Polychromatic Vertex Colorings of Cube Graphs

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In memory of Ralph G. Stanton

Abstract

Let $g(n,k)$ be the maximum number of colors for the vertices of the cube graph Q_n such that each subcube Q_k contains all colors. Some exact values of $g(n,k)$ are determined.

1 Introduction

A cube graph Q_n has 2^n vertices which are represented by all n-digit binary numbers. Two vertices are joined by an edge if the corresponding binary numbers differ in exactly one digit.

A vertex coloring of Q_n is called k-polychromatic, $1 \leq k \leq n$, if each subgraph Q_k has at least one vertex of each color [1]. Let $g(n,k)$ denote the maximum number of colors in a k-polychromatic coloring of Q_n. In [1] it is proved that

$$\lim_{n\to\infty} g(n,k) = k+1.$$

Here some first exact values of $g(n,k)$ are determined (see Table 1).

2 Results

The existence of $g(n,k)$ is guaranteed by the following general lower bound.

Lemma 1. $g(n,k) \geq k+1$.

Proof. Consider the coloring of Q_n with color $i = 1,\ldots,k{+}1$ for all vertices having j digits 1 in their binary representation with $j \equiv i \pmod{k{+}1}$. Since

$n\backslash k$	1	2	3	4	5	6	7	8	9
1	2								
2	2	4							
3	2	4	8						
4	2	3	8	16					
5	2	3	5	16	32				
6	2	3	5	10	32	64			
7	2	3	5			64	128		
8	2	3	4				128	256	
$\vdots$	$\vdots$	$\vdots$	$\vdots$					$\ddots$	$\ddots$

Table 1: $g(n, k)$.

in each Q_k there occur $k+1$ consecutive numbers j of digits 1 this coloring is as desired. □

For fixed k the numbers $g(n, k)$ are monotonically decreasing.

Lemma 2. $g(n+1, k) \leq g(n, k)$.

Proof. Consider a coloring of Q_{n+1} with $g(n+1, k)$ colors such that each Q_k has at least one vertex of each color. Then any subcube Q_n of this Q_{n+1} has the same property proving a lower bound for $g(n, k)$. □

For $k = 1$, $n-1$, and n the exact values of $g(n, k)$ are as follows.

Theorem 1. $g(n, n) = 2^n$, $g(n, n-1) = 2^{n-1}$, and $g(n, 1) = 2$.

Proof. For $k = n$ all 2^n vertices of Q_n can be colored pairwise differently.

By Lemma 2 it follows $g(n, n-1) \leq g(n-1, n-1) = 2^{n-1}$. The coloring of Q_n with 2^{n-1} colors and equally colored opposite vertices, that is, vertices with complementary binary representations, proves $g(n, n-1) \geq 2^{n-1}$, since each Q_{n-1} contains exactly one of two opposite vertices of Q_n.

By Lemma 1, Lemma 2, and $g(n, n) = 2^n$ from Theorem 1 it follows $2 = g(1, 1) \geq g(n, 1) \geq 1 + 1$. □

A general upper bound for $g(n, k)$ uses the vertex Turán numbers $h(n, k)$ for cube graphs considered in [2] where $h(n, k)$ is the minimum number of vertices to be chosen from Q_n such that each subcube Q_k contains at least one of the chosen vertices.

Lemma 3. $g(n, k) \leq \left\lfloor \dfrac{2^n}{h(n, k)} \right\rfloor$.

Proof. In each of the $g(n,k)$ colors there have to occur at least $h(n,k)$ of the 2^n vertices so that $g(n,k)h(n,k) \leq 2^n$. □

For $k = 2$ and 3 the values of $g(n,k)$ are determined completely.

Theorem 2. $g(n,2) = \begin{cases} 4 & \text{for } n = 2,3, \\ 3 & \text{for } n \geq 4. \end{cases}$

Proof. For $n = 2,3$ Theorem 1 can be used. With $h(4,2) = 5$ from [2] and with Lemma 3 it follows $g(4,2) \leq 3$. Then Lemmas 1 and 2 complete the proof of $g(n,2) = 3$ for $n \geq 4$. □

Theorem 3. $g(n,3) = \begin{cases} 8 & \text{for } n = 3,4, \\ 5 & \text{for } n = 5,6,7, \\ 4 & \text{for } n \geq 8. \end{cases}$

Proof. For $n = 3,4$ Theorem 1 can be used. With $h(5,3) = 6$ and with $h(8,3) = 52$ from [2] it follows $g(5,3) \leq 5$ and $g(8,3) \leq 4$, respectively, using Lemma 3. The values $g(n,3)$ for $n \geq 5$ follow by Lemmas 1 and 2 if $g(7,3) \geq 5$ which remains to be proved. This proof is given by the following partition of all 2^7 vertices of Q_7 into 5 color classes. The color class 1 consists of the 24 vertices given by the 12 rows of the matrix in Figure 1 together with their complements. Then every Q_3 in Q_7 contains

0	0	0	0	0	0	0
0	0	0	0	0	0	1
0	0	0	1	1	1	0
0	0	1	0	1	1	0
0	0	1	1	0	1	1
0	0	1	1	1	0	1
0	1	0	0	1	1	1
0	1	0	1	0	1	0
0	1	0	1	1	0	1
0	1	1	0	0	1	1
0	1	1	0	1	0	0
0	1	1	1	0	0	0

Figure 1: Vertices of color 1.

0	0	0	0	1	0	0
0	0	0	0	1	0	1
0	0	0	1	0	0	0
0	0	0	1	1	1	1
0	0	1	0	0	1	0
0	0	1	0	1	1	1
0	0	1	1	0	0	1
0	0	1	1	0	1	0
0	1	0	0	0	0	0
0	1	0	0	0	1	1
0	1	0	1	0	1	1
0	1	0	1	1	1	0
0	1	1	0	0	0	1
0	1	1	0	1	1	0
0	1	1	1	1	0	0
0	1	1	1	1	0	1

Figure 2: Vertices of color 5.

one vertex of color 1 if in every quadruple of coordinates each of all 2^4

binary quadruples occurs once in a vertex. This is the case since in every quadruple of columns of the matrix in Figure 1 there occur 8 rows being pairwise different and non-complementary.

The vertices of color class 2 can be obtained by the application of the mapping $(a, b, c, d, e, f, g) \mapsto (a, b, 1-c, 1-d, 1-e, g, f)$ to the vertices of color class 1. Since the mapping permutes the coordinates and takes the complements of some coordinates only, the desired properties concerning the quadruples is preserved for color class 2 so that every Q_3 in Q_7 contains one vertex of color 2. Correspondingly, the application of the mapping $(a, b, c, d, e, f, g) \mapsto (a, 1-c, 1-b, 1-d, 1-f, 1-e, g)$ to color classes 1 and 2 yields the vertices of color classes 3 and 4, respectively. It has to be checked only that the color classes 1 to 4 are pairwise disjoint.

In color class 5 there are the remaining 32 vertices being represented by the 16 rows of the matrix in Figure 2 together with their complements. Then every Q_3 contains a vertex of color 5 which can be checked in the same way as for color 1. □

Theorem 4. $g(6,4) = 10$.

Proof. By Lemma 3 and $h(6,4) = 6$ from [2] it follows $g(6,4) \leq 10$.

The proof of $g(6,4) \geq 10$ is given by the following partition of all 2^6 vertices of Q_6 into 10 color classes. The vertices of color classes 1 and 2 are given by the 8 and 6 rows of the matrices in Figures 3 and 4, respectively. Iterated application of the mapping $(a, b, c, d, e, f) \mapsto (a, b, e, f, 1-c, 1-d)$

0	0	0	0	1	0
0	0	1	1	0	1
0	1	0	1	0	0
0	1	1	0	1	1
1	0	0	1	1	1
1	0	1	0	0	0
1	1	0	0	0	1
1	1	1	1	1	0

Figure 3: Color class 1.

0	0	0	0	0	0
0	0	0	1	1	1
0	1	1	0	0	1
1	0	1	0	1	0
1	1	0	1	0	0
1	1	1	1	1	1

Figure 4: Color class 2.

to the vertices of color class 2 successively produces the vertices of color classes 3, 4, and 5. Then the application of the mapping $(a, b, c, d, e, f) \mapsto (a, 1-b, c, d, e, f)$ to the vertices of color classes 1 to 5 yields the vertices of color classes 6 to 10.

It can be checked that no vertex of Q_6 occurs twice. To see that every Q_4 in Q_6 contains one vertex of every color, due to the mappings, it is sufficient to check that in both matrices of Figures 3 and 4 every pair of columns contains each of the 4 binary pairs once. □

Finally, it may be mentioned that the following question remains totally open: What are reasonable bounds of the smallest n for fixed k such that $g(n,k) = k+1$?

References

[1] N. Alon, A. Krech, and T. Szabó: *Turán's theorem in the hypercube.* SIAM J. Discrete Math. **21** (2007), 66–72.

[2] H. Harborth and H. Nienborg: *Some further vertex Turán numbers for cube graphs.* Utilitas Math. **75** (2008), 83–87.