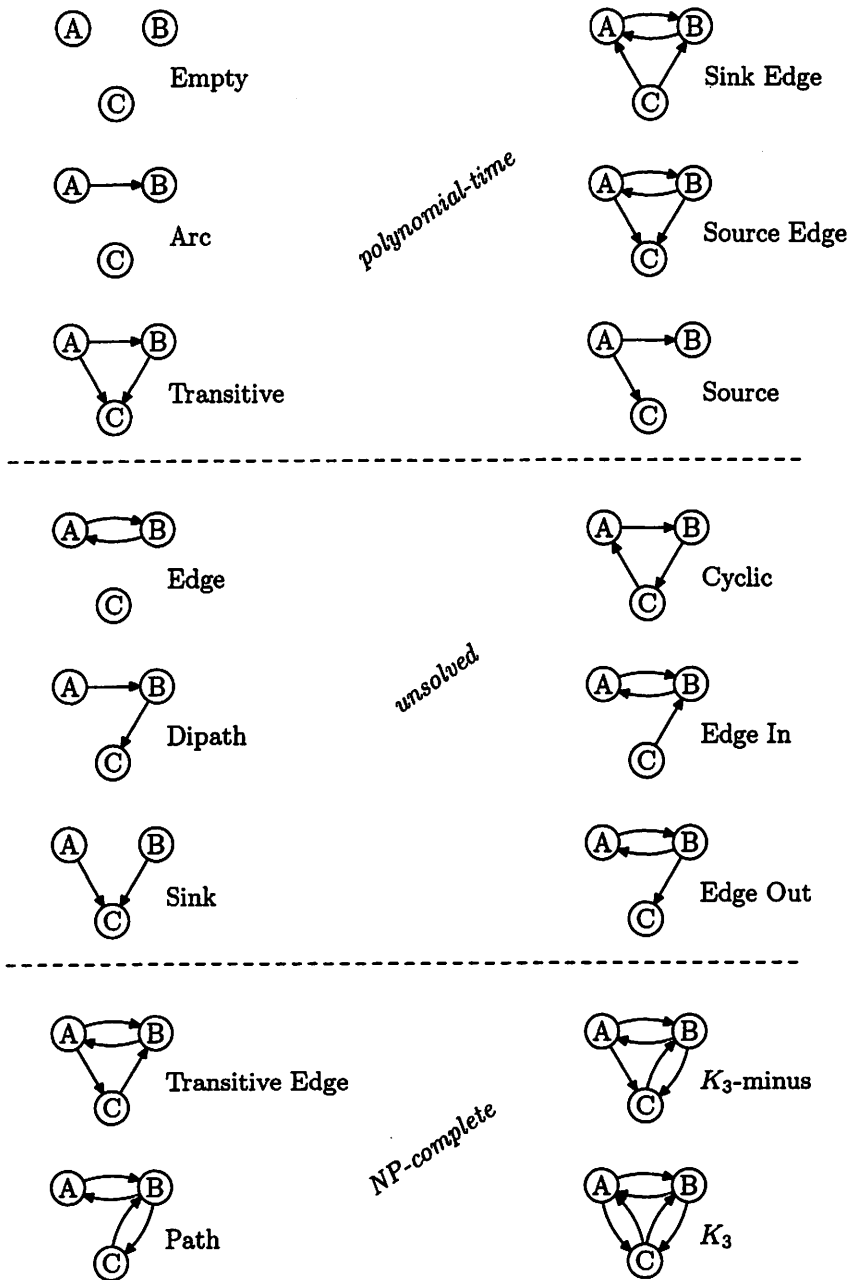


Figure 1: The 16 triads

4 Polynomial Triads

4.1 SINK EDGE

Theorem 3 Let $G = (V, E)$ be a graph of order n . $SINK\ EDGE \in \mathcal{D}(G)$ if and only if $n \geq 4$, and there exists an edge $uv \in E$ such that $2 \leq \deg(u), \deg(v) \leq n - 2$.

Proof. $SINK\ EDGE \in \mathcal{D}(G)$ if and only if the vertices of G can be partitioned into three sets $\{A, B, C\}$ such that: $A \Rightarrow B$, $B \Rightarrow A$, $C \Rightarrow A$, and $C \Rightarrow B$, but $A \not\Rightarrow C$ and $B \not\Rightarrow C$.

Assume $SINK\ EDGE \in \mathcal{D}(G)$. Then there exists an edge uv with $u \in A$ and $v \in B$. Since neither u nor v dominates C , it must be that $\deg(u), \deg(v) \leq n - 2$. Since C dominates both u and v , it must be that $\deg(u), \deg(v) \geq 2$.

Conversely, assume G contains such an edge uv (neither vertex is a leaf or a dominating vertex). Then let $A = \{u\}$, $B = \{v\}$, and $C = V - \{u, v\}$. It is easily checked that this is a $SINK\ EDGE$ -partition. \square

Corollary 4 (a) $SINK\ EDGE \in \mathcal{D}(G)$ if and only if G has a $SINK\ EDGE$ -partition of the form $A = \{u\}$, $B = \{v\}$ and $C = V - \{u, v\}$.
(b) $SINK\ EDGE$ can be decided in $O(m)$ time, where $m = |E|$.

4.2 SOURCE

Theorem 5 Let $G = (V, E)$ be a graph of order n . $SOURCE \in \mathcal{D}(G)$ if and only if $n \geq 4$ and there exist two vertices $u, v \in V$ such that (i) u and v are not adjacent, and (ii) $1 \leq \deg(u), \deg(v) \leq n - 3$.

Proof. Assume that $SOURCE \in \mathcal{D}(G)$. That is, there is a partition of the vertices of G into three sets $\{A, B, C\}$ such that $A \Rightarrow B$ and $A \Rightarrow C$, but neither sets B nor C dominate any other set. It follows that there exists nonadjacent vertices $u \in B$ and $v \in C$. Since $A \Rightarrow B$, we know that $\deg(u) \geq 1$; similarly $\deg(v) \geq 1$. Also, there must be at least one vertex $w \in A$ that is not adjacent to $u \in B$, since $B \not\Rightarrow A$. Thus, $\deg(u) \leq n - 3$. Similarly, $\deg(v) \leq n - 3$.

Conversely, assume that $n \geq 4$ and there exist two vertices $u, v \in V$ such that u and v are not adjacent and $1 \leq \deg(u), \deg(v) \leq n - 3$. Then the partition $\{V - \{u, v\}, \{u\}, \{v\}\}$ is a $SOURCE$ -partition. \square

Corollary 6 (a) $SOURCE \in \mathcal{D}(G)$ if and only if G has a $SOURCE$ -partition of the form $A = V - \{u, v\}$, $B = \{u\}$ and $C = \{v\}$, where u is not adjacent to v .

(b) $SOURCE \in \mathcal{P}$.

4.3 SOURCE EDGE

Recall that if $\{A, B, C\}$ is a $SOURCE$ EDGE-partition of a graph $G = (V, E)$, then A and B dominate each other, A and B both dominate C , but C dominates neither A nor B . The decision problem $SOURCE$ EDGE has a polynomial solution.

Lemma 7 If $G = (V, E)$ is a graph for which $SOURCE$ EDGE $\in \mathcal{D}(G)$, then G has a $SOURCE$ EDGE-partition $\{A, B, C\}$ in which $|C| = 1$.

Proof. Let $\{A, B, C\}$ be a $SOURCE$ EDGE-partition of a graph G and let x be any vertex in the set C . Then it is easy to see that the partition $\{A \cup (C - \{x\}), B, \{x\}\}$ is also a $SOURCE$ EDGE-partition of G . \square

Theorem 8 For any connected graph $G = (V, E)$ of order n , it holds that $SOURCE$ EDGE $\in \mathcal{D}(G)$ if and only if the following two conditions hold:

- 1) G has no isolated vertices.
- 2) There exists a vertex $x \in V$, such that
 - (i) $1 < \deg(x) < n - 2$,
 - (ii) $N(x)$ contains no vertices of degree 1 (x is not adjacent to a leaf),
 - (iii) $|N(V - N[x])| \geq 2$,
 - (iv) $|N(N(x))| \geq 3$.

Proof. (a) Assume $SOURCE$ EDGE $\in \mathcal{D}(G)$. By the above lemma, we know that G has a $SOURCE$ EDGE-partition $\{A, B, C\}$ in which $|C| = 1$. Let $C = \{x\}$.

Clearly, G cannot have any isolated vertices, since every vertex must be dominated by at least one vertex from a set other than its own. It remains to show condition (2):

- (i) Since both sets A and B dominate x , we know that $\deg(x) > 1$. And since C dominates neither A nor B , there must be a vertex in A that x does not dominate and a vertex in B that x does not dominate. Therefore, $\deg(x) < n - 2$.

- (ii) If vertex x were adjacent to a leaf y , then $y \notin C$ since it could not be dominated by either A or B , $y \notin A$ since it could not be dominated by B , and $y \notin B$ since it could not be dominated by A .
- (iii) Let $S = V - N[x]$. At least one vertex in A must be in S and at least one vertex in B must be in S , otherwise C will dominate either A or B . If $|N(S)| = 1$ then S must be an independent set, every vertex of which is a leaf and every such leaf is adjacent to the same vertex, say $y \in N(x)$. Now if $y \in A$, then it follows that $S \subseteq B$, and if $y \in B$ then $S \subseteq A$, either of which contradicts the fact that S contains at least one vertex in A and one vertex in B .
- (iv) Let $R = N(x)$. If $|N(R)| = 1$, then every vertex in R must be a leaf. This contradicts condition 2(ii). Thus, assume that $|N(R)| = 2$. In this case, every vertex in R has degree 2 and all vertices in R are adjacent to the same two vertices: x and a vertex $y \in S$. Since $x \in C$, we know that at least one neighbor of x is in A and at least one neighbor of x is in B . But in this case vertex y cannot be in C , else the vertices in R only have neighbors in C ; y cannot be in A , else there will be a vertex in R that is also in A and have no neighbor in B ; and y cannot be in B else there will be a vertex in R that is in B having no neighbor in A . Thus, $|N(R)| \geq 3$.

(b) Conversely, assume conditions 1) and 2) hold. Let $R = N(x)$ and $S = V - N[x]$. Note that the subgraph $G' = G[R \cup S]$ has no isolated vertices, since G has no isolated vertices and each vertex in R has degree at least two. Then by Ore's Theorem 1, the vertices of G' can be partitioned into two sets $\{A, B\}$ such that A dominates B and B dominates A .

But in order to create a SOURCE EDGE-partition with $C = \{x\}$, we must guarantee that both A and B have a vertex in S , so that C dominates neither A nor B , and both A and B have a vertex in R , so that both A and B dominate C . We can do this in three steps, as follows.

Step 1. We find two special edges uv and wy in $G' = G[R \cup S]$. There are two cases.

Case 1. If neither R nor S is an independent set, then let uv be an edge between two vertices in R and wy be an edge between two vertices in S .

Case 2. If either R or S , or both, are independent sets, then let uv and wy be two nonadjacent edges where $u, w \in R$ and $v, y \in S$. Conditions 2(iii) and 2(iv) guarantee that these two edges exist in this case.

Step 2. Let G'' be the graph obtained from G' by removing the edge uy if it exists. Note that the graph G'' still does not have an isolated vertex.

Step 3. Let A be a maximal independent set of G'' containing $\{u, y\}$, and let $B = V(G'') - A$.

By construction, $v, w \in B$. Since A is an independent dominating set of G'' , it is also a minimal dominating set, and therefore, by Ore's Theorem, B is also a dominating set of G'' . Note that no matter how the two edges wv and wy are chosen, R will have a vertex in A and a vertex in B , and S will have a vertex in A and a vertex in B . \square

Since conditions 1 and 2 can be verified to exist in $O(n^2)$ time, we have the following corollary.

Corollary 9 $SOURCE\ EDGE \in \mathcal{P}$.

4.4 EMPTY

In this section we show that EMPTY is polynomial-time solvable, even if the digraph has more vertices. Let $EMPTY(k)$ denote the digraph having k vertices and no arcs.

Theorem 10 $EMPTY(k) \in \mathcal{P}$.

Proof. We claim that graph G has an $EMPTY(k)$ -partition if and only if there exist (not necessarily distinct) vertices v_{ij} for all $1 \leq i \neq j \leq k$ such that:

- (a) The sets $B_i = \{v_{ij} : j \neq i\}$ for $1 \leq i \leq k$ are disjoint (though it can happen that $v_{ij} = v_{i'j'}$, for $j \neq j'$).
- (b) Every vertex v_{ij} is nonadjacent to all of the set $C_j = \{v_{ij} : i \neq j\}$.
- (c) For every other vertex w in $V - \bigcup_i B_i$, there exists a value j_w such that w is nonadjacent to all of C_{j_w} .

For, assume that there is an $EMPTY(k)$ -partition $\{A_1, A_2, \dots, A_k\}$ of graph G . Then, for all $i \neq j$, there is a vertex $v_{ij} \in A_i$ that is not adjacent to any vertices in A_j . It can be checked that these v_{ij} have the properties described above. (For property (c) let j_w be the color of w .)

Conversely, if we have vertices v_{ij} satisfying the above conditions, then give every v_{ij} color i , and give every other vertex w color j_w . This is an $EMPTY(k)$ -partition of the graph G .

A polynomial-time algorithm can therefore be constructed for deciding if $EMPTY(k) \in \mathcal{D}(G)$ as follows. Consider all $n^{k(k-1)}$ possibilities that

arise by choosing for each $i \neq j$ a (not necessarily distinct) vertex v_{ij} . In each case one can readily check conditions (a), (b), and (c). Thus one can determine whether the graph has such a partition. \square

4.5 TRANSITIVE

In this section we show that TRANSITIVE is polynomial-time solvable, even if the digraph has more vertices. Let TRANSITIVE(k) denote the transitive complete digraph of order k .

Theorem 11 $TRANSITIVE(k) \in \mathcal{P}$.

Proof. Let $\{A_1, A_2, \dots, A_k\}$ be a TRANSITIVE(k)-partition of graph G , where A_1 is a source and A_k is a sink. Then we may assume that $|A_k| = 1$. For, if A_k has more than one vertex, we can move all but one of the vertices in A_k to A_1 and have another TRANSITIVE(k)-partition.

We can generalize this. Define vertices as follows: pick one vertex $v^k \in A_k$. For i decreasing from $k - 1$ to 2, define a subset $A'_i \subseteq A_i$ as follows: for each vertex $w \in A'_{i+1} \cup \dots \cup A'_k$, there exists a vertex $v^i_w \in A_i$ that dominates w . Furthermore, for each $j > i$, there exists a vertex $x^i_j \in A_i$ that is not dominated by A_j . Let A'_i be the set of all v^i_w and x^i_j so chosen. Note that the v^i_w and x^i_j might not be distinct. It follows that $\{V - \bigcup_{i=2}^k A'_i, A'_2, A'_3, \dots, A'_k\}$ is a TRANSITIVE(k)-partition.

Therefore, a polynomial-time algorithm for TRANSITIVE(k) can be constructed as follows. Consider all possibilities for $\{A'_2, \dots, A'_k\}$, where we color the vertices in A'_i with color i . All remaining vertices are colored 1. The colors 2 through k have the desired property, by construction. So this is a TRANSITIVE(k)-partition if and only if the set of vertices colored 1 has the desired property. This condition is easily checked.

Note that for fixed k , the size of A'_i is bounded. The bound $B(m)$ for $|A'_{k-m}|$ obeys the recurrence: $B(m) \leq m + \sum_{m' < m} B(m')$, with $B(0) = 1$. This sequence is 1, 2, 5, 11, 23, \dots , the i^{th} term x_i of which is $x_i = 2x_{i-1} + 1$ for $i > 1$. \square

4.6 ARC

Theorem 12 $ARC \in \mathcal{P}$.

Proof. Let $\{A, B, C\}$ be an ARC-partition of a graph $G = (V, E)$. Then A dominates B , but A does not dominate C , B dominates neither A

nor C , and C dominates neither A nor B . It follows that there exist vertices v_{AB} , v_{AC} , v_{BC} , v_{CA} , and v_{CB} , where v_{XY} is a vertex of set X that is not dominated by any vertex in set Y . Let $S = \{v_{AB}, v_{AC}, v_{BC}, v_{CA}, v_{CB}\}$, where it is possible that $v_{XY} = v_{XY'}$.

Define the following three sets:

$$\begin{aligned}\tilde{A} &= V - S - N(v_{CA}), \\ \tilde{B} &= V - S - N(v_{AB}) - N(v_{CB}), \text{ and} \\ \tilde{C} &= V - S - N(v_{AC}) - N(v_{BC}).\end{aligned}$$

Note that each set \tilde{X} is the set of vertices outside S that can be in X if we want to preserve the non-domination conditions on S . Note also that $\tilde{X} \supseteq X - S$.

Now define the following three sets:

$$\begin{aligned}A' &= \{v_{AB}, v_{AC}\} \cup \tilde{A}, \\ B' &= \{v_{BA}\} \cup (\tilde{B} - (\tilde{A} \cup \tilde{C})), \text{ and} \\ C' &= \{v_{CA}, v_{CB}\} \cup (\tilde{C} - \tilde{A}).\end{aligned}$$

In other words, A' contains every vertex that **can** be in A , and B' contains every vertex that **must** be in B . Since every vertex is in one of \tilde{A} , \tilde{B} , or \tilde{C} , $\{A', B', C'\}$ is a partition of $V(G)$.

We claim that in fact $\{A', B', C'\}$ is an ARC-partition of G . Note that the vertices of S retain their color. By construction, no vertex v_{XY} has a neighbor in Y' . So the non-domination condition is satisfied. All that remains is to prove that A' dominates B' . But note that $A' \supseteq A$ and $B' \subseteq B$. Since A dominates B , it follows that A' dominates B' .

Now, given vertices v_{AB} , v_{AC} , v_{BC} , v_{CA} , and v_{CB} , we can easily construct A' , B' and C' . Note that \tilde{A} , \tilde{B} , and \tilde{C} depend only on the neighborhoods of these vertices and not their colors. We can then check that $\{A', B', C'\}$ is an ARC-partition by verifying that $\{A', B', C'\}$ is a partition (every vertex outside S is in $\tilde{A} \cup \tilde{B} \cup \tilde{C}$), that the implied constraints within S are satisfied, for example, there is no edge from v_{AB} to v_{BC} , and that A' dominates B' .

Therefore, for an algorithm we can try every possible set of vertices S , each having three to five vertices. If any of the resulting $\{A', B', C'\}$ is a partition of $V(G)$ that satisfies the requirements of an ARC-partition, then $\text{ARC} \in \mathcal{D}(G)$; otherwise $\text{ARC} \notin \mathcal{D}(G)$. \square

5 NP-complete Triads

To date we have succeeded in showing that four triads have \mathcal{NP} -complete decision problems. The first result of this type was already proved by Garey and Johnson (cf. p. 190 of [1]) for the triad K_3 .

5.1 K_3

Theorem 13 (Garey and Johnson [2]) K_3 is \mathcal{NP} -complete.

In this case the vertices of the graph G must have a partition into three mutually disjoint dominating sets. This is equivalent to saying that the domatic number of G is at least three.

5.2 PATH

Theorem 14 *PATH* is \mathcal{NP} -complete.

Proof. We use a transformation from 3-COLORABILITY (cf. p. 191 of [1]).

3-COLORABILITY

INSTANCE: Graph $G = (V, E)$.

QUESTION: Is G 3-colorable, i.e. does there exist a function $f: V \rightarrow \{1, 2, 3\}$ such that $f(u) \neq f(v)$ whenever u is adjacent to v ?

Given an input graph G , construct the graph G' as follows: for each edge $e = uv \in E$, add three vertices x, y, z and five edges such that $N(x) = \{u, v\}$, $N(y) = \{u, v, z\}$, and $N(z) = \{y\}$. This is illustrated in the left figure of Figure 2. Note that the edge uv is retained in G' . Finally, add a single disjoint copy of the path P_3 of order 3.

Claim: G is 3-colorable if and only if $\text{PATH} \in \mathcal{D}(G')$.

(1) Assume that G is 3-colorable. Then for each edge uv , vertices u and v receive different colors. Extend this to a *PATH*-partition of G' by (i) coloring every vertex y with color B , (ii) coloring every vertex x with the color not used to color u or v , (iii) if either u or v is colored B , then color z with the color not used to color u and v , and (iv) if neither u nor v is colored B , then color z arbitrarily with either A or C . Finally, color the central vertex of the P_3 with color B and the two leaves of the P_3 with different colors A and C , arbitrarily.

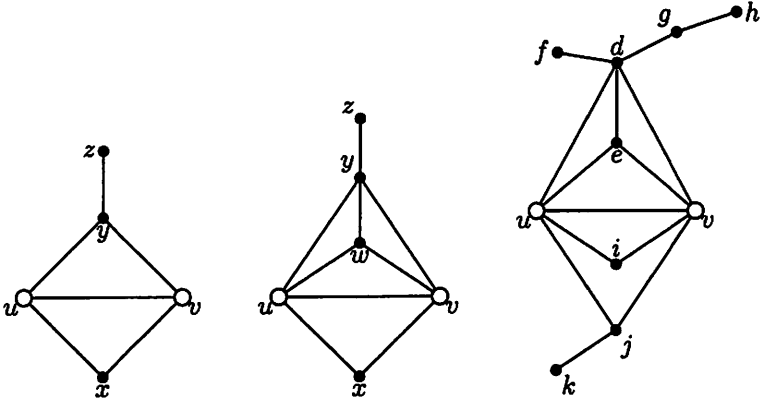


Figure 2: Gadgets for \mathcal{NP} -completeness proofs

Because of the coloring of the vertices of the P_3 , neither sets A nor C dominate each other. On the other hand, every vertex in A and in C has a neighbor in B , and every vertex in B has a neighbor in A and a neighbor in C . Therefore, $\{A, B, C\}$ is a PATH-partition of G' .

(2) Conversely, assume that $\{A, B, C\}$ is a PATH-partition of G' . Suppose that for some edge $e = uv$ of G , vertices u and v have the same color. If both u and v have color A , then there is a contradiction at vertex x ; x cannot be colored A since then it will not have a neighbor colored B , it cannot be colored B since it will not have a neighbor colored C , and it cannot be colored C since it will not have a neighbor colored B . Similarly, if both u and v are colored C , then there is again a contradiction at vertex x .

If u and v are both colored B , then there is a contradiction at vertex y , since to avoid a contradiction at z , it must be that z has color A or C and y has color B . But then y , having color B , is missing a vertex of color A or C in its neighborhood. Therefore, vertices u and v must receive different colors, and we have a proper coloring of G . \square

It is interesting to observe that for graphs having $\text{diam}(G) \leq 2$ the decision problem PATH has easy solutions. Any graph having $\text{diam}(G) = 1$ is a complete graph, for which $\text{PATH} \notin \mathcal{D}(G)$. For graphs with $\text{diam}(G) = 2$ we have the following theorem.

Theorem 15 For any connected graph $G = (V, E)$ having $\text{diam}(G) = 2$, $\text{PATH} \in \mathcal{D}(G)$.

Proof. Let $G = (V, E)$ be a connected graph having $\text{diam}(G) = 2$, and let $v \in V$ be any vertex with $\text{ecc}(v) = 2$. Define the following partition $\{A, B, C\}$:

$$C = V - N[v], \quad A = N[v] - N(C), \quad B = N(v) \cap N(C).$$

For this partition it is easy to show that $A \Rightarrow B, B \Rightarrow A, B \Rightarrow C, C \Rightarrow B, C \not\Rightarrow A$, and $A \not\Rightarrow C$. It follows that $\text{PATH} \in \mathcal{D}(G)$. \square

5.3 K_3 -MINUS

Theorem 16 K_3 -MINUS is \mathcal{NP} -complete.

Proof. Recall that K_3 -MINUS is the triad having all arcs except one, say from C to A . We again use a transformation from 3-COLORABILITY. The proof is similar to the proof of Theorem 14 for PATH, except that the construction is the following.

Given input graph G , construct the graph G' as follows: For each edge $e = uv \in E$, add four vertices x, y, z, w and eight edges such that $N(x) = \{u, v\}$, $N(y) = \{u, v, z, w\}$, $N(z) = \{y\}$, and $N(w) = \{u, v, y\}$. Note that the edge uv is retained in G' . This is illustrated in the middle figure of Figure 2.

Claim: G is 3-colorable if and only if K_3 -MINUS $\in \mathcal{D}(G')$.

Proof summary. If G is 3-colorable, then one can extend this 3-coloring to a K_3 -MINUS-partition of G' . One gives x and w the color missing on edge uv , y color B , and z color A .

Conversely, if we have a K_3 -MINUS-partition $\{A, B, C\}$ of G' , then it must be a proper 3-coloring of G . For, if the colors of the two vertices of edge $e = uv$ are both A or both C , then x cannot be correctly colored in the K_3 -MINUS-partition; and if the two vertices of e are both colored B , then either vertex w or vertex y cannot be correctly colored in the K_3 -MINUS-partition. \square

5.4 TRANSITIVE EDGE

Theorem 17 TRANSITIVE EDGE is \mathcal{NP} -complete.

Proof. We use a modification of the above proofs for PATH and K_3 -MINUS. Recall that TRANSITIVE EDGE is the triad with vertices A, B and C , where $A \Rightarrow B, B \Rightarrow A, A \Rightarrow C$ and $C \Rightarrow B$. We again use a transformation from 3-COLORABILITY.

Given an input graph G , construct the graph G' as follows: for each edge $e = uv \in E$, add a component F_{uv} having eight vertices d, e, f, g, h, i, j, k , and the edges $ud, vd, ue, ve, ui, vi, uj, vj, jk, ed, fd, gd$, and hg . As before, the edge uv in G is retained in G' . This is illustrated in the right figure of Figure 2.

Claim: G is 3-colorable if and only if TRANSITIVE EDGE $\in \mathcal{D}(G')$.

(1) Assume that $\{A, B, C\}$ is a proper 3-coloring of G . Then for each edge uv , vertices u and v receive different colors. We extend this to a TRANSITIVE EDGE-partition of G' as follows:

- (i) if edge uv is colored AB or BA , then give vertices f, g , and j color A ; give vertex d color B ; and give vertices e, i, h and k color C .
- (ii) if edge uv is colored AC or CA , then give vertices f, g , and k color A ; give vertices d, e, i and j color B ; and give vertex h color C .
- (iii) if edge uv is colored BC or CB , then give vertices e, f, g, i and j color A ; give vertex d color B ; and give vertices h and k color C .

In each of the edge components F_{uv} constructed, vertex f is colored A , and its only neighbor d is colored B . Therefore, set C does not dominate A . Each vertex h is colored C , and has only one neighbor g , which is colored A . Therefore set B does not dominate C . On the other hand, every vertex in B and every vertex in C has a neighbor in A , every vertex in A has a neighbor in B , and every vertex in B has a neighbor in C . Therefore, G' has a TRANSITIVE EDGE-partition.

(2) Assume that G' has a TRANSITIVE EDGE-partition $\{A, B, C\}$. We will show that $\{V \cap A, V \cap B, V \cap C\}$ is a proper 3-coloring of the vertices of G . This requires several observations about a TRANSITIVE EDGE-partition of G' :

1. No leaf in G' can be colored B , because vertices colored B need both a neighbor in A and a neighbor in C . Further, if a leaf is colored A , then its neighbor has to have color B ; and if a leaf is colored C , then its neighbor has to have color A . It follows that a vertex cannot be colored C if it is adjacent to a leaf.
2. Claim: the vertices d, f, g and h in each edge component F_{uv} must be colored so that d has color B , f and g have color A , and h has color C .

Proof of Claim : Vertex d is adjacent to a vertex of degree 1, so its color cannot be C . Vertex g cannot be colored B , since it would have no neighbor of color C . Vertex g cannot be colored C since it is adjacent to a vertex of degree 1. So vertex g is colored A , and vertex h is colored C . Vertex g needs a neighbor colored B , forcing vertex d to be colored B . Therefore vertex f will be colored A . Note that in coloring of vertices u, v , and e , at least one of these must be colored C , so that vertex d is dominated by C .

Now, suppose that the vertices on some edge $e = uv$ in G have the same color in a TRANSITIVE EDGE-partition $\{A, B, C\}$ of G' . In this case, we can show there is a contradiction in the component F_{uv} that was added to G' for edge uv .

(i) If both u and v have color C , then there is a contradiction at vertex i : if i is colored A , it must be adjacent to a vertex colored B ; if i is colored B , it must be adjacent to a vertex in both A and C ; and if i is colored C , it must be adjacent to a vertex colored A .

(ii) If both u and v are colored B , then there is a contradiction at vertex e : vertex d needs a neighbor colored C , which must then be vertex e , but then there will be no vertex of color A to dominate vertex e .

(iii) If both u and v have color A , then there is a contradiction at vertex j : since vertex j is adjacent to a vertex of degree 1, it cannot be colored C , and if it is colored A , it cannot have a neighbor colored B , and if it is colored B , it cannot have a neighbor colored C .

Therefore, $\{V \cap A, V \cap B, V \cap C\}$ is a proper 3-coloring of G . \square

6 Partial Results

In this section we show that the two problems EDGE IN and EDGE OUT are trivial if the graph has diameter at least 3.

6.1 EDGE IN

Recall that $\text{EDGE IN} \in \mathcal{D}(G)$ if and only if the vertices of G can be partitioned into three sets $\{A, B, C\}$ such that A dominates B , B dominates

A , but neither A nor C dominate each other, and C dominates B , but B does not dominate C .

Theorem 18 For any connected graph $G = (V, E)$, if $\text{diam}(G) \geq 3$ then $\text{EDGE IN} \in \mathcal{D}(G)$.

Proof. Let $G = (V, E)$ be a connected graph having $\text{diam}(G) \geq 3$, and let $u \in V$ be any vertex having maximum eccentricity $\text{ecc}(u) = \text{diam}(G) \geq 3$. Define the vertex partition $\{A, B, C\}$ where $A = \{u\}$, $B = N(u)$ and $C = V - N[u]$.

Now define $B(C) = N(C) \cap B$ to be the set of vertices in $B = N(u)$ that are adjacent to at least one vertex in $V - N[u]$. Consider then the vertices $B(N[u]) = B - B(C)$ in B that are not adjacent to any vertices in $C = V - N[u]$. Notice that if a vertex $w \in B(N[u])$ is not adjacent to any vertex in $B(C)$, then $\text{ecc}(w) = \text{ecc}(u) + 1$, contradicting our assumption that u has maximum eccentricity. Thus, every vertex in $B(N[u])$ is adjacent to at least one vertex in $B(C)$. Therefore, define the following new partition of V . Let $A' = \{u\} \cup B(N[u])$, $B' = B(C)$ and $C' = C$.

For this partition we can readily show that $A' \Rightarrow B'$, $B' \Rightarrow A'$, $A' \not\Rightarrow C'$, $C' \not\Rightarrow A'$, $B' \not\Rightarrow C'$, and $C' \Rightarrow B'$. It follows that $\text{EDGE IN} \in \mathcal{D}(G)$. \square

It is trivial that $\text{EDGE IN} \notin \mathcal{D}(G)$ if G has diameter one. We have not yet resolved the complexity of the decision problem $\text{EDGE IN} \in \mathcal{D}(G)$ for graphs G having $\text{diam}(G) = 2$.

6.2 EDGE OUT

Recall that $\text{EDGE OUT} \in \mathcal{D}(G)$ if and only if the vertices of G can be partitioned into three sets $\{A, B, C\}$ such that A dominates B , B dominates A , but neither A nor C dominate each other, and B dominates C , but C does not dominate B .

Theorem 19 For any connected graph $G = (V, E)$, if $\text{diam}(G) \geq 3$ then $\text{EDGE OUT} \in \mathcal{D}(G)$.

Proof. Let $G = (V, E)$ be a connected graph having $\text{diam}(G) \geq 3$ and let $u \in V$ be any vertex having maximum eccentricity $\text{ecc}(u) = \text{diam}(G) \geq 3$. Define the vertex partition $\{A, B, C\}$ where

$$\begin{aligned} C &= \{u\}. \\ B &= \{v \mid d(u, v) \equiv 1 \pmod{2}\}. \\ A &= \{w \mid 0 < d(u, w) \equiv 0 \pmod{2}\}. \end{aligned}$$

Now define C' to be the set of vertices in $N(u)$ that are not adjacent to any vertices in $V - N[v]$. Notice that if a vertex $w \in C'$ is not adjacent to any vertex in $N(v) - C'$, then $\text{ecc}(w) = \text{ecc}(u) + 1$, contradicting our assumption that u has maximum eccentricity. Thus, every vertex in C' is adjacent to at least one vertex in $(N(u) - C') \subset B$. Therefore, define the following new partition of V . Let $C'' = \{u\} \cup C'$, $B'' = B - C'$ and $A'' = A$.

For this partition we can readily show the following: $B'' \Rightarrow A''$, $A'' \Rightarrow B''$, $A'' \not\Rightarrow C''$, $C'' \not\Rightarrow A''$, $B'' \Rightarrow C''$, and $C'' \not\Rightarrow B''$. It follows that $\text{EDGE OUT} \in \mathcal{D}(G)$. \square

It is trivial that $\text{EDGE IN} \notin \mathcal{D}(G)$ if G has diameter one. We have not yet resolved the complexity of the decision problem $\text{EDGE OUT} \in \mathcal{D}(G)$ for graphs G having $\text{diam}(G) = 2$.

7 Summary and Open Problems

1. It is likely that the problem of deciding if a graph G has a TRIAD-partition $\{A, B, C\}$ where sets A , B , and C are all independent is \mathcal{NP} -complete, since this requires a 3-coloring of the graph G . But two similar problems might not be \mathcal{NP} -complete: (i) does a given graph G have a TRIAD-partition in which one (or at least one) of the three sets A , B and C is independent? (ii) does a given graph G have a TRIAD-partition in which two (or at most two) of the three sets A , B and C are independent?
2. Can any more of the results above for diads and triads be modified to provide results for quatrads, i.e. domination digraphs of order $n = 4$?
3. If the diameter of a graph G is at least 3, does a TRIAD decision problem become simpler? Why are EDGE IN and EDGE OUT difficult for diameter 2?
4. We have not yet determined the complexity of the six triads EDGE , DIPATH , CYCLIC TRIPLE , SINK , EDGE IN , and EDGE OUT .
5. Do some of these problems become easier for specific families of graphs? For example, computing the domatic number is easier for some families (see [3, 6]).

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