The spectrum of non-polychromatic equitable edge colored Steiner Triple Systems

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Abstract

In [1], we showed that for $v \equiv 1$ or 3 (mod 6), there is an equitable k-edge coloring of K_v that does not admit any polychromatic $STS(v)$, when $k = 2$, 3, and $v - 2$. In this paper, we extend the results to all feasible values of k, where $2 \leq k \leq v - 2$

1 Introduction

An *edge coloring* of a graph G is an assignment of colors to the edges of G. A *k-edge coloring* of G is an edge coloring of G in which k

distinct colors are used, $c_1, c_2, \cdots, c_k$ say. We let $E(c_i)$ denote the set of edges that are assigned color c_i, for $i = 1, 2, \ldots, k$. Also, let $d(v)$ denote the degree of a vertex v.

A *Steiner triple system of order* v, denoted $STS(v)$, is an ordered pair (V, T), where V is a v-set of symbols and T is a set of 3-element subsets of V called *triples* such that any pair of symbols in V occurs together in exactly one triple. It is known that a Steiner triple system (STS) of order v exists if and only if $v \equiv 1$ or $3 \pmod 6$; see [12].

Let G and H be graphs. An H-decomposition of G is a set $\mathcal{H} = \{H_1, H_2, \ldots, H_p\}$ such that H_i is isomorphic to H for $1 \leq i \leq p$ and $\mathcal{H}$ partitions the edge set of G. A K_3-decomposition of K_v is equivalent to an $STS(v)$, and we will use the terms interchangeably. Similarly, we will often refer to K_3 as a triple.

We color the triple $\{x, y, z\}$ with the color triple $< c_i, c_j, c_k >$ by assigning the colors c_i, c_j and c_k to the edges $\{x, y\}$, $\{y, z\}$ and $\{x, z\}$ respectively. Note that when no confusion is likely to arise, we denote the edge $\{x, y\}$ by xy. If i, j, k are distinct, then we say that $\{x, y, z\}$ is *polychromatic*. An $STS(v)$ or K_3-decomposition of some graph G is said to be polychromatic if every triple in the $STS(v)$ or decomposition is polychromatic.

Consider the graphs H and G, with an edge coloring of G. The graph H is said to be a polychromatic subgraph of G, if G contains a subgraph isomorphic to H, all of whose edges are assigned distinct colors. Most of the effort regarding edge coloring problems has been devoted to determining the least number of colors used to color $E(K_v)$ that forces a specified polychromatic subgraph to occur. Those problems are called *Anti-Ramsey* problems. For more details about Anti-Ramsey problems and polychromatic subgraphs, the reader is referred to [4, 5, 10, 13].

In [2], Bate studied complete graphs without polychromatic C_n for $n = 3$ and $n = 4$. A good survey about monochromatic and polychromatic subgraphs in edge colored graphs can be found in [11].

For each $v \in V(G)$ let $n_v(c_i)$ denote the number of edges in $E(G)$ of color c_i incident with vertex v, for $i = 1, 2, \ldots, k$. A k-edge coloring of G is said to be *equitable* if $|n_v(c_i) - n_v(c_j)| \leq 1$ for all $v \in V(G)$ and $1 \leq i < j \leq k$. That is, the number of edges of each color is as close as possible at any vertex of the graph G. If there exists an equitable k-edge coloring of G, then G is said to be *equitably k-edge colorable.*

Hilton and de Werra [9] found a sufficient condition for a simple graph to admit an equitable edge coloring:

Theorem 1.1. [9] *Let G be a simple graph and let $k \geq 2$. If $k \nmid d(v)$ ($\forall\ v \in V(G)$) then G has an equitable k-edge coloring.*

Specialized colorings of cycle systems (C_3 and C_4), in which some condition on the coloring is satisfied, have received recent attention. We refer the interested reader to [3, 6, 7, 8].

In [1] we asked two questions regarding the relationship between an equitable k-edge coloring of the complete graph, K_v, and the decomposition of K_v into triangles (that is, the construction of an $STS(v)$). The first question was: does every $STS(v)$ admit a polychromatic k-edge coloring which is also an equitable k-edge coloring of the complete graph of order v, K_v? We showed that the answer is no when $k = 3$.

The second question was: For $v \equiv 1$ or $3 \pmod 6$, does every equitable k-edge coloring of K_v admit a polychromatic $STS(v)$? In [1] we showed that the answer to this question is also no for $k = 2$, 3, and $v - 2$. We also conjectured that the answer would be no for the spectrum $2 \leq k \leq v - 2$.

In this paper, we show that our conjecture is true. That is, we prove that for $2 \leq k \leq v - 2$ and $v \equiv 1$ or $3 \pmod 6$, there exists an equitable k-edge coloring of K_v that does not admit any polychromatic $STS(v)$. This is done by finding an equitable k-edge coloring for which there exists an edge that appears in no polychromatic triangle. Note that if $k \geq v - 1$ then the edge coloring is proper and all the triangles are polychromatic.

A latin square of order n is an $n \times n$ array on n symbols, say $1, 2, \cdots, n$, in which every symbol occurs exactly once in each row and column of the array. A latin square is said to be *reduced* if its first row and first column have the symbols $1, 2, \cdots, n$ in that order. If the condition that each symbol occurs exactly once in each row and column is replaced by the condition that it appears the same number of times in each row and column, then such squares are called *frequency squares.* Similarly, if the condition is modified such that each symbol occurs at least once in each row and column, then the array is called a *generalized frequency square*.

Let L be a latin square of odd order n on the symbols $1, 2, \cdots, n$. A *k-fixed-cell-transversal* is a set of n cells with the property that one cell lies in each row, one in each column, and all the cells contain the same symbol k. For example, the cells $(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)$ and $(7,7)$ make a 7-fixed-cell-transversal in the following latin square:

1	5	2	6	3	7	4
5	2	6	3	7	4	1
2	6	3	7	4	1	5
6	3	7	4	1	5	2
3	7	4	1	5	2	6
7	4	1	5	2	6	3
4	1	5	2	6	3	7

Note that the above latin square is also idempotent and commutative. It is known that idempotent commutative latin squares exists for all odd orders.

2 The Lemmas

Lemmas 2.1 and 2.2 consider equitable k-edge colorings of the complete graph for odd and even k, respectively.

Lemma 2.1. *Let t, k and m be integers such that $t, k \geq 1$, k is odd, $0 \leq m \leq k-1$, and if $m = k-1$ then t is odd. Then there exists an equitable k-edge coloring of K_{tk+m+2} for which there exist two vertices x and y such that for each $z \in V(K_{tk+m+2}) \setminus \{x, y\}$ the edges xz and yz have the same color.*

Proof. Let t, k and m be as specified by the conditions of the lemma, and let the colors be $c_1, c_2, \ldots, c_k$. We arrange the vertices of K_{tk+m+2} into t layers of k vertices each, one layer of m vertices, and the two extra vertices x and y. That is, let $V(K_{tk+m+2}) = \bigcup_{i=1}^{t}\{1_i, 2_i, \ldots, k_i\} \cup \{1_{t+1}, 2_{t+1}, \ldots, m_{t+1}\} \cup \{x, y\}$.

Let $(Q, \cdot)$ be a commutative idempotent quasigroup of order k. Since k is odd, we know that a quasigroup with such properties exists. We use this to define an initial coloring as follows: for any distinct $i_r, j_s \in V(K_{tk+m+2})$, color the edge $i_r j_s$ with $c_{i \cdot j}$; for each $i_r \in V(K_{tk+m+2})$, color the edges $i_r x$ and $i_r y$ with c_i; and finally, color the edge xy with c_{m+1}.

Suppose that t is odd. For each $i \in \{1, 2, \ldots, m\}$, change the color of edges $i_1 i_2, i_3 i_4, \ldots, i_t i_{t+1}$ from $c_{i \cdot i} = c_i$ to $c_{i \cdot (m+1)}$. The resultant coloring is equitable. This leaves the case where t is even, and hence by assumption $m \leq k - 2$. For each $i \in \{1, 2, \ldots, m\}$, change the color of edges $i_1 i_2, i_3 i_4, \ldots, i_{t-1} i_t$ from c_i to $c_{i \cdot (m+1)}$, and change the color of edge $i_t i_{t+1}$ from c_i to $c_{i \cdot (m+2)}$. Again, this gives an equitable coloring.

□

Lemma 2.2. *Let k and n be integers such that k is even, n is odd, and $4 \leq k \leq n - 1$. Suppose also that $n \not\equiv k + 1 \pmod{2k}$. Then there exists an equitable k-edge coloring of K_n for which there exist two vertices x and y such that for each $z \in V(K_n) \setminus \{x, y\}$ the edges xz and yz have the same color.*

Proof. Let the colors be $c_0, c_1, \ldots, c_{k-1}$, let $m = n - 2$, and let $V(K_n) = \{0, 1, 2, \ldots, m - 1\} \cup \{x, y\}$. We construct the coloring by using an $m \times m$ commutative generalized frequency square on the symbols $\{0, 1, 2, \ldots, m - 1\}$, with the property that each symbol occurs either $\lfloor \frac{m}{k} \rfloor$ or $\lceil \frac{m}{k} \rceil$ times in each row, each column and on the main diagonal, and in addition the symbol in cell (i, i) occurs $\lfloor \frac{m}{k} \rfloor$ times in row i and column i. Letting $i \cdot j = j \cdot i$ denote the symbol in cell (i, j) of the generalized frequency square, the required coloring is obtained by coloring, for $i, j \in \{0, 1, \ldots, m - 1\}$, edge ij with $c_{i \cdot j}$, edges xi and yi with $c_{i \cdot i}$, and the edge xy with c_l, where l is any symbol which occurs $\lfloor \frac{m}{k} \rfloor$ times on the main diagonal of the generalized frequency square. Thus the problem is reduced to constructing such a square.

Let a, b, r be non-negative integers satisfying $m = rk + a$ and $k = a + b$. Note that a and b are odd. First suppose that $n \not\equiv 1 \pmod{k}$. Construct an initial square S by letting $i \cdot j = (f(i) + f(j) \bmod m) \bmod k$, where

$$f(i) = \begin{cases} i/2, & i \text{ even}, \\ (m + i)/2, & i \text{ odd}. \end{cases}$$

Note that $i \cdot i = i \bmod k$, and since $n \not\equiv 1 \pmod{k}$, $b \geq 3$. Each of the symbols $c_0, c_1, \ldots, c_{a-1}$ and $c_a, c_{a+1}, \ldots, c_{k-1}$ occur respectively $r+1$ and r times in every row and column and on the main diagonal of S. This square has the required properties, with the exception that if $i \cdot i \in \{0, 1, \ldots, a-1\}$, then $i \cdot i$ occurs $r+1$ times in row i and column i. We proceed by changing selected off-diagonal cells to reduce this number to r, while preserving the other required properties of the square.

We begin with the case where r is odd. Let $i \in \{0, 1, \ldots, a-1\}$, $j \in \{0, 1, \ldots, (r-1)/2\}$, $p = i + kj$, $q = i + k(j + (r+1)/2)$. Note that $p \cdot (p+b) = (p+b) \cdot p = p \cdot p = i$, and $q \cdot (q-b) = (q-b) \cdot q = q \cdot q = i$, but $(p+b) \cdot (p+b) \neq i$ and $(q-b) \cdot (q-b) \neq i$. Replace the symbol i in cells $(p, p+b)$ and $(p+b, p)$ with one of the symbols $a, a+1, \ldots, k-1$, ensuring that the symbol used is not $(p+b) \cdot (p+b)$. Similarly replace the symbol i in cells $(q, q-b)$ and $(q-b, q)$ with one of the symbols $a, a+1, \ldots, k-1$, ensuring that the symbol used is not $(q-b) \cdot (q-b)$. We make these substitutions in order of increasing p then q, and further ensure that the new color chosen for a cell has not already been added to that row or column; this is possible since $b \geq 3$ (because of the order in which we make the substitutions, for each cell in turn there may have already been a substitution in the same row, or in the same column, but not both). Following these substitutions, S has the required properties, except in one case: if substitutions are made in two cells in the same row (column) which had the same original color, then this color will not appear sufficiently often in that row (column) after the substitution. This occurs if and only if $k = 2b$. A separate construction is given below for this case.

Now consider the case where r is even. Let $i \in \{0, 1, \ldots, a-1\}$, $j \in \{0, 1, \ldots, r/2 - 1\}$, $p = i + kj$, $q = i + k(r - j)$. Note that $p \cdot p = q \cdot q = p \cdot q = q \cdot p = i$. In cells (p, q) and (q, p), replace symbol i with a. Finally, consider $i \in \{0, 1, \ldots, a-1\}$ and $p = kr/2 + i$. Note that $p \cdot p = (p+a) \cdot p = p \cdot (p+a) = i$. In cells $(p, p+a)$ and $(p+a, p)$, replace the symbol i with $a+1$, or with $a+2$ if $(p+a) \cdot (p+a) = a+1$.

We are left with two special cases, for which we use different constructions to obtain a $m \times m$ commutative generalized frequency square with the required properties. First, let r be odd and $k = 2b$ (that is, $a = b = k/2$). Note that $m = rk + a = (2r+1)a$. Let A be an $a \times a$ idempotent commutative latin square on the symbols

$\{0, 1, \ldots, a-1\}$. Let B be the square formed by replacing the main diagonal of A with the symbols $\{a, a+1, \ldots, m-1\}$ (that is, adding a to each number on the diagonal of A). Let C be an $a \times a$ idempotent commutative latin square on the symbols $\{a, a+1, \ldots, m-1\}$. Let D be the square formed by replacing the main diagonal of C with the symbols $\{0, 1, \ldots, a-1\}$. Let γ be a function which shifts the bottom row of a latin square to the top; that is, if X is a latin square, cell (i, j) of $\gamma(X)$ contains the symbol in cell $(i-1, j)$ of X, taking $i-1$ modulo the order of X. Form a generalized frequency square by taking a $(2r+1) \times (2r+1)$ array with rows and columns labeled $\{0, 1, \ldots, 2r\}$, and placing a latin square of order a in each position. Consider position (i, j), $0 \leq i, j \leq 2r$. If $i = j$ and i is even, place a copy of B in this position; otherwise, if $i + j$ is even place a copy of A in this position. If j is even and $i = j + 1$, place a copy of $\gamma(D)$ in this position. If i is even and $j = i + 1$, place a copy of $\gamma(D)^T$ in this position. If $(i, j) = (2r-1, 2r)$, place a copy of $\gamma^2(D)$ in this position. If $(i, j) = (2r, 2r-1)$, place a copy of $\gamma^2(D)^T$ in this position. Otherwise, if $i+j$ is odd, place a copy of C in this position. The result is a $m \times m$ generalized frequency square with the required properties.

Finally, consider the special case $n \equiv 1 \pmod{k}$. By assumption, $n \not\equiv k+1 \pmod{2k}$. Hence $m = 2sa + 2s - 1$, where $a = k - 1$ and s is a positive integer. Let A be an $a \times a$ idempotent commutative latin square on the symbols $\{0, 1, \ldots, a-1\}$, which contains at least one transversal T which is disjoint from the main diagonal. Let B be the array formed by replacing the main diagonal entries in A with the symbol a. Let C be the array formed from B by replacing all the entries in T by a. Let R_1 and C_1 be a $1 \times a$ array and a $a \times 1$ array respectively, formed by placing the entry in cell (i, j) of T in position j of R_1 and position i of C_1. Let R be the $1 \times 2sa$ array formed by the concatenation of s copies of R_1 followed by s copies of C_1^T, and let $C = R^T$. Let R' be a $1 \times 2sa$ array with i mod a in position i, and let $C' = R'^T$. Form a $2sa \times 2sa$ generalized frequency square S' by taking a $2s \times 2s$ array in which each cell is a copy of A, B, or C. Consider position (i, j), $0 \leq i, j \leq 2s-1$. If $i = j$, place a copy of A in this position. If $i + j = 2s - 1$, then if $i > j$ place a copy of C in this position, while if $i < j$ place a copy of C^T in this position. Otherwise place a copy of B in position (i, j). We now extend S' to an $m \times m$ generalized frequency square S, by placing $2s-2$ copies of R' and one copy of R below S', and $2s-2$ copies of C' and one copy of C to the right of S', and then completing S by placing symbol a in (i,j) for every $i,j \in \{2sa, 2sa+1, \ldots, m-1\}$. This square has the required properties. □

3 The main result

We are ready now to present the main result:

Theorem 3.1. *Let $n \equiv 1,3 \pmod 6$, $n \geq 7$ and $2 \leq k \leq n-2$. If k is even and $n \equiv k+1 \pmod{2k}$, then there is no equitable k-edge coloring of K_n. Otherwise there exists an equitable k-edge coloring of K_n that contains an edge that is not in any polychromatic triangle.*

Proof. Suppose that k is even, $n \equiv k+1 \pmod{2k}$, and there exists an equitable k-edge coloring of K_n. Then $n = pk+1$, p odd, and each vertex is incident with p edges of each color. This implies that there are $np/2$ edges of each color in total, which is not possible since n and p are odd. Henceforth we assume that if k is even, $n \not\equiv k+1 \pmod{2k}$.

For odd k, we write $n = tk + m + 2$, for some integer t and $0 \leq m \leq k-1$. Note that since n is odd, if $m = k-1$ then t is odd. Apply Lemma 2.1 to get the required equitable k-edge coloring of K_n that contains an edge (xy) which is not in any polychromatic triangle.

If k is even, $k \geq 4$, then Lemma 2.2 gives the required k-equitable edge coloring of K_n that contains an edge (xy) which is not in any polychromatic triangle. □

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