Extending matchings to 2-factors

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Dedicated to the memory of Ralph G. Stanton

Abstract

Let G be a finite 4-regular cyclically $2k$-edge-connected simple graph for some integer $k \geq 1$. Let $E(k)$ be a set of k independent edges in G and (E_1, E_2) be a partition of $E(k)$. We consider when there exists a 2-factor which excludes all edges of E_1 and includes all the edges of E_2. A complete characterization is provided.

1 Introduction

Matching extension has been widely studied since Plummer [2] first introduced the notion of m-extendability in 1980. For m a non-negative integer, a graph G with edge set $E(G)$ is said to be n-*extendable* if G has at least $2m + 2$ vertices and for each independent set of edges $M \subset E(G)$ with $|M| = m$ we can find a 1-factor F of G with $M \subset F$. Porteous and Aldred [3] generalized this concept by requiring that we can also specify edges for our 1-factors to avoid. Specifically, for non-negative integers m, n, a graph G with at least $2m + 2n + 2$ vertices is said to be $E(m, n)$ if for each pair of disjoint independent sets $M, N \subset E(G)$ with $|M| = m$ and $|N| = n$ we can find a 1-factor F such that $M \subset F$ and $N \cap F = \emptyset$. Many papers have subsequently investigated this property.

In this paper we consider the analogous problem of finding a 2-factor including various specified edges while avoiding other specified edges. In particular we will consider this problem for regular graphs. Of course, if G is a 2-regular graph, then G certainly admits a 2-factor but exactly one and there is no room to avoid any edges. If G is 3-regular, then removing a 1-factor from G leaves a 2-regular spanning subgraph of G, i.e. a 2-factor. Thus if G is 3-regular and $E(m, n)$, then G admits a 2-factor which avoids any matching M of size m whilst including any matching N disjoint from M and of size n. So in this sense the problem of selective 2-factors in 3-regular graphs has been widely studied. Here we will focus on 4-regular graphs. With a 4-regular graph we can always decompose the edge set into two disjoint 2-factors, F_1 and F_2, say. If we have disjoint matchings M, N in G and F_1 avoids all of the edges in M while including all of the edges in N, then F_2 includes all of M while avoiding all of N. This represents a duality of 2-factors in 4-regular graphs analogous to the 1-factor/2-factor duality in 3-regular graphs.

Consider the following 4-regular graph. Let H be a 4-regular bipartite graph with bipartition $V(H) = (X, Y)$ and let $D = \{x_1y_1, x_2y_2, \ldots, x_{2d}y_{2d}\}$ be a matching of size $2d$ in H. Form a new 4-regular graph H^* with $V(H^*) = V(H)$ and $E(H^*) = (E(H) \backslash D) \cup D_X \cup D_Y$, where $D_X = \{x_ix_{2d+1-i} : 1 \leq i \leq d\}$ and $D_Y = \{y_iy_{2d+1-i} : 1 \leq i \leq d\}$. Then H^* is 'nearly bipartite' apart from the edges in K_X and K_Y. Clearly, any 2-factor in H^* must use precisely as many edges from K_X as from K_Y and consequently we cannot find a 2-factor of H^* that avoids a specified set of q edges from D_X and includes a specified set of $d - q + 1$ edges from D_Y. A similar imbalance can be obtained by deleting a vertex from one side of the bipartition and adding two independent edges joining the resulting vertices of degree three to form a 4-regular graph. Clearly, any 2-factor must include precisely one of the new edges, so we cannot include both or avoid both of these edges. In what follows we show that, for cyclically $2k$-edge-connected 4-regular graphs, this 'near biparticity' is essentially all that prevents our desired 2-factors. To facilitate a precise statement of our result and the subsequent proof, we introduce some definitions, notation and a well-known result in the next section.

2 Preliminaries

Let G be a finite simple graph with edge and vertex sets $E(G)$ and $V(G)$ respectively. Suppose that $X \subseteq V(G)$. $G[X]$ denotes the induced subgraph of G with vertex set X. We write $E(X)$ and $V(X)$ rather than $E(G[X])$ and $V(G[X])$. Set $q(X) = |E(X)|$. If $X, Y \subseteq V(G)$, set $E(X,Y) = \{xy \in E(G) : x \in X, y \in Y\}$ and $q(x,y) = |E(X,Y)|$. We use the usual convention of writing $q_G(X)$, and so on, if we wish to emphasize the dependency on G.

Definition 2.1 *G is said to be k-*pseudo-bipartite *if there exists a partition (S,T) of $V(G)$ such that $q(S) + q(T) \leq 2(k-1)$.*

Definition 2.2 *Let $E(k)$ be a subset of k independent edges of G. Let (E_1, E_2) be a partition of $E(k)$ with $|E_1| = i$ and $|E_2| = k - i$. Then we say that (E_1, E_2) is a* potential $(i, k-i)$-factor of G *if there exists a 2-factor F of G such that $E_1 \cap E(F) = \emptyset$ and $E_2 \subseteq E(F)$.*

Our investigation will rely on the theory of f-factors and we present some of the pertinent details extracted from the treatment in [1] below.

Let $f : V(G) \longrightarrow \mathbb{N} \cup 0$ be a function from the vertex set of a graph G to the set of non-negative integers. An *f-factor* of G is a spanning subgraph F of G in which $\deg_F(v) = f(v)$ for each vertex $v \in V(G)$.

A *graph triple* $B = (S,T,U)$ for G is an ordered triple (S,T,U) such that $\{S,T,U\}$ is a partition of $V(G)$. A component C of $G[U]$ is said to be odd with respect to the function f if $f(V(C)) + q(V(C),T)$ is odd. For a graph triple $B = (S,T,U)$, we denote by $h(B)$ the number of odd components with respect to f in $G[U]$ and define the *deficiency* $\delta(B)$ of B with respect to f as follows.

$$\delta(B) = h(B) - f(S) + f(T) - \deg(T) + q(S,T).$$

A graph triple $B(S,T,U)$ is said to be an *f-barrier* if $\delta(B) > 0$ (by parity considerations this is equivalent to $\delta(B) \geq 2$).

Tutte's f-factor Theorem: *Let G be a graph and let $f : V(G) \longrightarrow \mathbb{N} \cup 0$ be a function from the vertex set of a graph G to the set of non-negative integers. Then G has either an f-factor or an f-barrier but not both.* □

3 The main result

Theorem 3.1 *Let G be a 4-regular and cyclically $2k$-edge-connected simple graph for some integer $k \geq 1$ and let $E(k)$ be a subset of k independent edges of G. If (E_1, E_2) is a partition of $E(k)$ with $|E_1| = i$ and $|E_2| = k - i$, then*

(E_1, E_2) is a potential $(i, k-i)$-factor of G unless G is k-pseudo-bipartite and there exists a partition (S,T) of $V(G)$ such that $|E_1 \cap E(T)| + |E_2 \cap E(S)| \geq (q(S) + q(T) + 2)/2$.

Proof. Let G, $E(k)$, E_1 and E_2 be as in the statement of the theorem and suppose that (E_1, E_2) is not a potential $(i, k-i)$-factor of G.

Let $G' = G - E(k)$ and define $f : V(G') \longrightarrow \{1, 2\}$ by

$$f(v) = \begin{cases} 1 & \text{if } v \text{ is incident with an edge of } E_2, \\ 2 & \text{otherwise}. \end{cases}$$

Thus (E_1, E_2) is a potential $(i, k-i)$-factor of G if and only if G' admits an f-factor.

By Tutte's f-factor theorem, our assumption that (E_1, E_2) is not a potential $(i, k-i)$-factor of G is equivalent to saying that G' has an f-barrier, that is to say, we can find a graph triple $B = (S, T, U)$ of $V(G')$ such that:

$$\delta(B) = h(B) - f(S) + f(T) - \deg_{G'}(T) + q_{G'}(S,T) \geq 2 \tag{1}$$

Among all f-barriers fix one $B = (S, T, U)$ such that $|U|$ is as small as possible. In this f-barrier every component in $G'[U]$ is cyclic. To see this we use the minimality of $|U|$ and note that for $x \in U$, $B' = (S \cup \{x\}, T, U \backslash \{x\})$ and $B'' = (S, T \cup \{x\}, U \backslash \{x\})$ cannot be f-barriers. Hence

$$\delta(B') = h(B') - f(S) - f(x) + f(T) - \deg_{G'}(T) + q_{G'}(S,T) + q_{G'}(x,T) \leq 0 \tag{2}$$

and

$$\begin{aligned}\delta(B'') = {} & h(B'') - f(S) + f(T) + f(x) - \deg_{G'}(T) - \deg_{G'}(x) + q_{G'}(S,T) \\ & + q_{G'}(x,S) \\ \leq {} & 0.\end{aligned} \tag{3}$$

By **(1)** and **(2)** we have

$$f(x) \geq 2 + q_{G'}(x,T) - h(B) + h(B'). \tag{4}$$

By **(1)** and **(3)** we have

$$f(x) \leq q_{G'}(x,T) + q_{G'}(x,U) - h(B'') + h(B) - 2. \tag{5}$$

Since $h(B') \geq h(B) - 1$ and $h(B'') \geq h(B) - 1$, we have

$$q_{G'}(x,T) + 1 \leq f(x) \leq q_{G'}(x,T) + q_{G'}(x,U) - 1$$

or

$$q_{G'}(x, U) \geq 2. \tag{6}$$

Thus, every vertex $x \in U$ has $\deg_{G'[U]}(x) \geq 2$ and every component of $G'[U]$ is cyclic.

To facilitate our investigation of the nature of our f-barrier $B = (S, T, U)$, we introduce the following notation. For a vertex set $X \subseteq V(G)$, we set $|X| = x$, $\theta_1(X) = |X \cap V(E_1)|$ and $\theta_2(X) = |X \cap V(E_2)|$.

We observe that, with these definitions

(*) $\theta_1(S) + \theta_1(T) + \theta_1(U) = 2i$; $\theta_2(S) + \theta_2(T) + \theta_2(U) = 2(k - i)$

(**) $|E_G(U, S \cup T) \cap E(k)| \leq \theta_1(U) + \theta_2(U)$.

Now $\deg_{G'}(T) = 4t - \theta_1(T) - \theta_2(T)$ and, since $B = (S, T, U)$ is an f-barrier,

$\delta(B) = h(B) - f(S) + f(T) - \deg_{G'}(T) + q_{G'}(S, T) \geq 2.$
So

$$\begin{aligned} h(B) &\geq 2 + f(S) - f(T) + 4t - \theta(T) - \theta_2(T) - q_{G'}(S, T) \\ &= 2 + (2s\theta_2(S)) - (2t - \theta_2(T)) + 4t - \theta_1(T) - \theta_2(T) - q_{G'}(S, T) \\ &= 2(s + t) - \theta_1(T) - \theta_2(S) - q_{G'}(S, T) + 2 \end{aligned}$$

i.e.

$$h(B) \geq 2(s + t) - \theta_1(T) - \theta_2(S) - q_{G'}(S, T) + 2. \tag{7}$$

Since each of component of $G'[U]$ is cyclic and G is cyclically $2k$-edge-connected

$q_{G'}(U, S \cup T) \geq 2kh(B) - \theta_1(U) - \theta_2(U)$

and, by definition

$$q_{G'}(U, S \cup T) \leq 4(s + t) - (\theta_1(S) + \theta_2(S) + \theta_1(T) + \theta_2(T)) - 2q_{G'}(S, T). \tag{8}$$

Hence $\theta_2(S) + \theta_1(T) + \theta_1(U) + \theta_2(U) \leq 2(k - 1)h(B) + 4$ **(9)**

By (*) and **(9)**, $2k \geq 2(k - 1)h(B) + 4$, i.e $k \geq (k - 1)h(B) + 2$ so $h(B) = 0$ and $k \geq 2$.

With $h(B) = 0$, **(7)** becomes

$\theta_2(S) + \theta_1(T) + q_{G'}(S, T) \geq 2(s + t) + 2.$

Combined with **(8)** this gives

$$q_{G'}(U, S\cup T) \le 4(s+t) + \theta_2(S) + \theta_1(T) - \theta_1(S) - \theta_2(T) - 2(\theta_2(S) + \theta_1(T) + q_{G'}(S,T))$$
$$\le \theta_2(S) + \theta_1(T) - \theta_1(S) - \theta_2(T) - 4. \quad \textbf{(10)}$$

Now

$$q_G(U, S\cup T) \le q_{G'}(U, S\cup T) + \theta_1(U) + \theta_2(U). \quad \textbf{(11)}$$

Combining **(10)** and **(11)** we get

$$q_G(U, S\cup T) \le \theta_2(S) + \theta_1(T) - \theta_1(S) - \theta_2(T) + \theta_1(U) + \theta_2(U) - 4$$
$$\le \theta_2(S) + \theta_1(T) + \theta_1(S) + \theta_2(T) + \theta_1(U) + \theta_2(U) - 4 = 2k - 4.$$

Since G is cyclically $2k$-edge-connected, this implies $G[U]$ is acyclic. But earlier we established that all components of $G'[U]$ are cyclic. From this we conclude that $U = \emptyset$.

Consequently (S,T) is a partition of $V(G)$. Moreover, when we consider the foregoing analysis in this light we see that

(i) $q_G(S) + q_G(T) \le 2k - 2$ i.e. G is k-pseudo-bipartite

(ii) $|E_1 \cap E_G(T)| + |E_2 \cap E_G(S)| \ge (q_G(S) + q_G(T) + 2)/2.$ □

References

[1] D. Holton and J. Sheehan, The Petersen Graph, *AMS Lecture Series*, CUP 1993.

[2] M.D. Plummer, On n-extendability in graphs, *Discrete Math.* **31** (1980) 201–210.

[3] M.I. Porteous and R.E.L. Aldred, Matching extensions with prescribed and forbidden edges, *Austral. J. Combin.* **13** (1996) 163-174.