A Proof of
The Modular Edge-Graceful Trees Conjecture

Ryan Jones, Kyle Kolasinski and Ping Zhang

Department of Mathematics
Western Michigan University
Kalamazoo, MI 49008-5248

In Memory of Professor Ralph Stanton (1923–2010)

ABSTRACT

Let G be a connected graph of order $n \geq 3$ and size m and let $f : E(G) \to \mathbb{Z}_n$ be an edge labeling of G. Define an induced vertex labeling $f' : V(G) \to \mathbb{Z}_n$ in terms of f by $f'(v) = \sum_{u \in N(v)} f(uv)$ where the sum is computed in $\mathbb{Z}_n$. If f' is one-to-one, then f is called a modular edge-graceful labeling and G is a modular edge-graceful graph. It is known that no connected graph of order $n \geq 3$ with $n \equiv 2 \pmod 4$ is modular edge-graceful. A 1991 conjecture states that every tree of order n where $n \not\equiv 2 \pmod 4$ is modular edge-graceful. In this work, we show that this conjecture is true and furthermore that a nontrivial connected graph of order n is modular edge-graceful if and only if $n \not\equiv 2 \pmod 4$. The modular edge-gracefulness meg(G) of a connected graph G order $n \geq 3$ is the smallest integer $k \geq n$ for which there exists an edge labeling $f : E(G) \to \mathbb{Z}_k$ such that the induced vertex labeling $f' : V(G) \to \mathbb{Z}_k$ is one-to-one. It is shown that meg$(G) = n + 1$ for every connected graph G that is not modular edge-graceful.

Key Words: modular edge-graceful graphs, modular edge-gracefulness.

AMS Subject Classification: 05C05, 05C78.

1 Introduction

Over the past few decades the subject of graph labelings has been growing in popularity. Gallian [4] has compiled a periodically updated survey of many kinds of labelings and numerous results, obtained from well over a thousand referenced research articles. The origin of the study of graph labelings as a major area of graph theory can be traced to a research paper

JCMCC **80** (2012), pp. 445-455

by Rosa [12]. Among the labelings he introduced was a vertex labeling he referred to as a β-valuation. Let G be a graph of order n and size m. A one-to-one function $f : V(G) \to \{0, 1, 2, \ldots, m\}$ is called a *β-valuation* (or a *β-labeling*) of G if

$$\{|f(u) - f(v)| : uv \in E(G)\} = \{1, 2, \ldots, m\}.$$

In order for a graph to possess a β-labeling, it is necessary that $m \geq n-1$. In 1972 Golomb [6] referred to a β-labeling as a *graceful labeling* and a graph possessing a graceful labeling as a *graceful graph*. Eventually, it was this terminology that became standard. While every connected graph G of order n and size m satisfies $m \geq n-1$, not every connected graph is graceful. Many graphs have been shown to be graceful, however, including all cycles C_n where $n \equiv 0 \pmod 4$ or $n \equiv 3 \pmod 4$ and all paths. In addition, all stars, all doubles stars and all caterpillars (trees the deletion of whose end-vertices produces a path) have been shown to be graceful. In fact, one of the best known conjectures in this area is due to Ringel and Kotzig.

The Graceful Tree Conjecture *Every tree is graceful.*

In 1985 Lo [9] introduced a dual type of labeling – this one an edge labeling. Let G be a connected graph of order $n \geq 2$ and size m. For a vertex v of G, let $N(v)$ denote the neighborhood of v (the set of vertices adjacent to v). An *edge-graceful labeling* of G is a bijective function $f : E(G) \to \{1, 2, \ldots, m\}$ that gives rise to a bijective function $f' : V(G) \to \{0, 1, 2, \ldots, n-1\}$ given by

$$f'(v) = \sum_{u \in N(v)} f(uv),$$

where the sum is computed in $\mathbb{Z}_n$. A graph that admits an edge-graceful labeling is called an *edge-graceful graph.* In the definition of an edge-graceful labeling of a connected graph G of order $n \geq 2$ and size m, the edge labeling f is required to be one-to-one. Since, however, the induced vertex labels $f'(v)$ are obtained by addition in $\mathbb{Z}_n$, the function f is actually a function from $E(G)$ to $\mathbb{Z}_n$ and is in general not one-to-one. Dividing m by n, we obtain

$$m = nq + r, \text{ where } q = \lfloor m/n \rfloor \text{ and } 0 \leq r \leq n-1.$$

Hence in an edge-graceful labeling of G, $q+1$ edges are labeled i for each i with $1 \leq i \leq r$ and q edges are labeled i for each i with $r+1 \leq i \leq n$ (in $\mathbb{Z}_n$). Thus this edge labeling $f : E(G) \to \mathbb{Z}_n$ is an one-to-one function only when $m = n-1$ or $m = n$.

In 2008 a vertex coloring of a graph was introduced in [10] in connection with finding a solution to a coin placement problem on a checkerboard. For a graph G without isolated vertices, let $c : V(G) \to \mathbb{Z}_k$ $(k \geq 2)$ be a vertex coloring of G where adjacent vertices may be colored the same. Then a vertex coloring c' of G is defined such that $c'(v)$ is the sum in $\mathbb{Z}_k$ of the colors of the vertices in the neighborhood of v for each $v \in V(G)$. The coloring c is called a *modular k-coloring* of G if $c'(u) \neq c'(v)$ in $\mathbb{Z}_k$ for every pair u, v of adjacent vertices of G. The *modular chromatic number* of G is the minimum k for which G has a modular k-coloring. This coloring was studied further in [11], which led to a complete solution of the checkerboard problem under investigation.

The modular coloring described above led to an edge version, introduced in [7]. For a graph G without isolated vertices, let $c : E(G) \to \mathbb{Z}_k$ $(k \geq 2)$ be an edge coloring of G where adjacent edges may be colored the same. Then a vertex coloring c' is defined such that $c'(v)$ is the sum in $\mathbb{Z}_k$ of the colors of the edges incident with v for each $v \in V(G)$. An edge coloring c is a *modular k-edge coloring* of G if $c'(u) \neq c'(v)$ in $\mathbb{Z}_k$ for all pairs u, v of adjacent vertices of G. The *modular chromatic index* of G is the minimum k for which G has a modular k-edge coloring.

Combining the concepts of graceful labeling and modular edge coloring gives rise to a modular edge-graceful labeling. Let G be a connected graph of order $n \geq 3$ and size m and let $f : E(G) \to \mathbb{Z}_n$, where f need not be one-to-one. Let $f' : V(G) \to \mathbb{Z}_n$ such that

$$f'(v) = \sum_{u \in N(v)} f(uv), \tag{1}$$

where the sum is computed in $\mathbb{Z}_n$. If f' is one-to-one, then f is called a *modular edge-graceful labeling* and G is a *modular edge-graceful graph.* Consequently, every edge-graceful graph is a modular edge-graceful graph. This concept was introduced independently in the 1991 by Jothi [5] under the terminology of *line-graceful graphs* (also see [4]). A necessary condition for a graph to be modular edge-graceful is known. We provide an independent proof of this result here for completeness.

Proposition 1.1 [4] *Let G be a connected graph of order $n \geq 3$. If G is modular edge-graceful, then $n \not\equiv 2 \pmod 4$.*

Proof. Suppose that there exists a modular edge-graceful graph of order n with $n \equiv 2 \pmod 4$ and let $f : E(G) \to \mathbb{Z}_n$ be a modular edge-graceful labeling of G. Let $f' : V(G) \to \mathbb{Z}_n$ be the induced vertex labeling. Hence $\{f'(v) : v \in V(G)\} = \mathbb{Z}_n$ and so $\sum_{v \in V(G)} f'(v) \equiv n/2 \pmod n$, where $n/2$ is odd since $n \equiv 2 \pmod 4$. On the other hand, observe that

$\sum_{v\in V(G)} f'(v) = 2\sum_{uv\in E(G)} f(uv)$, implying that $\sum_{v\in V(G)} f'(v)$ is even, a contradiction. ∎

As described in [4], a number of classes of graphs have been determined to be modular edge-graceful. In order to state these results, we present additional definitions. A vertex v in a graph is *odd* if $\deg v$ is odd while v is *even* if $\deg v$ is even. The *corona* $cor(G)$ of a graph G is that graph obtained from G by adding a new vertex v' to G for each vertex v of G and joining v' to v.

Theorem 1.2 [4] *The following graphs of order at least* 3 *are modular edge-graceful:*

(a) *all stars* $K_{1,n-1}$ *for which* $n \not\equiv 2 \pmod 4$,

(b) *all paths* P_n *for which* $n \not\equiv 2 \pmod 4$,

(c) *all cycles* C_n *for which* $n \not\equiv 2 \pmod 4$,

(d) *all trees of order* n *containing exactly one even vertex and for which* $n \not\equiv 2 \pmod 4$,

(e) *all* k*-ary trees for which* k *is even,*

(f) *all trees of order* $n \leq 9$ *and* $n \neq 6$,

(g) *all coronas* $cor(P_n)$ *of paths* P_n *for which* n *is even,*

(h) *all coronas* $cor(C_n)$ *of cycles* C_n *for which* n *is even.*

Modular edge-graceful graphs are studied extensively in [8]. In fact, each known result stated in Theorem 1.2, except for (d), is a consequence of the more general results obtained in [8]. In 1991, Jothi made the following conjecture (see [4]).

The Modular Edge-Graceful Tree Conjecture *If* T *is a tree of order* $n \geq 3$ *for which* $n \not\equiv 2 \pmod 4$, *then* T *is modular edge-graceful..*

In this work, we show that the Modular Edge-Graceful Tree Conjecture is true and a nontrivial connected graph of order n is modular edge-graceful if and only if $n \not\equiv 2 \pmod 4$. The *modular edge-gracefulness* $\text{meg}(G)$ of a graph G order $n \geq 3$ is the smallest integer $k \geq n$ for which there exists a labeling $f : E(G) \to \mathbb{Z}_k$ such that the induced vertex labeling $f' : V(G) \to \mathbb{Z}_k$ defined in (1) is one-to-one. We show that $\text{meg}(G) = n+1$ for every connected graph G that is not modular edge-graceful.

We refer to the books [2, 3] for any graph theory notation and terminology not described in this paper. Henceforth, we assume all graphs under consideration are connected graphs of order at least 3.

2 Modular Edge-Graceful Graphs Theorem

In this section, we show that a nontrivial connected graph of order n is modular edge-graceful if and only if $n \not\equiv 2 \pmod 4$. First, we present some preliminary results. Among the results obtained in [8] are the following results.

Theorem 2.1 [8] *A tree of order $n \geq 3$ having diameter at most 5 is modular edge-graceful if and only if $n \not\equiv 2 \pmod 4$.*

Proposition 2.2 [8] *If H is a modular edge-graceful connected graph, then every graph containing H as a spanning subgraph is also modular edge-graceful.*

We next present a result dealing with modular edge-graceful graphs that has the same flavor as the Bondy and Chvátal theorem on Hamiltonian graphs and closures (see [1]). First, we present a lemma.

Lemma 2.3 *Let G be a connected graph of order at least 3 containing two nonadjacent vertices u and v that are connected by a path of odd length. Then the graph $G + uv$ is modular edge-graceful if and only if G is modular edge-graceful.*

Proof. Since G is a connected spanning subgraph of $G+uv$, it then follows by Proposition 2.2 that if G is modular edge-graceful, then so is $G + uv$. For the converse, assume that $G + uv$ is modular edge-graceful and let $f : V(G+uv) \to \mathbb{Z}_n$ be a modular edge-graceful labeling of $G+uv$. Suppose that P is a $u - v$ path of odd length in G, say $P = (u = v_1, v_2, \ldots, v_p = v)$ where $p \geq 4$ is even. Now define the edge labeling $g : V(G) \to \mathbb{Z}_n$ of G by

$$g(e) = \begin{cases} f(e) & \text{if } e \notin E(P) \\ f(e) + f(uv) & \text{if } e = v_iv_{i+1},\ 1 \leq i \leq p-1 \text{ and } i \text{ is odd} \\ f(e) - f(uv) & \text{if } e = v_iv_{i+1},\ 2 \leq i \leq p-2 \text{ and } i \text{ is even.} \end{cases}$$

Since $g'(x) = f'(x)$ in $\mathbb{Z}_n$ for all $x \in V(G)$, it follows that g is a modular edge-graceful labeling of G. Thus G is modular edge-graceful. ∎

Let G be a connected graph of order at least 3 and let $\mathcal{P}$ be a partition of $V(G)$ into two or more independent sets. Define the *odd path closure of G with respect to $\mathcal{P}$*, denoted by $C_o(G, \mathcal{P})$ (or simply by $C_o(G)$ if the partition $\mathcal{P}$ under consideration is clear), to be the graph obtained from G by recursively joining pairs of nonadjacent vertices that belong to different independent sets in $\mathcal{P}$ and that are connected by a path of odd length in G. Repeated applications of Lemma 2.3 give us the following result on modular edge-graceful graphs and odd path closures.

Proposition 2.4 *Let G be a connected graph of order at least 3, let $\mathcal{P}$ be a partition of $V(G)$ into two or more independent sets, and let $C_o(G)$ be the odd path closure of G with respect to $\mathcal{P}$. Then $C_o(G)$ is modular edge-graceful if and only if G is modular edge-graceful.*

Of course, every nontrivial tree is a connected bipartite graph. We now show that the odd path closure of a connected bipartite graph of order at least 3 with respect to given partite sets is a complete bipartite graph.

Lemma 2.5 *Let G be a connected bipartite graph with partite sets U and W where $|U| = r$ and $|W| = s$ and $r + s \geq 3$. Then the odd path closure $C_o(G)$ of G with respect to the partition $\{U, W\}$ is $K_{r,s}$.*

Proof. First, observe that $C_o(G)$ is a bipartite graph with partite sets U and W. If $C_o(G) \neq K_{r,s}$, then there are vertices $u \in U$ and $w \in W$ such that $uw \notin E(C_o(G))$. Since $C_o(G)$ is bipartite,

$$\begin{aligned} U &= \{v \in V(C_o(G)) : \ d_{C_o(G)}(u, v) \text{ is even}\} \\ W &= \{v \in V(C_o(G)) : \ d_{C_o(G)}(u, v) \text{ is odd}\}. \end{aligned}$$

Since $w \in W$, it follows that $d_{C_o(G)}(u, w)$ is odd. Thus $uw \in E(C_o(G))$, which is a contradiction. ∎

For positive integers a and b, let $S_{a,b}$ be the double star of order $a + b$ whose central vertices have degrees a and b, respectively. By Theorem 2.1, every double star $S_{a,b}$ is modular edge-graceful if $a+b \not\equiv 2 \pmod 4$. We are now prepared to present the following modular edge-graceful trees theorem.

Theorem 2.6 *Let T be a tree of order $n \geq 3$. Then T is modular edge-graceful if and only if $n \not\equiv 2 \pmod 4$.*

Proof. We have seen that if $n \equiv 2 \pmod 4$, then T is not modular edge-graceful. For the converse, assume that $n \not\equiv 2 \pmod 4$. Let U and W be the partite sets of T with $|U| = r$ and $|W| = s$. By Lemma 2.5, the odd path closure $C_o(G)$ of G with respect to the partition $\{U, W\}$ is $K_{r,s}$. By Proposition 2.4, it suffices to show that $G = K_{r,s}$ is modular edge-graceful. If $r = 1$ or $s = 1$, then $K_{r,s}$ is a star and so it is modular edge-graceful by Theorem 2.1. If $r \geq 2$ and $s \geq 2$, then the double star $S_{r,s}$ of order $r + s$ is a modular edge-graceful spanning subgraph of $K_{r,s}$. It then follows by Proposition 2.2 that $K_{r,s}$ is modular edge-graceful. Therefore, T is modular edge-graceful by Proposition 2.4. ∎

The following is a consequence of Proposition 2.2 and Theorem 2.6,

Corollary 2.7 *Let G be a connected graph of order $n \geq 3$. Then G is a modular edge-graceful if and only if $n \not\equiv 2 \pmod 4$.*

3 Modular Edge-Gracefulness of Graphs

In this section we consider connected graphs that are not modular edge-graceful. For every connected graph G of order n, there is a smallest integer $k \geq n$ for which there exists an edge labeling $f : E(G) \to \mathbb{Z}_k$ such that the induced vertex labeling $f' : V(G) \to \mathbb{Z}_k$ defined by

$$f'(v) = \sum_{u \in N(v)} f(uv),$$

where the sum is computed in $\mathbb{Z}_n$, is one-to-one. This number k is referred to as the *modular edge-gracefulness* meg(G) of G. Thus meg$(G) \geq n$ and meg$(G) = n$ if and only if G is a modular edge-graceful graph of order n. Thus, if G is not modular edge-graceful, then meg$(G) \geq n+1$. As in the case of the gracefulness of a graph, the modular edge-gracefulness of a graph G is a measure of how close G is to being modular edge-graceful. In this section, we show that meg$(G) = n+1$ for every connected graph G of order n that is not modular edge-graceful. By Corollary 2.7, if G is a nontrivial connected graph of order n that is not modular edge-graceful, then $n \equiv 2 \pmod 4$. Thus we show that meg$(G) = n+1$ for every connected graph G of order n with $n \equiv 2 \pmod 4$. We begin with two lemmas.

Lemma 3.1 *If H is a connected spanning subgraph of a graph G of order at least* 3*, then* meg$(G) \leq$ meg(H).

Proof. Suppose that meg$(H) = k$. Let $f_H : E(H) \to \mathbb{Z}_k$ be an edge labeling of H such that the induced vertex labeling $f'_H : V(H) \to \mathbb{Z}_k$ is one-to-one. Define an edge labeling $f_G : E(G) \to \mathbb{Z}_k$ by $f_G(e) = f_H(e)$ if $e \in E(H)$ and $f_G(e) = 0$ if $e \in E(G) - E(H)$. Since the induced vertex labeling $f'_G : V(G) \to \mathbb{Z}_k$ has the property that $f'_G(v) = f'_H(v)$ for all $v \in V(G)$, it follows that f'_G is one-to-one. Thus meg$(G) \leq k =$ meg(H). ∎

Lemma 3.2 *Let G be a connected graph of order at least* 3*, let $\mathcal{P}$ be a partition of $V(G)$ into two or more independent sets and let $C_o(G)$ be the odd path closure of G with respect to $\mathcal{P}$. Then* meg$(G) =$ meg$(C_o(G))$.

Proof. Since G is a connected spanning subgraph of a graph $C_o(G)$, then meg$(G) \leq$ meg$(C_o(G))$ by Lemma 3.1. On the other hand, an argument similar to the proof of Lemma 2.3 shows that meg$(C_o(G)) \leq$ meg(G) and so meg$(C_o(G)) =$ meg(G). ∎

In view of Lemmas 2.5, 3.1 and 3.2, we first determine the modular edge-gracefulness of a star or a double star.

Theorem 3.3 *If G is a star or a double star of order $n \geq 6$ with $n \equiv 2$ (mod 4), then* meg$(G) = n+1$.

Proof. First suppose that $G = K_{1,n-1}$ is a star with its central vertex v that is adjacent to $v_1, v_2, \ldots, v_{n-1}$. Define a labeling $f : E(G) \to \mathbb{Z}_{n+1}$ by

$$f(vv_i) = \begin{cases} 0 & \text{if } i = 1 \\ -\frac{i}{2} & \text{if } i \text{ is even and } 2 \le i \le n-1 \\ \frac{i+1}{2} & \text{if } i \text{ is odd and } 3 \le i \le n-3 \\ \frac{n+2}{2} & \text{if } i = n-1. \end{cases}$$

Thus $\{f(vv_i) : 1 \le i \le n-1\} = \left\{0, -1, \pm 2, \pm 3, \ldots, \pm\frac{n-2}{2}, \frac{n+2}{2}\right\}$. Since

$$f'(v) = \frac{n}{2}$$

$$f'(v_i) = \begin{cases} 0 & \text{if } i = 1 \\ -\frac{i}{2} & \text{if } i \text{ is even and } 2 \le i \le n-1 \\ \frac{i+1}{2} & \text{if } i \text{ is odd and } 3 \le i \le n-3 \\ \frac{n+2}{2} & \text{if } i = n-1, \end{cases}$$

it follows that $f' : V(G) \to \mathbb{Z}_{n+1}$ is one-to-one and so f is a modular edge-graceful labeling. Therefore, G is modular edge-graceful.

Next, suppose that G is a double star with central vertices u and v where u is adjacent to $u_1, u_2, \ldots, u_r$ and v is adjacent to $v_1, v_2, \ldots, v_s$. Thus $n = r + s + 2$ and so $r + s \equiv 0 \pmod 4$. We consider two cases.

Case 1. *Either* $r \equiv 0 \pmod 4$ *and* $s \equiv 0 \pmod 4$ *or* $r \equiv 2 \pmod 4$ *and* $s \equiv 2 \pmod 4$. Define an edge labeling $f : E(G) \to \mathbb{Z}_{n+1}$ by

$$f(uu_i) = \begin{cases} 0 & \text{if } i = 1 \\ 1 & \text{if } i = 2 \\ \frac{i+1}{2} & \text{if } i \text{ is odd and } 3 \le i \le r-1 \\ -\frac{i}{2} & \text{if } i \text{ is even and } 4 \le i \le r \end{cases}$$

$$f(vv_i) = \begin{cases} \frac{r+i+1}{2} & \text{if } i \text{ is odd and } 1 \le i \le s-1 \\ -\frac{r+i}{2} & \text{if } i \text{ is even and } 2 \le i \le s \end{cases}$$

$$f(uv) = \frac{r+s+2}{2}.$$

Observe that

$$\{f(uu_i) : 1 \le i \le r\} = \left\{0, 1, \pm 2, \pm 3, \ldots, \pm\frac{r}{2}\right\}$$

$$\{f(vv_i) : 1 \le i \le s\} = \left\{\pm\frac{r+2}{2}, \pm\frac{r+4}{2}, \ldots, \pm\frac{r+s}{2}\right\}$$

Hence $\{f'(x) : x \in V(G)\} = \{0, 1, \pm 2, \pm 3, \ldots, \pm\frac{r+s}{2}, \frac{r+s}{2}+1, \frac{r+s}{2}+2\}$. Thus the induced vertex labeling $f' : V(G) \to \mathbb{Z}_{n+1}$ is one-to-one.

Case 2. *Either* $r \equiv 1 \pmod 4$ *and* $s \equiv 3 \pmod 4$ *or* $r \equiv 3 \pmod 4$ *and* $s \equiv 1 \pmod 4$*, say the former; that is, we assume that* $r \equiv 1 \pmod 4$ *and* $s \equiv 3 \pmod 4$. Then $r \geq 1$ and $s \geq 3$. We consider two subcases, according to whether $r = 1$ or $r \geq 5$.

Subcase 2.1. $r = 1$. Define an edge labeling $f : E(G) \to \mathbb{Z}_{n+1}$ by

$$\begin{aligned} f(uu_1) &= \frac{1+s}{2} \\ f(vv_i) &= \begin{cases} \frac{1+i}{2} & \text{if } i \text{ is odd and } 1 \leq i \leq s-2 \\ -\frac{i}{2} & \text{if } i \text{ is even and } 2 \leq i \leq s-1 \\ \frac{1+s}{2}+1 & \text{if } i = s \end{cases} \\ f(uv) &= 2. \end{aligned}$$

Figure 1 shows the edge labeling f in each case when $s = 3$ and $s = 7$. Observe that $\{f'(x) : x \in V(G)\} = \{\pm 1, \pm 2, \ldots, \pm\frac{s-1}{2}, \frac{s+1}{2}, \frac{s+1}{2}+1\}$. Thus the induced vertex labeling $f' : V(G) \to \mathbb{Z}_{n+1}$ is one-to-one.

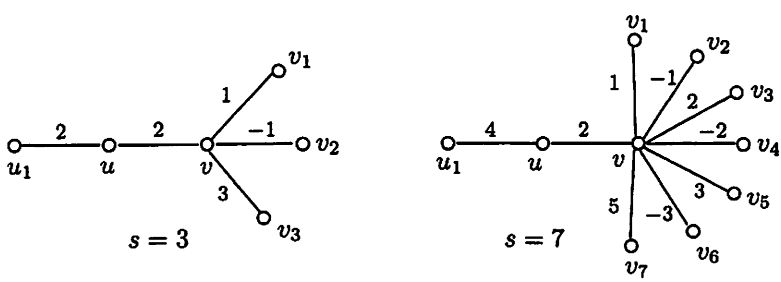

Figure 1: The labelings in Subcase 2.1 for $s = 3$ and $s = 7$

Subcase 2.2. $r \geq 5$. Define an edge labeling $f : E(G) \to \mathbb{Z}_{n+1}$ by

$$f(uu_i) = \begin{cases} \frac{i+1}{2} & \text{if } i \text{ is odd and } 1 \leq i \leq r-2 \\ -\frac{i}{2} & \text{if } i \text{ is even and } 2 \leq i \leq r-1 \\ \frac{r+s}{2} & \text{if } i = r \end{cases}$$

$$f(vv_i) = \begin{cases} \frac{r+i}{2} & \text{if } i \text{ is odd and } 1 \leq i \leq s-2 \\ -\frac{r+i-1}{2} & \text{if } i \text{ is even and } 2 \leq i \leq s-1 \\ \frac{r+s+2}{2} & \text{if } i = s \end{cases}$$

$$f(uv) = 2.$$

Observe that

$$\{f(uu_i) : 1 \leq i \leq r\} = \left\{\pm 1, \pm 2, \ldots, \pm\frac{r-1}{2}, \frac{r+s}{2}\right\}$$

$$\{f(vv_i) : 1 \leq i \leq s\} = \left\{\pm\frac{r+1}{2}, \pm\frac{r+3}{2}, \ldots, \pm\frac{r+s-2}{2}, \frac{r+s+2}{2}\right\}$$

Hence $\{f'(x) : x \in V(G)\} = \{\pm 1, \pm 2, \ldots, \pm\frac{r+s-2}{2}, \frac{r+s}{2}, \frac{r+s}{2} + 1\}$. Thus the induced vertex labeling $f' : V(G) \to \mathbb{Z}_{n+1}$ is one-to-one.

In each case, f is a modular edge-graceful labeling of G and so G is modular edge-graceful. ■

We are now prepared to show that $\text{meg}(T) = n+1$ for every tree T that is not modular edge-graceful.

Theorem 3.4 *If T is a tree of order $n \geq 6$ with $n \equiv 2 \pmod 4$, then* $\text{meg}(T) = n+1$.

Proof. Suppose that the partite sets of T are U and W with $|U| = r$ and $|W| = s$. Then $n = r+s \equiv 2 \pmod 4$. By Lemma 2.5, the odd path closure $C_o(G)$ of G with respect to the partition $\{U, W\}$ is $K_{r,s}$. If $r = 1$ or $s = 1$, then $\text{meg}(K_{r,s}) = n+1$ by Theorem 3.3. Thus we may assume that $r \geq 2$ and $s \geq 2$. Then the double star $S_{r,s}$ is a spanning subgraph of $K_{r,s}$. Since $K_{r,s}$ is not modular edge-graceful, $\text{meg}(K_{r,s}) \geq n+1$. On the other hand, $\text{meg}(S_{r,s}) = n+1$ by Theorem 3.3. It then follows by Lemma 3.1 that $\text{meg}(K_{r,s}) \leq \text{meg}(S_{r,s}) = n+1$ and so $\text{meg}(K_{r,s}) = n+1$. Therefore, $\text{meg}(T) = n+1$ by Lemma 3.2. ■

As a consequence of Lemma 3.2 and Theorem 3.4, we have the following.

Corollary 3.5 *If G is a nontrivial connected graph of order $n \geq 6$ with $n \equiv 2 \pmod 4$, then* $\text{meg}(G) = n+1$.

4 Acknowledgments

We are grateful to the referee whose valuable suggestions resulted in an improved paper.

References

[1] J. A. Bondy and V. Chvátal, A method in graph theory. *Discrete Math.* **15** (1976) 111-136.

[2] G. Chartrand, L. Lesniak and P. Zhang, *Graphs & Digraphs, Fifth Edition*, Chapman & Hall/CRC, Boca Raton, FL (2010).

[3] G. Chartrand and P. Zhang, *Chromatic Graph Theory.* Chapman & Hall/CRC Press, Boca Raton (2009).

[4] J. A. Gallian, A dynamic survey of graph labeling. *Electron. J. Combin.* **16** (2009) #DS6.

[5] R. B. Gnana Jothi, *Topics in Graph Theory*, Ph.D. Thesis, Madurai Kamaraj University (1991).

[6] S. W. Golomb, How to number a graph, in *Graph Theory and Computing.* Academic Press, New York (1972) 23-37.

[7] R. Jones, K. Kolasinski, F. Okamoto and P. Zhang, Modular neighbor-distinguishing edge colorings of graphs. *J. Combin. Math. Combin. Comput.* To appear.

[8] R. Jones, K. Kolasinski, F. Okamoto and P. Zhang, On modular edge-graceful graphs, Preprint.

[9] S. P. Lo, On edge-graceful labelings of graphs. *Congr. Numer.* **50** (1985) 231-241.

[10] F. Okamoto, E. Salehi and P. Zhang, A checkerboard problem and modular colorings of graphs. *Bull. Inst. Combin Appl.* **58** (2010) 29-47.

[11] F. Okamoto, E. Salehi and P. Zhang, A solution to the checkerboard problem. *Intern. J. Comput. Appl. Math.* **5** (2010) 447-458.

[12] A. Rosa, On certain valuations of the vertices of a graph, in *Theory of Graphs, Pro. Internat. Sympos. Rome 1966.* Gordon and Breach, New York (1967) 349-355.