Complete sets of metamorphoses: paired stars into 4-cycles

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In memory of Ralph G. Stanton

Abstract

If an edge-disjoint decomposition of a complete graph of order n into copies of a 3-star (i.e., the graph $K_{1,3}$ on 4 vertices) is taken, and if these 3-stars can be paired up in three distinct ways to form a graph on 6 vertices consisting of a 4-cycle with two opposite pendant edges, such that: (1) in each of the three pairings, there exists a metamorphosis into a 4-cycle system; (2) taking precisely those 4-cycles formed from the two pendant edges from each pair of 3-stars, in each of the three metamorphoses, we again have a 4-cycle system of the complete graph, then this is called a complete set of metamorphoses from paired 3-stars into 4-cycles.

We show that such a complete set of metamorphoses from paired 3-stars into 4-cycles exists if and only if the order of the complete graph is 1 or 9 (mod 24), and greater than 9.

JCMCC **80** (2012), pp. 457-466

1 Introduction

A lot of work has been done on so-called *metamorphoses* problems, so let us begin with some definitions to explain such things.

First, a G-design of order n is an edge-disjoint decomposition of a complete graph K_n into copies of a graph G. We shall also use certain edge-disjoint decompositions of graphs other than complete graphs, such as some bipartite graphs. A λ-fold G-design of order n is an edge-disjoint decomposition of the graph λK_n (which has each pair of vertices joined by precisely λ edges), into copies of the graph G. The copies of G in the edge-disjoint decomposition are frequently called *blocks*, using design-theoretic terminology. We also use $E(G)$ to denote the edges of the graph G.

Next, suppose that H is a (usually connected) proper subgraph of G, and suppose we have a G-design of order n. If we can take each copy of G (each block), retain a copy of the subgraph H from each block, and rearrange all the edges in the collection $\{E(G) \setminus E(H)\}$ for all blocks G in the design, into further copies of H, then this is called a *metamorphosis* of a G-design of order n into an H-design of the same order n.

Metamorphoses problems have been considered in many papers; see for instance [1, 3, 4, 5, 6, 7, 8, 9, 11, 12].

Very recently two papers [2, 10] have dealt with "complete sets" of metamorphoses. These require the graph G, having a metamorphosis into a graph H, to consist of some pair of smaller graphs, and then an additional property which we describe below. We warn the reader that the terminology "complete" is used in a different sense from that in [11], where "complete" means that *simultaneous* metamorphoses are found into all possible subgroups of the kite graph G under consideration in that paper.

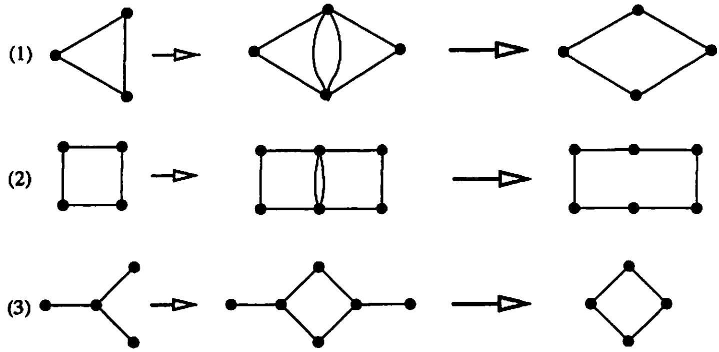

Figure 1. Three "complete" metamorphoses problems.

In Figure 1, case (1) indicates how pairs of triangles from a twofold triple system have been used to obtain a metamorphosis into 4-cycles ([6]; also [10] for complete set); case (2) indicates how the 4-cycles in a twofold 4-cycle system have been paired to give a metamorphosis into a twofold 6-cycle system ([12]; also [2] for complete set); and in case (3), which we consider here, we pair 3-stars together to yield a 4-cycle system of the same order, and we find a "complete" set of metamorphoses. So we now describe what is meant by a complete set of metamorphoses, with reference to the three problems illustrated in Figure 1.

In [10], (see case (1) in Fig. 1) *three* metamorphoses were found, say (A), (B) and (C), using the same *fixed* twofold triple system each time, but paired in different ways, in such a way that the removed double edges from the three metamorphoses (A), (B) and (C) together covered $2K_n$ and so formed a single twofold 4-cycle system. In [2], (see case (2) in Fig. 1) *four* metamorphoses were found, using a fixed twofold 4-cycle system, by pairing the 4-cycles in four different ways, so that the collection of 6-cycles formed from the removed double edges in each of the four metamorphoses together formed a twofold 6-cycle system.

Here we deal with case (3) of Fig. 1. For relevant orders n (dealt with below), we take a fixed 3-star decomposition of K_n; we pair up these 3-stars as indicated in Fig. 1 (3), and take *three* such pairings and metamorphoses, so that the collection of 4-cycles formed from the pendant edges which are removed from the paired 3-stars, for the three metamorphoses, precisely form a 4-cycle system of order n.

We now consider the necessary requirements on the order n for a complete set of metamorphoses of paired 3-stars into 4-cycles to exist. The total number of 3-stars must be divisible by 4 in order to have both a pairing of the 3-stars and then an even number of these pairs, so that the pendant edges number 0 (mod 4); thus these pendant edges have the potential to be rearranged into 4-cycles. So we require the order n to satisfy $n(n-1) \equiv 0 \pmod{24}$. But for a 4-cycle system of order n to exist, we also require $n(n-1) \equiv 0 \pmod{8}$ and n odd. Therefore a necessary requirement for our case (3) is that $n \equiv 1$ or $9 \pmod{24}$.

Although a metamorphosis of order 9 from paired 3-stars into a 4-cycle system exists, exhaustive computer searches show that it is not possible to find *one* 3-star decomposition of K_9 which has *three* pairings giving three metamorphoses into 4-cycle systems with the 4-cycles from the pendant edges themselves forming a 4-cycle system of K_9. In other words, there is no *complete set* of metamorphoses from a paired 3-star decomposition of order 9 into a 4-cycle system of order 9.

So in what follows we construct a complete set of three metamorphoses, using one fixed 3-star decomposition of order n, from paired 3-stars into 4-cycle systems of order n, for all orders $n \equiv 1$ or 9 (mod 24), $n > 9$.

Henceforth we shall denote the 3-star with vertex set $\{a, b, c, d\}$ and edge set
$\{\{a, b\}, \{a, c\}, \{a, d\}\}$ by $\langle a : b, c, d\rangle$. Moreover, the 4-cycle with vertex set $\{a, b, c, d\}$ and edge set $\{\{a, b\}, \{b, c\}, \{c, d\}, \{a, d\}\}$ will be denoted by (a, b, c, d) (or any cyclic shift of this).

In Section 2 we give some necessary examples; Section 3 gives the constructions for 1 and 9 (mod 24); we then conclude in Section 4 with the main result and further questions.

2 Some necessary examples

We need the following four examples for the general construction.

Example 2.1 A complete set of metamorphoses of K_{25}.

We take the vertex set of K_{25} to be $\mathbb{Z}_{25}$. This example is illustrated in Figure 2. The following four starter stars (mod 25) form a 3-star system of order 25; this will be our fixed 3-star system.

$$\langle 0 : 1, 2, 4\rangle, \quad \langle 0 : 3, 5, 6\rangle, \quad \langle 0 : 7, 8, 10\rangle, \quad \langle 0 : 9, 11, 12\rangle.$$

Using this 3-star system, we form three pairings, (A), (B) and (C), and for each, we give a metamorphosis into a 4-cycle system, as follows.

Metamorphosis (A):
Pair $\langle 0 : 1, 2, 4\rangle$ with $\langle 18 : 2, 4, 5\rangle$; also pair $\langle 0 : 3, 5, 6\rangle$ with $\langle 20 : 3, 5, 2\rangle$ (mod 25). The pendant edges here are $\{0, 1\}, \{18, 5\}, \{0, 6\}, \{20, 2\}$, using differences 1, 12, 6, 7. These form a starter 4-cycle $(0, 1, 13, 6)$, giving the metamorphosis with the remaining 4-cycles from the paired 3-stars, $(0, 2, 18, 4)$ and $(0, 3, 20, 5)$ (mod 25).

Metamorphosis (B):
Pair $\langle 0 : 1, 2, 4\rangle$ with $\langle 23 : 1, 4, 3\rangle$; also pair $\langle 0 : 8, 7, 10\rangle$ with $\langle 23 : 7, 10, 9\rangle$ (mod 25). The pendant edges here are $\{0, 2\}, \{23, 3\}, \{0, 8\}, \{23, 9\}$, using differences 2, 5, 8, 11. These form a starter 4-cycle $(0, 11, 3, 5)$, giving the metamorphosis with the remaining 4-cycles from the paired 3-stars, $(0, 1, 23, 4)$ and $(0, 7, 23, 10)$ (mod 25).

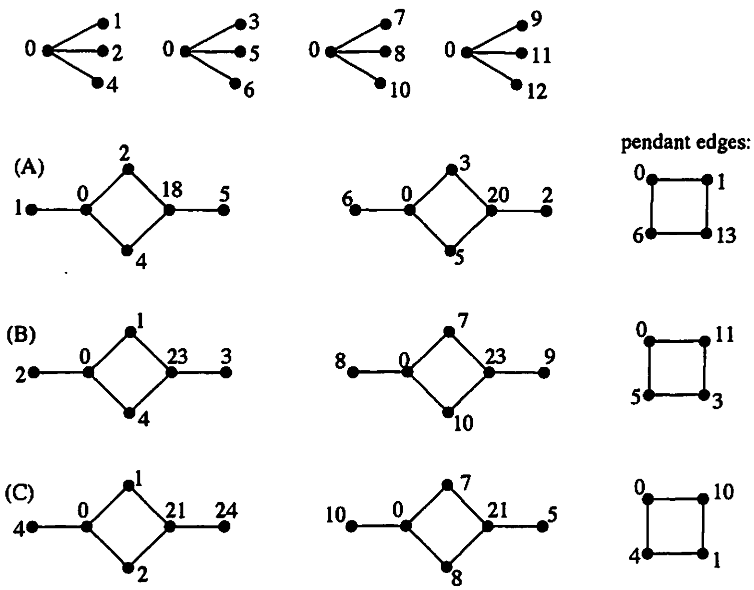

Figure 2. A complete set of metamorphoses of order 25, working mod 25.

Metamorphosis (C):
Pair $\langle 0:4,1,2\rangle$ with $\langle 21:1,2,24\rangle$; also pair $\langle 0:10,7,8\rangle$ with $\langle 21:7,8,5\rangle$ (mod 25). The pendant edges here are $\{0,4\},\{21,24\},\{0,10\},\{21,5\}$, using differences 3, 4, 9, 10. These form a starter 4-cycle $(0,10,1,4)$, giving the metamorphosis with the remaining 4-cycles from the paired 3-stars, $(0,1,21,2)$ and $(0,7,21,8)$ (mod 25).

Note that the pendant edges themselves, from all three metamorphoses, form a 4-cycle system of order 25. □

Example 2.2 A complete set of metamorphoses of K_{33}.

Let the vertex set of K_{33} be $\{0_1,1_1,\ldots,10_1\}\cup\{0_2,1_2,\ldots,10_2\}\cup\{0_3,1_3,\ldots,10_3\}$. Working modulo 11, we have sixteen starter 3-stars:

$$\begin{array}{llll}
\langle 0_3:0_1,9_2,1_3\rangle, & \langle 9_1:7_1,9_2,1_3\rangle, & \langle 2_2:0_1,3_1,7_2\rangle, & \langle 4_3:10_1,3_1,7_2\rangle,\\
\langle 10_1:0_1,2_2,7_3\rangle, & \langle 3_2:8_1,2_2,7_3\rangle, & \langle 2_2:1_1,6_2,5_3\rangle, & \langle 4_2:10_1,6_2,5_3\rangle,\\
\langle 7_3:0_1,8_1,0_2\rangle, & \langle 4_1:1_1,8_1,0_2\rangle, & \langle 6_1:0_1,10_2,1_3\rangle, & \langle 4_3:2_1,10_2,1_3\rangle,\\
\langle 8_2:0_1,0_2,8_3\rangle, & \langle 6_3:2_1,0_2,8_3\rangle, & \langle 9_3:0_1,4_3,5_3\rangle, & \langle 6_2:8_1,4_3,5_3\rangle.
\end{array}$$

These are paired in three different ways, for three metamorphoses, as follows:

Metamorphosis (A):

Take the above sixteen starters and pair them with respect to the last pair of vertices in each 3-star. The sixteen pendant edges form four starter 4-cycles (mod 11) as follows:

$$(0_1,1_2,3_1,5_2),(0_1,6_2,9_1,0_3),(0_1,5_3,1_1,2_1),(0_1,7_3,9_1,3_1).$$

Metamorphosis (B):
Take the sixteen starters below and pair them with respect to the last pair of vertices in each 3-star.

$\langle 2_3:3_3,2_1,0_2\rangle$,	$\langle 6_3:8_3,2_1,0_2\rangle$,	$\langle 2_1:2_2,0_1,5_3\rangle$,	$\langle 9_3:4_3,0_1,5_3\rangle$,
$\langle 2_2:3_1,0_1,7_2\rangle$,	$\langle 5_2:6_3,0_1,7_2\rangle$,	$\langle 7_3:6_1,2_1,10_2\rangle$,	$\langle 4_3:1_3,2_1,10_2\rangle$,
$\langle 10_1:2_2,0_1,7_3\rangle$,	$\langle 9_2:8_3,0_1,7_3\rangle$,	$\langle 7_2:0_3,1_1,6_2\rangle$,	$\langle 2_2:5_3,1_1,6_2\rangle$,
$\langle 7_3:8_1,0_1,0_2\rangle$,	$\langle 8_2:8_3,0_1,0_2\rangle$,	$\langle 4_1:8_1,1_1,0_2\rangle$,	$\langle 7_1:2_3,1_1,0_2\rangle$.

The sixteen pendant edges form four starter 4-cycles (mod 11) as follows:

$$(0_1,4_1,3_2,6_3),(1_1,1_2,5_3,0_3),(0_1,3_2,4_3,1_3),(0_2,0_3,9_3,10_3).$$

Metamorphosis (C):
Take the sixteen starters below and pair them with respect to the last pair of vertices in each 3-star.

$\langle 0_3:9_2,0_1,1_3\rangle$,	$\langle 6_1:10_2,0_1,1_3\rangle$,	$\langle 2_1:5_3,0_1,2_2\rangle$,	$\langle 10_1:7_3,0_1,2_2\rangle$,
$\langle 2_2:7_2,0_1,3_1\rangle$,	$\langle 10_3:3_2,0_1,3_1\rangle$,	$\langle 4_3:7_2,10_1,3_1\rangle$,	$\langle 6_1:2_2,10_1,3_1\rangle$,
$\langle 8_2:7_2,2_1,1_3\rangle$,	$\langle 4_3:10_2,2_1,1_3\rangle$,	$\langle 2_2:6_2,1_1,5_3\rangle$,	$\langle 10_3:6_3,1_1,5_3\rangle$,
$\langle 7_2:9_2,2_1,8_3\rangle$,	$\langle 6_3:0_2,2_1,8_3\rangle$,	$\langle 8_2:0_2,0_1,8_3\rangle$,	$\langle 9_2:7_3,0_1,8_3\rangle$.

The sixteen pendant edges form four starter 4-cycles (mod 11) as follows:

$$(0_1,4_2,0_2,8_3),(0_1,7_2,8_2,3_3),(0_2,2_2,0_3,7_3),(0_2,3_2,8_3,6_2).$$

□

Example 2.3 A complete set of metamorphoses of $K_{12,12}$.

We take the vertex set of $K_{12,12}$ to be $\{0_1,1_1,\dots,11_1\}\cup\{0_2,1_2,\dots,11_2\}$. Working modulo 12, we have four starter 3-stars:

$$\langle 0_1:0_2,1_2,3_2\rangle,\ \langle 0_1:2_2,4_2,5_2\rangle,\ \langle 0_1:6_2,7_2,9_2\rangle,\ \langle 0_1:8_2,10_2,11_2\rangle.$$

These are paired in three different ways, for three metamorphoses, as follows:

Metamorphosis (A):
Pair $\langle 0_1 : 1_2, 3_2, 0_2 \rangle$ with $\langle 5_1 : 1_2, 3_2, 4_2 \rangle$; and pair $\langle 0_1 : 2_2, 4_2, 5_2 \rangle$ with $\langle 7_1 : 2_2, 4_2, 1_2 \rangle$. The four pendant edges form a starter 4-cycle (mod 12) as follows: $(0_1, 0_2, 6_1, 11_2)$.

Metamorphosis (B):
Pair $\langle 0_1 : 0_2, 3_2, 1_2 \rangle$ with $\langle 10_1 : 0_2, 3_2, 2_2 \rangle$; and pair $\langle 0_1 : 6_2, 9_2, 7_2 \rangle$ with $\langle 10_1 : 6_2, 9_2, 8_2 \rangle$. The four pendant edges form a starter 4-cycle (mod 12) as follows: $(0_1, 1_2, 6_1, 4_2)$.

Metamorphosis (C):
Pair $\langle 0_1 : 0_2, 1_2, 3_2 \rangle$ with $\langle 8_1 : 0_2, 1_2, 10_2 \rangle$; and pair $\langle 0_1 : 6_2, 7_2, 9_2 \rangle$ with $\langle 8_1 : 6_2, 7_2, 4_2 \rangle$. The four pendant edges form a starter 4-cycle (mod 12) as follows: $(0_1, 3_2, 6_1, 2_2)$.

Note that the 4-cycles from the pendant edges from all three metamorphoses themselves form a 4-cycle decomposition of $K_{12,12}$. □

Example 2.4 A complete set of metamorphoses of $K_{8,12}$.

Let the vertex set of $K_{8,12}$ be $\{i_j \mid 0 \leqslant i \leqslant 3,\ j = 1,2\} \cup \{i_j \mid 0 \leqslant i \leqslant 3,\ j = 3,4,5\}$. Working modulo 4, we have eight starter 3-stars:

$$\langle 1_2 : 0_3, 0_5, 1_4 \rangle,\quad \langle 2_2 : 0_4, 0_5, 1_4 \rangle,\quad \langle 0_2 : 0_3, 1_3, 0_5 \rangle,\quad \langle 3_2 : 0_4, 1_3, 0_5 \rangle,$$
$$\langle 0_1 : 1_3, 3_5, 2_4 \rangle,\quad \langle 1_1 : 1_4, 3_5, 2_4 \rangle,\quad \langle 3_1 : 1_3, 2_3, 3_5 \rangle,\quad \langle 2_1 : 1_4, 2_3, 3_5 \rangle.$$

These are paired in three different ways, for three metamorphoses, as follows:

Metamorphosis (A):
Take the sixteen starters above and pair them with respect to the last pair of vertices in each 3-star. The eight pendant edges form the following two starter 4-cycles (mod 4).

$$(0_1, 1_3, 3_1, 3_4),\ (0_2, 3_3, 3_2, 1_4).$$

Metamorphosis (B):
Take the sixteen starters below and pair them with respect to the last pair of vertices in each 3-star.

$$\langle 1_2 : 1_4, 0_3, 0_5 \rangle,\quad \langle 0_2 : 1_3, 0_3, 0_5 \rangle,\quad \langle 2_2 : 1_4, 0_4, 0_5 \rangle,\quad \langle 3_2 : 1_3, 0_4, 0_5 \rangle,$$
$$\langle 0_1 : 2_4, 1_3, 3_5 \rangle,\quad \langle 3_1 : 2_3, 1_3, 3_5 \rangle,\quad \langle 1_1 : 2_4, 1_4, 3_5 \rangle,\quad \langle 2_1 : 2_3, 1_4, 3_5 \rangle.$$

The eight pendant edges form the following two starter 4-cycles (mod 4).

$$(0_1, 0_3, 1_1, 2_4), (0_2, 1_3, 3_2, 3_4).$$

Metamorphosis (C):
Take the sixteen starters below and pair them with respect to the last pair of vertices in each 3-star.

$$\begin{array}{llll} \langle 2_2 : 1_5, 1_3, 2_4 \rangle, & \langle 0_1 : 3_5, 1_3, 2_4 \rangle, & \langle 3_2 : 1_5, 1_4, 2_4 \rangle, & \langle 1_1 : 3_5, 1_4, 2_4 \rangle, \\ \langle 1_2 : 1_5, 1_3, 2_3 \rangle, & \langle 3_1 : 3_5, 1_3, 2_3 \rangle, & \langle 0_2 : 1_5, 1_4, 2_3 \rangle, & \langle 2_1 : 3_5, 1_4, 2_3 \rangle. \end{array}$$

The eight pendant edges form the following two starter 4-cycles (mod 4).

$$(0_1, 0_5, 1_1, 2_5), (0_2, 0_5, 2_2, 1_5).$$

□

3 The construction

3.1 The case of order 1 (mod 24)

Let the vertex set of K_{24x+1} be $\{\infty\} \cup \{(i,j) \mid 1 \leqslant i \leqslant 2x,\ 1 \leqslant j \leqslant 12\}$.

On each set $\{\infty\} \cup \{(2i-1,j),(2i,j) \mid 1 \leqslant j \leqslant 12\}$, for each i with $1 \leqslant i \leqslant x$, we place a copy of a complete set of metamorphoses of order 25 (see Example 2.1).

Then on each set $\{(i_1,j) \mid 1 \leqslant j \leqslant 12\} \cup \{(i_2,j) \mid 1 \leqslant j \leqslant 12\}$ (for all $4\binom{x}{2}$ pairs $i_1 \neq i_2$ with $\{i_1,i_2\} \neq \{2k-1,2k\}$ for any k) we place a complete set of metamorphoses of $K_{12,12}$ (see Example 2.3). This completes the construction for order 1 (mod 24).

3.2 The case of order 9 (mod 24)

Now let the vertex set of K_{24x+9} be

$$\{\infty\} \cup \{a_1, a_2, \ldots, a_8\} \cup \{(i,j) \mid 1 \leqslant i \leqslant 2x,\ 1 \leqslant j \leqslant 12\}.$$

On the set $\{\infty\} \cup \{a_1,a_2,\ldots,a_8\} \cup \{(i,j) \mid i = 1,2,\ 1 \leqslant j \leqslant 12\}$, we take a complete set of metamorphoses of K_{33} (see Example 2.2).

Then on each set $\{\infty\} \cup \{(2i-1,j),(2i,j) \mid 1 \leqslant j \leqslant 12\}$, for each i with $2 \leqslant i \leqslant x$, we place a copy of a complete set of metamorphoses of order 25 (see Example 2.1).

Next, we use copies of Example 2.3 for each set $\{(i_1, j) \mid 1 \leqslant j \leqslant 12\} \cup \{(i_2, j) \mid 1 \leqslant j \leqslant 12\}$ (for all $4\binom{x}{2}$ pairs $i_1 \neq i_2$ with $\{i_1, i_2\} \neq \{2k-1, 2k\}$ for any k).

Finally, we use copies of Example 2.4 on the vertex sets

$$\{a_1, a_2, \ldots, a_8\} \cup \{(i, j) \mid 1 \leqslant j \leqslant 12\}$$

for each i with $3 \leqslant i \leqslant 2x$.

This completes the case of order 9 (mod 24), greater than 9.

4 Concluding comments

We have now shown the following.

THEOREM 4.1 *There exists a 3-star decomposition of K_n such that there are three different pairings of the stars yielding metamorphoses into 4-cycles, with the 4-cycles from the pendant edges in all of these three pairings together forming a 4-cycle system of K_n, if and only if $n \equiv 1$ or 9* (mod 24), $n > 9$.

The reader may like to consider other ways of pairing two 3-stars to form a graph containing a 4-cycle subgraph. There are two such graphs on 5 vertices, and one (with index 2, that is, repeated edges allowed) having 4 vertices. The problem of finding a complete set of metamorphoses in these cases is currently being investigated.

Acknowledgements

The first author (E.J.B.) thanks both Auburn University and the third author (C.C.L.) for great hospitality during the work on this paper.

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