On Mod(2) and Mod(3)-Edge-Magic Maximal Outerplanar Graphs

Gee-Choon Lau[a]
Faculty of Computer & Mathematical Sciences
Universiti Teknologi MARA (Segamat Campus)
85000 Segamat, Johor, Malaysia.

Sin-Min Lee
Department of Computer Science
San Jose State University
San Jose, California 95192 U.S.A.

In Memory of Professor Ralph Stanton

ABSTRACT. Let G be a (p,q)-graph. Suppose an edge labeling of G given by $f : E(G) \rightarrow \{1, 2, \ldots, q\}$ is a bijective function. For a vertex $v \in V(G)$, the induced vertex labeling of G is a function $f^*(v) = \sum f(uv)$ for all $uv \in E(G)$. We say $f^*(v)$ the vertex sum of v. If, for all $v \in V(G)$, the vertex sums equal to a constant (mod k) where $k \geq 2$, then we say G admits a Mod(k)-edge-magic labeling, and G is called a Mod(k)-edge-magic graph. In this paper, we show that (i) all maximal outerplanar graphs (or MOPs) are mode(2)-EM, and (ii) many Mod(3)-EM labelings of MOPs can be constructed (a) by adding new vertices to a MOP of smaller size, or (b) by taking the edge-gluing of two MOPs of smaller size, with a known Mod(3)-EM labeling. These provide us with infinitely many Mod(3)-EM MOPs. We conjecture that all MOPs are Mod(3)-EM.

Key words and phrases: Mod(k)-edge-magic, maximal outerplanar graph.

AMS 2000 MSC: 05C78, 05C25

1. Introduction

All graphs in this paper are simple graphs with no loops or multiple edges. A (p,q)-graph G in which the edges are labeled 1, 2, 3, ..., q so that the vertex sums are constant, is called supermagic (see [1]). B. M. Stewart [15, 16] showed that the complete graphs K_3, K_4, K_5 are not supermagic and for $n > 5$, K_n is supermagic if and only if $n \neq 0$ (mod 4). Shiu, Lam and Lee [11] considered a class of supermagic graphs which are the composition of regular supermagic graph with a null graph. Lee, Seah and Tan [7] introduced the following concept of edge-magic graphs which is the generalization of supermagic graphs.

[a]Corresponding author: geeclau@yahoo.com

Definition 1.1 Let G be a (p,q)-graph. Suppose an edge labeling of G given by $f: E(G) \to \{1, 2, \ldots, q\}$ is a bijective function. For a vertex $v \in V(G)$, the induced vertex labeling of G is a function $f^*(v) = \sum f(uv)$ for all $uv \in E(G)$. We say $f^*(v)$ the vertex sum of v. If, for all $v \in V(G)$, the vertex sums are constant, mod p, then we say G admits an edge-magic labeling, and G is called an edge-magic (in short EM) graph.

Example 1. Figure 1 shows a graph G with 6 vertices and 8 edges that is EM with different constant sums.

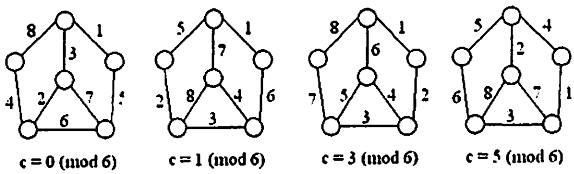


Figure 1.

Example 2. The following graphs with 6 vertices are EM.

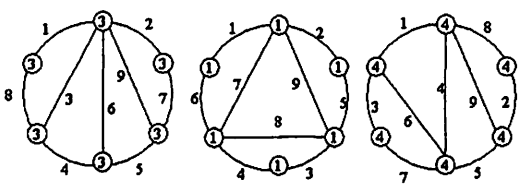


Figure 2.

A necessary condition for a (p,q)-graph to be edge-magic is $q(q+1) \equiv 0 \pmod{p}$. However, this condition is not sufficient. There are infinitely many connected graphs, such as trees and cycles, satisfy this condition that are not edge-magic.

Now we introduce the following concept.

Definition 1.2 Let $k \geq 2$ and G be a (p,q)-graph. Suppose there exists an edge labeling $f: E(G) \to \{1, 2, \ldots, q\}$ which is bijective, and an induced vertex labeling, $f^*(v) = \sum f(uv)$ for all $uv \in E(G)$ of G. If the vertex sums are constant, mod k, then we say G admits a Mod(k)-edge-magic labeling, and G is called a Mod(k)-edge-magic (in short Mod(k)-EM) graph.

In the case $k = p$, then a Mod(k)-EM graph is also an edge-magic graph. A necessary condition for a graph to be Mod(k)-edge-magic is given in the following theorem.

Theorem 1.1. ([3]) If $p \equiv 0 \pmod{k}$, then a necessary condition for G to be Mod(k)-EM is that $q(q+1) \equiv 0 \pmod{k}$.

Example 3. The graph G in Figure 3 is Mod(k)-EM for $k = 2, 3, 4, 6$ but not 5.

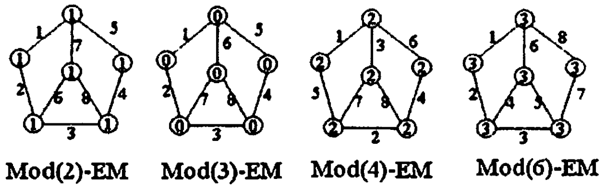


Figure 3.

Example 4. The cubic graph $TW(8)$ in Figure 4 is Mod(2)-EM with vertex label 1.

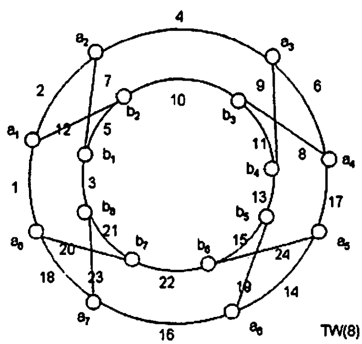


Figure 4.

Example 5. Figure 5 shows that the following graph is Mod(3)-EM with vertex label 0.

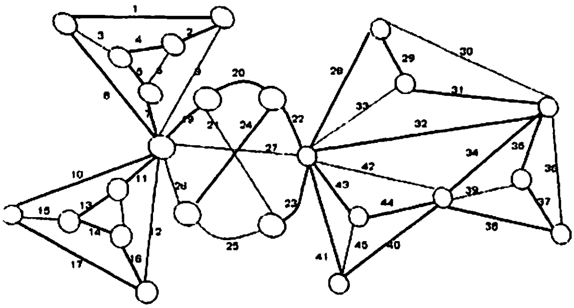


Figure 5.

For more general results and unsolved problems on edge-magic graphs, the reader can refer to [1, 2, 5, 6, 8, 10, 12, 13, 14, 17].

A planar graph is a graph which can be drawn without edge-crossing. A graph is called outerplanar if it has an embedding in the plane such that the vertices lie on a fixed circle and the edges lie inside the disk of the circle and do not intersect. This face outside of the circle is called the outer face. The edges on the boundary of an outerplanar graph are called outer edges and other edges are called inner edges or chords. If we consider a planar graph with no loops or faces bounded by two edges (digons), it may be possible to add a new edge to the presentation of G such that every vertex lies on the exterior face. When no such adjunction can be made, the graph is called a maximal outerplanar graph (or MOP) since additional edge will destroy its outerplanar property. A maximal outerplanar graph can be viewed as a triangulation of a convex polygon (see Figure 2).

Lemma 1.1. ([4]) Let G be a maximal outerplanar graphs with n vertices, $n \geq 3$. Then, G has
(i) $2n$-3 edges, of which there are n outer-edges and n-3 are chords;
(ii) n-2 inner faces, each of which is a triangle;
(iii) at least two vertices with degree 2.

In this paper, we show that (i) all MOPs are mode(2)-EM, (ii) many Mod(3)-EM labelings of MOPs can be constructed (a) by adding new vertices to a MOP of smaller size, or (b) by taking the edge-gluing of two MOPs of smaller size, with a known Mod(3)-EM labeling. These provide us with infinitely many Mod(3)-EM MOPs. We conjecture that all MOPs are Mod(3)-EM.

2. Maximal outerplanar graphs which are Mod(2)-EM

In [3], Chopra, Dios and Lee showed that

Theorem 2.1. A necessary condition for a (p,q)-graph G to be Mod(2)-EM is that $q(q+1) \equiv ps \pmod 2$, where s is the common vertex sum under a Mod(2)-EM labeling. Possible values, mod 2, for s are given in the following table:

	$p \equiv 0 \pmod 2$	$p \equiv 1 \pmod 2$
$q \equiv 0 \pmod 2$	0,1	0
$q \equiv 1 \pmod 2$	0,1	0

A sufficient condition for G to be Mod(2)-EM is given below without proof.

Theorem 2.2. If a (p,q)-graph G has an eulerian subgraph of size $\lceil q/2 \rceil$, then G is Mod(2)-EM.

We have the following result.

Theorem 2.3. All maximal outerplanar graphs are Mod(2)-EM.

Proof. It suffices to show that every maximal outerplanar graph G has an eulerian subgraph of size $\lceil q/2 \rceil$. From Lemma 1.1, we know that G has two non-adjacent vertices of degree 2 of which the two neighboring vertices are adjacent respectively. Since G has odd number of edges, we see that $\lceil q/2 \rceil = p - 1$.

Label the vertices of the outer cycle of G by $v_1, v_2, \ldots, v_p$. Suppose v_k $(1 < k < p)$ is a vertex of degree two, then $v_1 v_2 \ldots v_{k-1} v_{k+1} \ldots v_p$ is a cycle of size $p-1$. Now label the edges of this cycle by $1, 3, 5, \ldots, 2p-3$ and the remaining edges of G by $2, 4, 6, \ldots, 2p-4$. We see that each vertex of G has label 0 (mod 2). □

Example 6. Figure 6 shows that the graphs G_1 and G_2 are Mod(2)-EM.

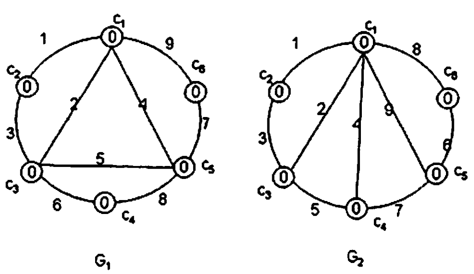


Figure 6

3. Constructing Mod(3)-EM maximal outerplanar graphs

We first give a necessary condition for MOPs to be Mod(3)-EM graphs with vertex label 0, 1 or 2.

Theorem 3.1. A necessary condition for a maximal outerplanar graph of order p and size q is Mod(3)-EM with

a) vertex label 0 is that $p \equiv 0$ or 1 (mod 3);
b) vertex label 1 is that $p \equiv 0$ or 2 (mod 3);
c) vertex label 2 is that $p \equiv 0$ (mod 3).

Proof. a) Suppose G is MOP graph of order p and size q which is Mod(3)-EM with vertex label 0, then by Theorem 1.1, we have

$$q(q+1) \equiv 0 \pmod 3$$
$$(2p-3)(2p-2) \equiv 0$$
$$4p^2 - 10p + 6 \equiv 0$$
$$p^2 - p \equiv 0.$$

So, $p \equiv 0$ or 1 (mod 3). Similarly, we can also prove (b) and (c). □

Let A_i ($i = 0, 1, 2$) be the family of all MOP graphs that are Mod(3)-EM with vertex label i. We now present methods on constructing Mod(3)-EM labeling for MOPs by adding new vertices to known Mod(3)-EM MOPs of smaller order.

We first investigate MOPs that are in A_0. A sufficient condition for G to be in A_0 is given below.

Theorem 3.2. If a (p, q)-graph G has a eulerian subgraph of total size $\lceil 2q/3 \rceil$ with each component has even length, then G is in A_0.

Proof. In mod (3), the edges must be labeled with 1, 2, or 0. Label the edges of the eulerian subgraph by 1 and 2 (mod 3) alternately. Label the remaining edges by 0 (mod 3). It is readily seen that each vertex of G has label 0 (mod 3). □

For $i = 0$, 1 and $p \geq 4$, let G_i be a MOP of order $p = 3k + \text{i}$. Suppose G_i is in A_0 with a known Mod(3)-edge-magic labeling. We now describe 5 distinct methods, denoted $N1$ to $N5$ as shown in Figure 7 below, of constructing a new MOP, denoted H_i, from a given MOP G_i by adding vertices $u_1, u_2, \ldots u_t$, $t \geq 3$, where the new edges are in mod 3 labeling.

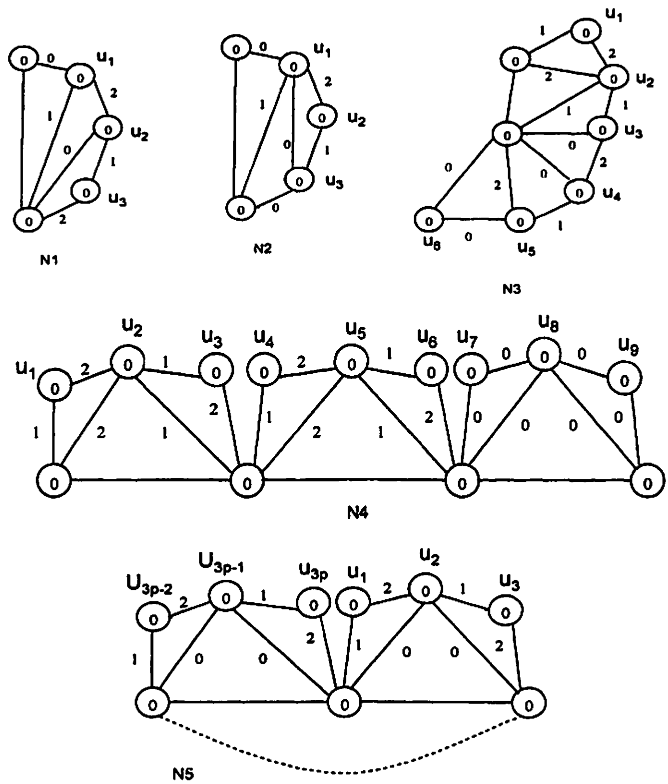


Figure 7.

In method $N1$, 3 new vertices and 6 new edges are added to G_0 of order $3k$ and G_1 of order $3k+1$, respectively. Note that the edges of G_0 and G_1 have been labeled with elements in $\{1, 2, \ldots, 6k-3\}$ and $\{1, 2, \ldots, 6k-1\}$, respectively. Hence, in H_0 of order $3k+3$ and H_1 of order $3k+4$, respectively, we now label the 6 new edges with elements in $\{6k-2, 6k-1, 6k, 6k+1, 6k+2, 6k+3\} = \{1, 2, 0, 1, 2, 0\}$ (mod 3), and $\{6k, 6k+1, 6k+2, 6k+3, 6k+4, 6k+5\} = \{0, 1, 2, 0, 1, 2\}$ (mod 3), respectively. Clearly, H_1 and H_2 are new MOPs in A_0.

In method $N2$, we also need to add 3 new vertices and 6 new edges to G_0 and G_1, respectively. By a similar argument as above, we see that H_1 and H_2 are also new MOPs in A_0. Note that if the graphs obtained by using method $N2$ are not isomorphic to the graphs obtained by using method $N1$, then we have new MOPs of order $3k+3$ and $3k+4$ that admit a Mod(3)-EM labeling.

In method $N3$, 6 new vertices and 12 new edges are added to G_0 and G_1, respectively. In H_0 and H_1, the new edges need to be labeled with elements in $\{6k-2, 6k-1, \ldots, 6k+9\} = \{1, 2, 0, 1, 2, 0, 1, 2, 0, 1, 2, 0\}$ (mod 3), and $\{6k, 6k+1, \ldots, 6k+11\} = \{0, 1, 2, 0, 1, 2, 0, 1, 2, 0, 1, 2\}$ (mod 3), respectively. If the graphs obtained by using method $N3$ are not isomorphic to the graphs obtained by using methods $N1$ or $N2$ repeatedly, then we have new MOPs of order $3k+6$ and $3k+7$ that admit a Mod(3)-EM labeling.

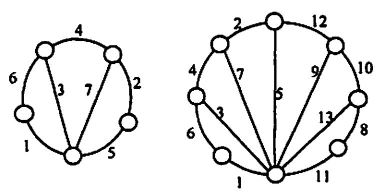
In method $N4$, 9 new vertices and 18 new edges are added. In H_0 and H_1, the new edges need to be labeled with elements in $\{6k-2, 6k-1, \ldots, 6k+15\} = \{1, 2, 0, 1, 2, 0, 1, 2, 0, 1, 2, 0, 1, 2, 0, 1, 2, 0\}$ and $\{6k, 6k+1, \ldots, 6k+17\} = \{0, 1, 2, 0, 1, 2, 0, 1, 2, 0, 1, 2, 0, 1, 2, 0, 1, 2\}$, respectively. Using this method, new MOPs of order $3k+9$ and $3k+10$ that admit a Mod(3)-EM labeling can be obtained.

In method $N5$, $3p$ new vertices and $6p$ new edges are added to all the outer edges of G_0 (and G_1). In H_0 (and H_1), the new edges need to be labeled with elements in $\{6k-2, 6k-1, \ldots, 6k+6p-3\}$ (and $\{6k, 6k+1, \ldots, 6k+6p-1\}$). Using this method, new MOP of order $4p = 12k$ (and $= 12k+4$) that admits a Mod(3)-EM labeling can be obtained.

It is easy to verify that each MOP of order 4, 6 and 7 admits a mod(3)-EM labeling with vertex label 0. Using method $N1$ to $N5$, we have the following theorem.

Theorem 3.3. There exist infinitely many MOPs of order $p \equiv 0$ or 1 (mod 3) that admit a mod(3)-EM labeling with vertex label 0.

We now investigate MOPs in A_1. We first note that the MOPs of order 5, 6 and 8 shown in Figure 8 are in A_1.



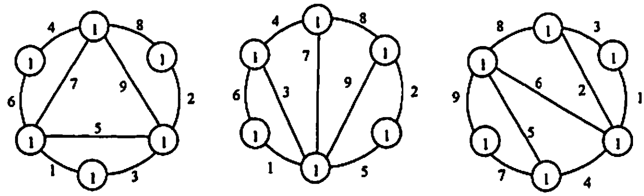


Figure 8.

For $p \geq 5$, let G_l be a MOP of order $p = 3k + 2$. Suppose G is in A_1. By an argument similar to that in describing the methods $N1$ to $N5$ in Figure 7 above, we also have 5 distinct methods, denoted $M1$ to $M5$ as shown in Figure 9 below, of constructing a new MOP, denoted H, from G by adding 3 vertices u_1, u_2, u_3, where the new edges are in mod 3 labeling. In each method, the 6 new edges need to be labeled with elements in $\{6k+2, 6k+3, 6k+4, 6k+5, 6k+6, 6k+7\}$.

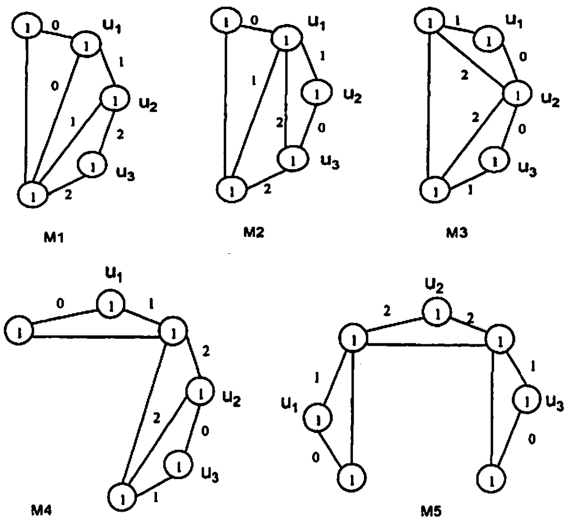


Figure 9.

Clearly, the graph H_l thus obtained is also in A_1. We have checked that all 12 non-isomorphic MOPs of order 8 can be obtained from the MOP graph of order 5 by using one of the five methods above. Hence, all the MOPs of order 8 are in A_1. Using the method $M1$ to $M5$, we have the following theorem.

<u>Theorem 3.4.</u> There exist infinitely many MOPs of order $p \equiv 2 \pmod 3$ that admit a mod(3)-EM labeling with vertex label 1.

Corollary 3.1. Each MOP of order $p \equiv 0$ or 2 (mod 3) with vertex set $\{v_1, v_2, \ldots, v_p\}$ and edge set $\{(v_1,v_2), (v_2,v_3), \ldots, (v_{p-1},v_p), (v_p,v_1)\} \cup \{(v_1,v_k) \mid k = 3, 4, \ldots, p-1\}$ admit a Mod(3)-EM labeling with vertex label 1.

Remark 3.1. Methods *M*3, *M*4 and *M*5 could give us new MOPs of order $p \equiv 0$ (mod 3) that admit a Mod(3)-EM labeling, but cannot be constructed by using methods *N*1 to *N*5 above, repeatedly.

Example 6. The MOP of order 9 in Figure 10 is a MOP that can be constructed from any MOP of order 6 by using method *M*3, *M*4 or *M*5 but not from any of methods *N*1 to *N*5. The given labeling is based on construction method *M*4.

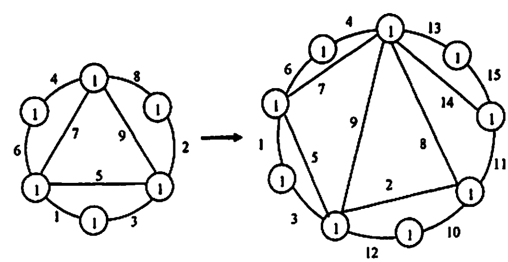


Figure 10

We now investigate MOPs in A_2. Figure 11 shows that all MOPs of order 6 are in A_2.

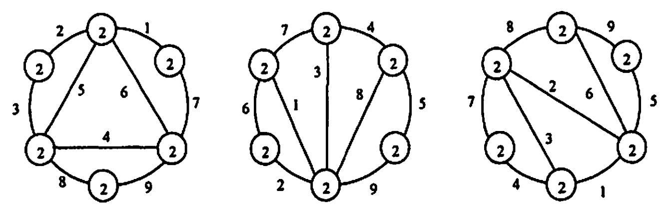


Figure 11.

For $p \geq 5$, let G be a MOP of order $p = 3k$. Suppose G is in A_2. By an argument similar to that in describing the methods *N*1 to *N*5 in Figure 7 above, we also have 4 distinct methods, denoted *R*1 to *R*4 as shown in Figure 12 below, of constructing a new MOP, denoted *H*, from the given MOP G_i by adding vertices $u_1, u_2, \ldots, u_t$ ($t = 3$ or 6), where the new edges are in mod 3 labeling. In each method, the $2t$ new edges need to be labeled with elements in $\{6k-2, 6k-1, \ldots, 6k-3+2t\}$.

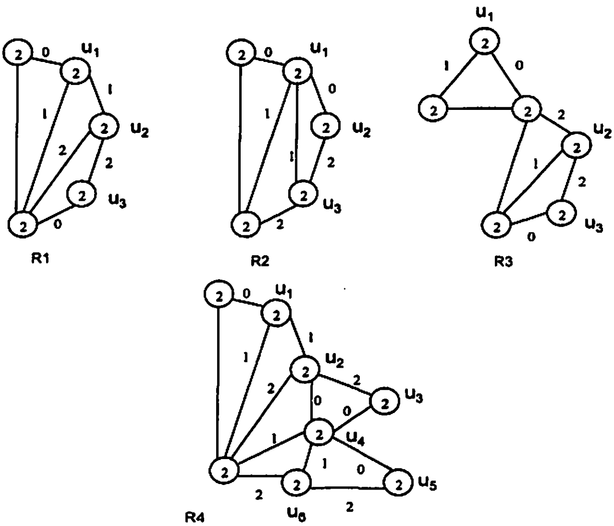


Figure 12.

Clearly, the graph H thus obtained is also in A_2. Using method $R1$ to $R4$, we have the following theorem.

Theorem 3.5. There exists infinitely many MOPs of order $p \equiv 0 \pmod 3$ that admit a mod(3)-EM labeling with vertex label 3.

Remark 3.2. Note that method $R4$ could give us new MOPs of order $p \equiv 0$ (mod 3) that admit Mod(3)-EM labeling, but cannot be constructed by using the methods $N1$ to $N5$, or $M1$ to M5, repeatedly.

Example 7. The MOP of order 12 in Figure 13 is a MOP with a Mod(3)-edge-magic labeling that can be constructed from a MOP of order 6 by using method $R4$ but not from any of the methods $N1$ to $N5$, or $M1$ to $M5$, or $R1$ to $R3$, repeatedly.

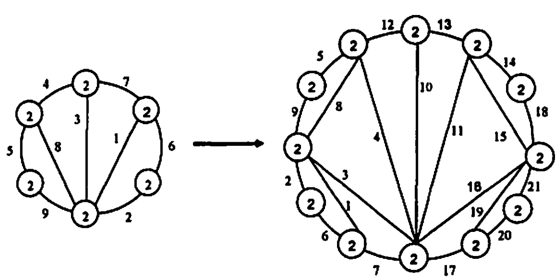


Figure 13.

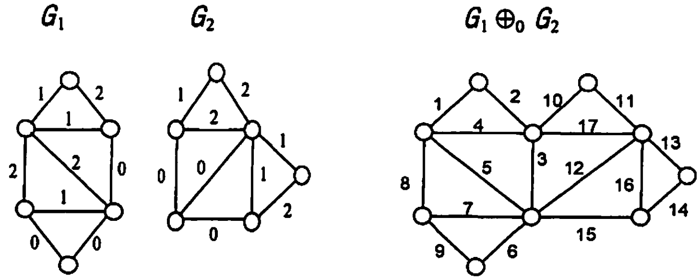
Observe that a MOP is an edge-gluing of an outer-edge of two MOPs with smaller order. Let both G and H be in A_i. Denote by $G \oplus_i H$ the graph obtained from G and H by gluing an outer edge of G to an outer edge of H where both edges has label i (mod 3). The following theorem gives another way of constructing infinitely many new Mod(3)-EM maximal outerplanar graphs.

Theorem 3.6. For $i = 1,2$, let G_i be MOP graphs with order p_i.
(a) If $p_i \equiv 0$ (mod 3), then $G_i \in A_0$ implies that $G_1 \oplus_0 G_2 \in A_0$.
(b) If $p_i \equiv 2$ (mod 3), then $G_i \in A_1$ implies that $G_1 \oplus_1 G_2 \in A_1$.
(c) If $p_1 \equiv 0$ (mod 3) and $p_2 \equiv 2$ (mod 3), then $G_i \in A_1$ implies that $G_1 \oplus_1 G_2 \in A_1$.

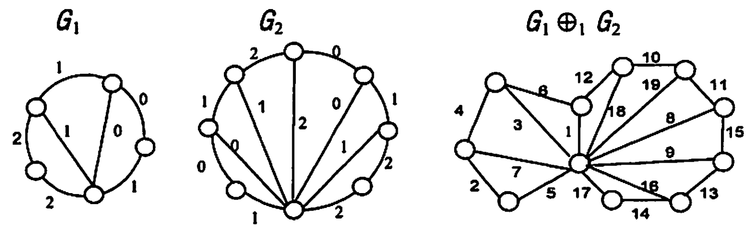
Proof. (a) Suppose $G_i \in A_0$ for $i = 1, 2$. Let $p_i = 3k_i$, then G_i has $6k_i - 3$ edges such that $2k_i - 1$, $2k_i - 1$ and $2k_i - 1$ edges are labeled with 1, 2, and 0 (mod 3), respectively. Suppose an edge-gluing of G_1 and G_2 has $3(k_1 + k_2) - 2$ vertices and $6(k_1 + k_2) - 7$ edges such that $2(k_1 + k_2) - 2$, $2(k_1 + k_2) - 2$ and $2(k_1 + k_2) - 3$ of the edges need to be labeled with 1, 2, and 0 (mod 3), respectively. Let e_1 and e_2 be an outer edge in G_1 and G_2 respectively, both with label 0 (mod 3). Clearly, the graph $G_1 \oplus_0 G_2$ obtained from G_1 and G_2 by overlapping on e_1 and e_2 is a graph in A_0.

The proofs for (b) and (c) are similar and are left to the readers. □

Example 9. We give examples for Theorem 2.6(a) and (b) above in Figure 14. Graphs G_1 and G_2 are given under mod(3)-labelings.



Case (a): $G_1 \oplus_0 G_2$ is a graph of order 10 having 17 edges in A_0.



Case (b): $G_1 \oplus_1 G_2$ is a graph of order 11 having 19 edges in A_1.

Figure 14.

An example of Theorem 2.6(c) above can be obtained from graphs of order 6 and 8 in Figure 8.

Remark 3.3. The graph of order 14 in Figure 15 is in A_1 with a labeling that cannot be obtained from any of the construction methods $M1$ – $M5$ in Figure 9, or the relevant methods describe in Theorem 3.6 above.

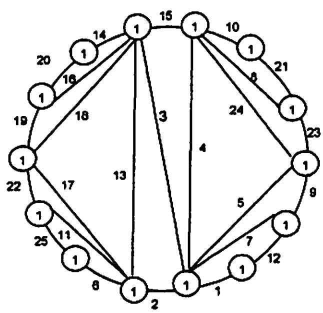


Figure 15.

We end this paper with the following conjecture.

Conjecture. All maximal outerplanar graphs are Mod(3)-EM.

References

[1] M. Ba˘ca and I. Holl¨ander and Ko-Wei Lih, Two classes of super-magic graphs, *J. Combin. Math. Combin. Comput.*, 23 (1997), 113 – 120.

[2] D. Chopra, H. Kwong and Sin-Min Lee, On edge-magic (p, 3p – 1) – graphs *Congr. Numer.*, 179 (2006), 49 – 63.

[3] D. Chopra, R. Dios and Sin-Min Lee, On Mod(2)-edge-magic graphs, manuscript.

[4] P.S. Kumar and C.E.V. Madhavan, A new class of separators and planarity of chordal graphs, *Lecture Notes in Computer Science*, 405 (1989), 30 – 41.

[5] Sin-Min Lee, M. Kitagaki, J. Young, and W. Kocay, On edge-graceful and edge-magic maximal outerplanar graphs, J. *Combin. Math. Combin. Comput.*, 59 (2006) 119 – 129.

[6] Sin-Min Lee, W.M. Pigg and T.J. Cox, On edge-magic cubic graphs conjecture, *Congr. Numer.*, 105 (1994) 214 – 222.

[7] Sin-Min Lee, Eric Seah and S.K. Tan, On edge-magic graphs, *Congr. Numer.*, 86 (1992) 179 – 191.

[8] Sin-Min Lee, E. Seah and Siu-Ming Tong, On the edge-magic and edge-graceful total graphs conjecture, *Congr. Numer.* 141 (1999) 37 – 48.

[9] Sin-Min Lee, Ling.Wang and Yihui Wen, On The Edge-magic Cubic Graphs and Multigraph, *Congr. Numer.*, 165 (2003) 145 – 160.

[10] Karl Schaffer and Sin Min Lee, Edge-graceful and edge-magic labellings of Cartesian products of graphs, *Congr. Numer.*, 141 (1999) 119 – 134.

[11] W.C. Shiu, P.C.B. Lam and Sin-Min Lee, On a construction of supermagic graphs, *J. Combin. Math. Combin. Comput.*, 42 (2002)147 – 160.

[12] W.C. Shiu, P.C.B. Lam and Sin-Min Lee, Edge-magicness of the composition of a cycle with a null graph, *Congr. Numer.*, 132 (1998) 9 – 18.

[13] W.C. Shiu, P.C.B. Lam and Sin-Min Lee, Edge - magic index sets of (p, p – 1)-graphs, *Electronic Notes in Discrete Mathematics*, Vol. 11 (2002) 443 – 458.

[14] W.C. Shiu and Sin-Min Lee, Some edge-magic cubic graphs, *J. Combin. Math. Combin. Comput.*, 40 (2002) 115 – 127.

[15] B.M. Stewart, Magic graphs, *Canadian Journal of Mathematics*, 18 (1966) 1031 – 1059.

[16] B.M. Stewart, Supermagic complete graphs, *Canadian Journal of Mathematics*, 19 (1967) 427 – 438.

[17] Yihui Wen, Sin-Min Lee, and Hugo Sun, On the Supermagic Edge-splitting Extension of Graphs, *Ars Combinatoria* 79 (2006) 115 – 128.