Truncated Tetrahedron, Octahedron and Cube Designs

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Dedicated to the memory of Ralph Gordon Stanton

Abstract

We prove that the complete graph K_v can be decomposed into truncated tetrahedra if and only if $v \equiv 1$ or 28 (mod 36), into truncated octahedra if and only if $v \equiv 1$ or 64 (mod 72), and into truncated cubes if and only if $v \equiv 1$ or 64 (mod 72).

1 Introduction

The spectrum of integers v for which the complete graph K_v can be decomposed into copies of the graph of one of the Platonic solids is determined for the tetrahedron, octahedron, cube and dodecahedron but only partial results are available for the icosahedron. The current state of knowledge, see also [3], appears to be as follows.

1. Tetrahedron designs are equivalent to Steiner systems $S(2, 4, v)$. The necessary and sufficient condition is $v \equiv 1$ or 4 (mod 12), [15].
2. Octahedron designs are equivalent to Steiner triple systems $S(2, 3, v)$ which can be decomposed into Pasch configurations. The necessary and sufficient condition is $v \equiv 1$ or 9 (mod 24), $v \neq 9$, [14], [1].
3. Cube designs exist if and only if $v \equiv 1$ or 16 (mod 24), [17], [16], [6].

4. Dodecahedron designs exist if and only if $v \equiv 1, 16, 25$ or $40 \pmod{60}$ and $v \neq 16$, [2], [3], [4].
5. Icosahedron designs exist if and only if $v \equiv 1, 16, 21$ or $36 \pmod{60}$ with possible exceptions $v = 21, 141, 156, 201, 261$ and 276, [2], [3], [8].

A natural extension of the above is to consider decompositions into the Archimedean graphs, of which there are two infinite families (the prisms and antiprisms) as well as thirteen further examples. Results have appeared for the cuboctahedron and the rhombicuboctahedron.

1. Cuboctahedron designs exist if and only if $v \equiv 1$ or $33 \pmod{48}$, [13].
2. Rhombicuboctahedron designs exist if and only if $v \equiv 1$ or $33 \pmod{96}$, [9].

These seem to be the only classes of Archimedean designs where the spectrum has been completely determined. In this paper we add three further classes. We state these results as Theorem 1, Theorem 2 and Theorem 3. Further partial results can be found in the dynamic survey [7]. The necessity of the conditions, $v \equiv 1$ or $28 \pmod{36}$ for the truncated tetrahedron decomposition and $v \equiv 1$ or $64 \pmod{72}$ for the truncated octahedron and truncated cube decompositions, are easy to establish by elementary counting. The sufficiency of Theorem 1 follows from Lemmas 11, 12 and 13 in the next section, of Theorem 2 from Lemmas 18 and 19 in section 3, and of Theorem 3 from Lemmas 23 and 31 in section 4.

Theorem 1 *Truncated tetrahedron designs exist if and only if* $v \equiv 1$ *or* $28 \pmod{36}$.

Theorem 2 *Truncated octahedron designs exist if and only if* $v \equiv 1$ *or* $64 \pmod{72}$.

Theorem 3 *Truncated cube designs exist if and only if* $v \equiv 1$ *or* $64 \pmod{72}$.

Our method of proof uses a standard technique (Wilson's fundamental construction). For this we need the concept of a *group divisible design* (GDD). Recall that a k-GDD of type u^t is an ordered triple (V, G, B) where V is a base set of cardinality $v = tu$, G is a partition of V into t subsets of cardinality u called *groups* and B is a family of subsets of cardinality k called *blocks* which collectively have the property that every pair of elements from different groups occurs in precisely one block but no pair of elements from the same group occurs at all. We will also need k-GDDs of type $u^t w^1$. These are defined analogously, with the base set V being of cardinality $v = tu + w$ and the partition G being into t subsets of cardinality u and one set of cardinality w.

2 Truncated tetrahedron constructions

The truncated tetrahedron has 12 vertices, 18 edges and 8 faces, and we will represent them by ordered 12-tuples (A, B, C, D, E, F, G, H, J, K, L, M), where the co-ordinates represent vertices as in Figure 1. We first

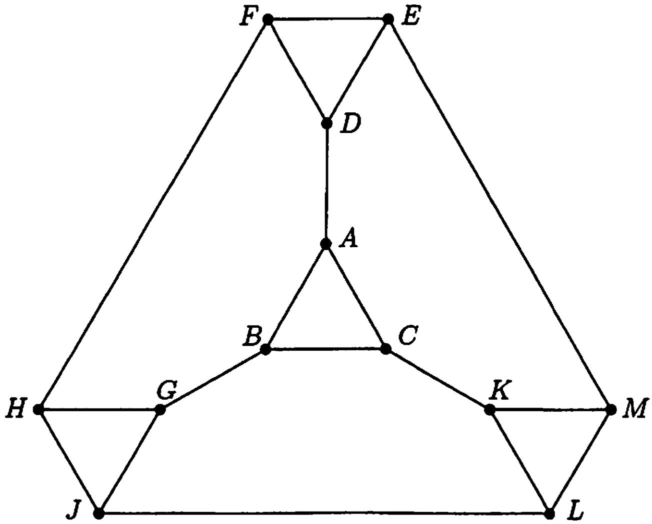

Figure 1: Truncated tetrahedron graph.

present truncated tetrahedron designs of orders 28, 37, 64, 73 and 100, all obtained by a computer search assuming appropriate cyclic automorphisms.

Lemma 4 *There exists a truncated tetrahedron design of order* 28.
Proof. Let the vertex set of the complete graph K_{28} be Z_{28}. The decomposition consists of the truncated tetrahedra

(0, 1, 2, 3, 4, 8, 5, 13, 7, 6, 9, 12),

(0, 6, 7, 9, 2, 14, 3, 19, 10, 22, 24, 13),

(0, 10, 18, 15, 1, 11, 25, 3, 12, 7, 24, 17)

under the action of the mapping $i \mapsto i+4 \pmod{28}$. □

Lemma 5 *There exists a truncated tetrahedron design of order* 37.
Proof. Let the vertex set of K_{37} be Z_{37}. The decomposition consists of the truncated tetrahedron

$$(0, 1, 3, 4, 9, 15, 8, 29, 17, 16, 34, 24)$$

under the action of the mapping $i \mapsto i+1 \pmod{37}$. □

Lemma 6 *There exists a truncated tetrahedron design of order* 64.
Proof. Let the vertex set of K_{64} be $Z_{63} \cup \{\infty\}$. The decomposition consists of the truncated tetrahedra

(29, 62, 20, 37, 3, 4, 28, 40, 0, 2, 41, 36),

(56, 62, 18, 31, 47, 34, 2, 16, 1, 22, 48, 36),

(0, 10, 52, 19, 62, 58, 17, 5, 21, 27, 41, 14),

(0, 2, 3, 6, 1, 21, 4, 12, 10, 11, 42, 18),

(0, 7, 26, 13, 4, 35, 47, 12, 57, 43, ∞, 32),

(0, 17, 22, 27, 6, 48, 43, 21, 38, 59, 1, 42)

under the action of the mapping $\infty \mapsto \infty$, $i \mapsto i+3 \pmod{63}$ for the first five, and $i \mapsto i+9 \pmod{63}$ for the sixth. □

Lemma 7 *There exists a truncated tetrahedron design of order* 73.
Proof. Let the vertex set of K_{73} be Z_{73}. The decomposition consists of the truncated tetrahedra

(0, 1, 3, 4, 9, 15, 8, 23, 32, 13, 44, 26),

(0, 14, 30, 21, 41, 67, 53, 31, 3, 63, 38, 9)

under the action of the mapping $i \mapsto i+1 \pmod{73}$. □

Lemma 8 *There exists a truncated tetrahedron design of order* 100.
Proof. Let the vertex set of K_{100} be Z_{100}. The decomposition consists of the truncated tetrahedra

(36, 13, 90, 41, 11, 26, 77, 4, 96, 69, 25, 62),

(93, 56, 74, 8, 61, 75, 11, 39, 6, 28, 10, 22),
(72, 85, 88, 0, 35, 27, 79, 54, 68, 12, 56, 18),
(49, 18, 58, 69, 47, 37, 26, 10, 60, 45, 17, 46),
(97, 1, 8, 42, 23, 72, 85, 50, 87, 49, 46, 11),
(32, 57, 15, 52, 95, 26, 43, 46, 12, 44, 3, 63),
(58, 71, 48, 51, 61, 52, 26, 78, 50, 65, 39, 18),
(46, 7, 4, 21, 69, 29, 28, 80, 49, 36, 83, 99),
(37, 36, 72, 61, 83, 27, 11, 31, 85, 70, 26, 68),
(0, 30, 40, 14, 46, 97, 83, 35, 85, 44, 39, 82),
(0, 33, 39, 7, 20, 65, 66, 77, 95, 63, 86, 81)

under the action of the mapping $i \mapsto i + 4 \pmod{100}$. □

Some of the main ingredients which we will need in applying Wilson's fundamental construction are given in the above lemmas. We also require decompositions of certain complete multipartite graphs into truncated tetrahedra. We present these next. Unlike the decompositions of the complete graphs in Lemmas 4 to 8, these were found by hand.

Lemma 9 *There exists a decomposition of the complete tripartite graph* $K_{6,6,6}$ *into* 6 *truncated tetrahedra.*

Proof. Let the three partitions of $K_{6,6,6}$ be $\{(i,0) : 0 \le i \le 5\}$, $\{(i,1) : 0 \le i \le 5\}$ and $\{(i,2) : 0 \le i \le 5\}$. The decomposition consists of the truncated tetrahedron

$$((0,0),(0,1),(0,2),(4,1),(2,0),(1,2),(4,2),(2,1),(1,0),(4,0),(2,2),(1,1))$$

under the action of the mapping $(i,j) \mapsto (i+1,j) \pmod 6$. We will refer to this design as a truncated tetrahedron GDD of type 6^3. □

Lemma 10 *There exists a decomposition of the complete* 4*-partite graph* $K_{3,3,3,3}$ *into* 3 *truncated tetrahedra.*

Proof. Let the four partitions of $K_{3,3,3,3}$ be $\{(i,0) : 0 \le i \le 2\}$, $\{(i,1) : 0 \le i \le 2\}$, $\{(i,2) : 0 \le i \le 2\}$ and $\{(i,3) : 0 \le i \le 2\}$. The decomposition consists of the truncated tetrahedron

$$((0,0),(0,1),(0,2),(1,1),(2,0),(0,3),(1,2),(2,1),(2,3),(1,0),(2,2),(1,3))$$

under the action of the mapping $(i,j) \mapsto (i+1,j) \pmod 3$. We will refer to this design as a truncated tetrahedron GDD of type 3^4. □

We are now in a position to present the main results.

Lemma 11 *There exists a truncated tetrahedron design of order* $v = 36t + 1$, $t \geq 1$.
Proof. Cases $t = 1$ and 2 are proved by Lemmas 5 and 7 respectively. So we may assume that $t \geq 3$.

There exists a 3-GDD of type $6^t, t \geq 3$, [15]; see also [10]. This is called the *master* GDD. Replace each element of the base set V by 6 elements (i.e. inflate by a factor 6) and adjoin a further element, ∞. On every inflated group of the 3-GDD, together with the element ∞, place the truncated tetrahedron design of order 37 from Lemma 5. Further, replace each block of the master GDD by the truncated tetrahedron GDD of type 6^3 from Lemma 9, called the *slave* GDD. □

Lemma 12 *There exists a truncated tetrahedron design of order* $v = 36t + 28$, $t \geq 0$, $t \neq 3$.
Proof. Cases $t = 0, 1$ and 2 are dealt with by Lemmas 4, 6 and 8 respectively. So we may assume that $t \geq 4$.

There exists a 4-GDD of type $12^t 9^1, t \geq 4$, [11]; see also [10]. Replace each element of the base set V by 3 elements and adjoin a further element, ∞. On every inflated group of the 4-GDD, together with the element ∞, place the truncated tetrahedron design of order 37 from Lemma 5 or, in the case of the inflated group of cardinality 27, the truncated tetrahedron design of order 28 from Lemma 4. Replace each block of the master GDD by the slave truncated tetrahedron GDD of type 3^4 from Lemma 10.

□

The above leaves just one outstanding case, $v = 136$, corresponding to $t = 3$ in Lemma 12, which we address in the final lemma of this section.

Lemma 13 *There exist a truncated tetrahedron design of order* 136.
Proof. There exists a 4-GDD of type 9^5, [5]; see also [10]. Replace each element of the base set V by 3 elements and adjoin a further element, ∞. On every inflated group of the 4-GDD, together with the element ∞, place the truncated tetrahedron design of order 28 from Lemma 4, and replace each block of the master GDD by the slave truncated tetrahedron GDD of type 3^4 from Lemma 10. □

This completes the proof of Theorem 1.

3 Truncated octahedron constructions

The truncated octahedron has 24 vertices, 36 edges and 14 faces, and we will represent them by ordered 24-tuples (A, B, C, D, E, F, G, H, J, K, L, M, N, P, Q, R, S, T, U, V, W, X, Y, Z), where the co-ordinates represent vertices as in Figure 2.

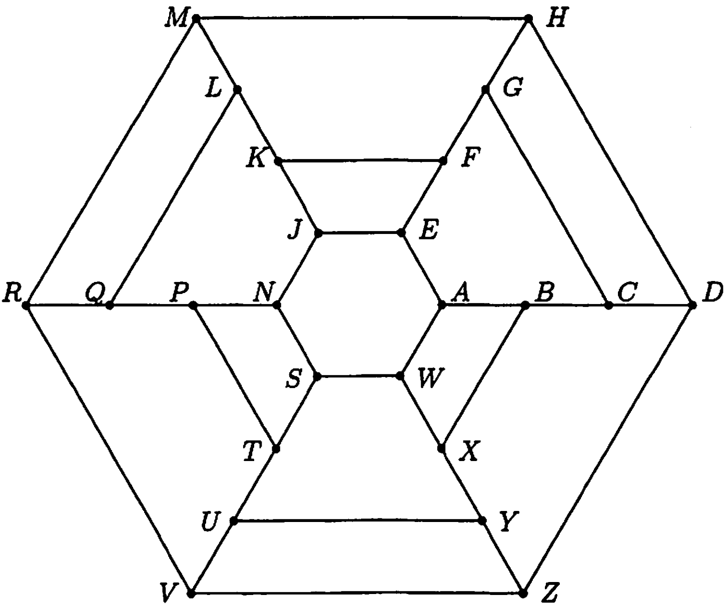

Figure 2: Truncated octahedron graph.

Lemma 14 *There exists a truncated octahedron design of order* 64.
Proof. Let the vertex set of K_{64} be $Z_{63} \cup \{\infty\}$. The decomposition consists of the truncated octahedra

$$(0,3,1,2,4,5,6,7,8,9,10,13,\\ 11,12,14,16,15,17,19,25,18,20,26,30),$$

$$(0,5,2,4,6,3,9,16,10,7,8,18,\\ 1,11,19,13,12,20,14,21,22,15,23,28),$$

$$(0,8,5,13,9,1,16,2,19,31,3,12,\\ 4,6,15,25,11,23,7,17,24,29,14,26),$$

$$(0,11,7,15,12,6,18,1,13,22,2,17,$$

$$8, 14, 27, 3, 20, 4, 21, 32, 36, 49, 5, 46),$$

$$(0, 14, 1, 19, 15, 4, 22, 3, 8, 24, 2, 21, \\ 23, 5, 25, 6, 40, 26, 7, 20, 16, 33, 44, 57),$$

$$(0, 17, 34, 1, 19, 2, 46, 24, 6, 26, 3, 42, \\ 27, 4, 18, \infty, 5, 30, 7, 21, 28, 52, 22, 43),$$

$$(0, 20, 2, 23, 21, 1, 28, 53, 41, \infty, 4, 26, \\ 14, 54, 35, 55, 45, 17, 47, 18, 25, 50, 11, 52),$$

$$(0, 30, 51, 21, 33, 62, 23, 55, 58, 24, 60, 20, \\ 32, 6, 34, 52, 4, 38, \infty, 25, 31, 8, 26, 57)$$

under the action of the mapping $\infty \mapsto \infty$, $i \mapsto i + 9 \pmod{63}$. □

Lemma 15 *There exists a truncated octahedron design of order* 73.
Proof. Let the vertex set of K_{73} be Z_{73}. The decomposition consists of the truncated octahedron

$$(0, 1, 3, 6, 4, 9, 15, 22, 12, 23, 2, 31, \\ 25, 5, 20, 48, 47, 24, 54, 7, 10, 43, 16, 55)$$

under the action of the mapping $i \mapsto i + 1 \pmod{73}$. □

Lemma 16 *There exists a decomposition of the complete bipartite graph* $K_{24,24}$ *into* 16 *truncated octahedra.*
Proof. Let the vertex set of $K_{24,24}$ be Z_{48} partitioned according to residue classes modulo 2. The decomposition consists of the truncated octahedron

$$(0, 1, 2, 3, 5, 8, 13, 6, 12, 21, 4, 17, \\ 7, 10, 25, 44, 14, 45, 34, 11, 23, 40, 15, 30)$$

under the action of the mapping $i \mapsto i + 3 \pmod{48}$. □

Lemma 17 *There exists a decomposition of the complete bipartite graph* $K_{21,24}$ *into* 14 *truncated octahedra.*
Proof. Let the vertex set of $K_{21,24}$ be $\{0, 1, \ldots, 44\}$ partitioned into $\{i : i < 42,\ i \equiv 0 \pmod 2\}$ and $\{i : i < 42,\ i \equiv 1 \pmod 2\} \cup \{42, 43, 44\}$. The decomposition consists of the truncated octahedra

$$(0, 1, 2, 3, 5, 4, 7, 10, 6, 9, 12, 19, \\ 15, 8, 23, 26, 16, 11, 18, 13, 27, 32, 41, 22),$$

$$(0, 13, 24, 1, 15, 2, 42, 16, 28, 35, 10, 31, \\ 17, 38, 23, 6, 30, 43, 34, 44, 21, 32, 9, 26)$$

under the action of the mapping $i \mapsto i+6 \pmod{42}$ for $i < 42$, $i \mapsto i$ for $i \geq 42$. □

Lemma 18 *There exists a truncated octahedron design of order $v = 72t+1$, $t \geq 1$.*

Proof. Case $t = 1$ follows from Lemma 15. So we assume that $t \geq 2$. Take the complete t-partite graph K_{3^t}. Replace each vertex by 24 elements and adjoin a further element, ∞. On every inflated partition, together with ∞, place the truncated octahedron design of order 73 from Lemma 15. Replace each edge of the t-partite graph by the truncated octahedron decomposition of $K_{24,24}$ from Lemma 16. □

Lemma 19 *There exists a truncated octahedron design of order $v = 72t + 64$, $t \geq 0$.*

Proof. Case $t = 0$ follows from Lemma 14. So we assume that $t \geq 1$. Take the complete $(t+1)$-partite graph $K_{3^{t+1}}$. In one partition replace each vertex by 21 elements, replace all other vertices by 24 elements, and adjoin a further element, ∞. On every inflated partition, together with ∞, place either the truncated octahedron design of order 64 from Lemma 14 or the truncated octahedron design of order 73 from Lemma 15. Replace each edge of the $(t+1)$-partite graph by either the truncated octahedron decomposition of $K_{21,24}$ from Lemma 17 or the truncated octahedron decomposition of $K_{24,24}$ from Lemma 16. □

This completes the proof of Theorem 2.

4 Truncated cube constructions

The truncated cube has 24 vertices, 36 edges and 14 faces, and we will represent them by ordered 24-tuples (A, B, C, D, E, F, G, H, J, K, L, M, N, P, Q, R, S, T, U, V, W, X, Y, Z), where the co-ordinates represent vertices as in Figure 3. Although the main parameters and design existence conditions of the truncated cube are the same as those of the truncated octahedron, the truncated cube graph has chromatic number 3. As a consequence, the bipartite graph decompositions that feature in the proof of Theorem 2 are not available for the truncated cube. Hence the details of our handling of the truncated cube differ significantly from those of section 3.

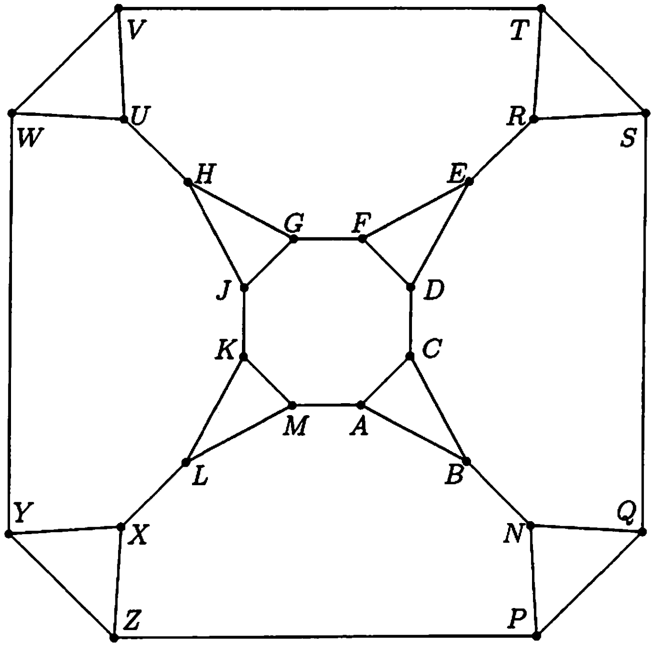

Figure 3: Truncated cube graph.

Lemma 20 *There exists a truncated cube design of order* 73.
Proof. Let the vertex set of K_{73} be Z_{73}. The decomposition consists of the truncated cube

$$(0,1,3,7,2,13,4,11,19,5,15,27,\\ 14,30,47,22,72,51,41,10,59,49,23,68)$$

under the action of the mapping $i \mapsto i+1 \pmod{73}$. □

Lemma 21 *There exists a truncated cube design of order* 145.
Proof. Let the vertex set of K_{145} be Z_{145}. The decomposition consists of the truncated cubes

$$(26,7,141,84,57,48,49,119,111,11,109,40,\\ 35,86,14,77,17,83,24,105,34,143,41,0),$$

$$(0,4,16,1,6,19,2,25,50,8,32,63,\\ 30,62,95,41,3,90,92,36,144,78,38,139)$$

under the action of the mapping $i \mapsto i+1 \pmod{145}$. □

Lemma 22 *There exists a decomposition of the complete tripartite graph $K_{12,12,12}$ into 12 truncated cubes.*
Proof. Let the vertex set of $K_{12,12,12}$ be Z_{36} partitioned according to residue classes modulo 3. The decomposition consists of the truncated cube

$$(0,1,2,3,7,11,4,6,17,12,5,19,15,25,35,30,10,26,34,9,32,13,18,8)$$

under the action of the mapping $i \mapsto i+3 \pmod{36}$. We will refer to this design as a truncated cube GDD of type 12^3. □

Lemma 23 *There exists a truncated cube design of order $v = 72t+1$, $t \geq 1$.*
Proof. Cases $t = 1$ and 2 follow from Lemmas 20 and 21 respectively. So we assume that $t \geq 3$.

For the master GDD, take a 3-GDD of type $6^t, t \geq 3$, [15]; see also [10]. Inflate each base element by a factor of 12 and adjoin a further element, ∞. On every inflated group of the 3-GDD, together with ∞, place the truncated cube design of order 73 from Lemma 20 and replace each block of the 3-GDD by the truncated cube GDD of type 12^3 from Lemma 22.

□

Lemma 24 *There exists a truncated cube design of order* 64.
Proof. Let the vertex set of K_{64} be $Z_{63} \cup \{\infty\}$. The decomposition consists of the truncated cubes

$$(\infty,54,55,0,10,45,58,62,36,51,21,59,$$
$$35,18,12,48,1,34,27,15,6,26,30,8),$$

$$(36,40,16,21,20,50,62,30,61,54,18,38,$$
$$6,17,4,25,31,11,33,47,10,5,52,7),$$

$$(43,6,62,40,29,5,47,55,26,37,0,46,$$
$$52,44,61,7,32,35,17,11,2,56,50,59),$$

$$(29,22,10,58,7,55,38,35,4,13,5,61,$$
$$25,37,2,59,40,3,26,49,32,46,53,14)$$

under the action of the mapping $\infty \mapsto \infty$, $i \mapsto i+3 \pmod{63}$ for the first two, and $i \mapsto i+9 \pmod{63}$ for the last two. □

Lemma 25 *There exists a truncated cube design of order* 136.
Proof. Let the vertex set of K_{136} be Z_{136}. The decomposition consists of

the truncated cubes

$(84, 120, 108, 67, 31, 2, 20, 107, 126, 122, 65, 97,$
$70, 116, 90, 38, 125, 76, 109, 101, 25, 8, 18, 12),$

$(40, 79, 20, 15, 90, 59, 107, 108, 9, 43, 134, 78,$
$2, 60, 16, 102, 88, 51, 98, 55, 44, 69, 41, 81),$

$(130, 33, 56, 74, 87, 12, 103, 85, 62, 28, 79, 26,$
$43, 107, 70, 54, 113, 109, 21, 94, 92, 3, 9, 53),$

$(9, 72, 57, 112, 16, 24, 113, 10, 124, 5, 66, 55,$
$63, 36, 118, 46, 4, 98, 133, 26, 99, 8, 77, 91),$

$(103, 60, 0, 52, 71, 113, 119, 29, 50, 53, 133, 115,$
$110, 126, 109, 41, 32, 78, 120, 86, 77, 114, 66, 123),$

$(66, 67, 87, 7, 31, 101, 120, 107, 91, 113, 15, 72,$
$84, 59, 56, 96, 122, 30, 11, 25, 9, 8, 109, 24),$

$(48, 81, 71, 8, 123, 13, 33, 86, 122, 46, 6, 95,$
$50, 67, 75, 2, 114, 29, 41, 68, 12, 11, 43, 121),$

$(88, 124, 112, 71, 35, 6, 24, 111, 130, 126, 69, 101,$
$74, 120, 94, 42, 129, 80, 113, 105, 29, 12, 22, 16),$

$(44, 83, 24, 19, 94, 63, 111, 112, 13, 47, 2, 82,$
$6, 64, 20, 106, 92, 55, 102, 59, 48, 73, 45, 85),$

$(134, 37, 60, 78, 91, 16, 107, 89, 66, 32, 83, 30,$
$47, 111, 74, 58, 117, 113, 25, 98, 96, 7, 13, 57),$

$(13, 76, 61, 116, 20, 28, 117, 14, 128, 9, 70, 59,$
$67, 40, 122, 50, 8, 102, 1, 30, 103, 12, 81, 95),$

$(107, 64, 4, 56, 75, 117, 123, 33, 54, 57, 1, 119,$
$114, 130, 113, 45, 36, 82, 124, 90, 81, 118, 70, 127),$

$(70, 71, 91, 11, 35, 105, 124, 111, 95, 117, 19, 76,$
$88, 63, 60, 100, 126, 34, 15, 29, 13, 12, 113, 28),$

$(52, 85, 75, 12, 127, 17, 37, 90, 126, 50, 10, 99,$
$54, 71, 79, 6, 118, 33, 45, 72, 16, 15, 47, 125),$

$(119, 35, 104, 105, 129, 0, 68, 61, 37, 36, 103, 51,$
$58, 86, 17, 63, 91, 121, 131, 53, 23, 126, 85, 18)$

under the action of the mapping $i \mapsto i + 8 \pmod{136}$. □

Lemma 26 *There exists a truncated cube design of order* 208.
Proof. Let the vertex set of K_{208} be $Z_{207} \cup \{\infty\}$. The decomposition

consists of the truncated cubes

$$(\infty, 53, 198, 171, 139, 116, 192, 103, 32, 44, 114, 157,\\ 204, 92, 105, 20, 138, 202, 99, 18, 130, 47, 109, 34),$$

$$(191, 86, 15, 194, 40, 153, 20, 68, 13, 118, 52, 71,\\ 147, 158, 37, 203, 152, 106, 140, 189, 95, 132, 63, 154),$$

$$(120, 188, 144, 189, 83, 149, 146, 94, 145, 46, 168, 66,\\ 31, 138, 45, 100, 184, 129, 79, 122, 40, 115, 8, 9),$$

$$(179, 149, 201, 130, 168, 132, 147, 153, 106, 41, 10, 184,\\ 1, 197, 0, 159, 164, 85, 57, 39, 73, 28, 64, 140),$$

$$(105, 193, 59, 183, 172, 26, 141, 129, 85, 176, 67, 107,\\ 20, 44, 121, 99, 51, 64, 168, 37, 65, 205, 101, 191),$$

$$(87, 129, 152, 193, 34, 46, 105, 48, 191, 66, 181, 61,\\ 206, 110, 77, 106, 28, 82, 76, 180, 2, 179, 10, 198),$$

$$(49, 59, 139, 181, 152, 189, 131, 137, 146, 38, 206, 24,\\ 79, 197, 144, 13, 16, 109, 15, 159, 32, 26, 107, 99),$$

$$(99, 148, 96, 178, 16, 38, 1, 31, 171, 80, 64, 134,\\ 142, 204, 107, 166, 22, 108, 157, 74, 174, 141, 181, 120),$$

$$(4, 201, 185, 57, 83, 87, 146, 167, 23, 137, 117, 155,\\ 66, 0, 156, 41, 15, 45, 125, 104, 188, 192, 162, 72),$$

$$(152, 168, 178, 122, 102, 140, 26, 105, 30, 89, 173, 110,\\ 108, 118, 92, 51, 171, 111, 180, 24, 114, 80, 42, 62)$$

under the action of the mapping $\infty \mapsto \infty$, $i \mapsto i+3 \pmod{207}$ for the first eight, and $i \mapsto i+9 \pmod{207}$ for the last two. □

Lemma 27 *There exists a decomposition of the complete 4-partite graph $K_{12,12,12,12}$ into 24 truncated cubes.*

Proof. Let the vertex set of $K_{12,12,12,12}$ be Z_{48} partitioned according to residue classes modulo 4. The decomposition consists of the truncated cubes

$$(0, 1, 2, 3, 4, 9, 6, 8, 11, 5, 7, 10,\\ 12, 18, 25, 13, 23, 16, 19, 30, 36, 14, 27, 32),$$

$$(0, 15, 25, 2, 11, 24, 5, 12, 34, 13, 46, 27,\\ 33, 7, 22, 36, 9, 26, 30, 47, 21, 23, 4, 37)$$

under the action of the mapping $i \mapsto i+4 \pmod{48}$. We will refer to this design as a truncated cube GDD of type 12^4. □

Lemma 28 *There exists a decomposition of the complete 4-partite graph* $K_{12,12,12,15}$ *into 27 truncated cubes.*
Proof. Let the vertex set of $K_{12,12,12,15}$ be $\{0, 1, \ldots, 50\}$ partitioned into $\{i : i < 36,\ i \equiv 0 \pmod 3\}$, $\{i : i < 36,\ i \equiv 1 \pmod 3\}$, $\{i : i < 36,\ i \equiv 2 \pmod 3\}$ and $\{36, 37, \ldots, 50\}$. The decomposition consists of the truncated cubes

$$(9, 10, 39, 31, 45, 8, 12, 38, 7, 29, 42, 4,\ 27, 25, 14, 15, 36, 11, 1, 47, 18, 6, 34, 43),$$

$$(15, 31, 44, 30, 46, 26, 27, 38, 28, 9, 41, 32,\ 3, 47, 5, 29, 21, 34, 35, 45, 24, 17, 39, 16),$$

$$(27, 1, 5, 44, 24, 10, 0, 43, 26, 28, 20, 46,\ 40, 14, 21, 47, 31, 2, 9, 45, 16, 34, 32, 3)$$

under the action of the mapping $i \mapsto i + 4 \pmod{36}$ for $i < 36$, $i \mapsto 36 + (i - 36 + 5 \bmod 15)$ for $i \geq 36$. We will refer to this design as a truncated cube GDD of type $12^3 15^1$. □

Lemma 29 *There exists a decomposition of the complete 4-partite graph* $K_{24,24,24,24}$ *into 96 truncated cubes.*
Proof. Let the vertex set of $K_{24,24,24,24}$ be Z_{96} partitioned according to residue classes modulo 4. The decomposition consists of the truncated cube

$$(0, 1, 3, 8, 2, 15, 4, 13, 23, 5, 19, 34,\ 18, 39, 61, 29, 92, 66, 55, 20, 78, 58, 33, 88)$$

under the action of the mapping $i \mapsto i + 1 \pmod{96}$. We will refer to this design as a truncated cube GDD of type 24^4. □

Lemma 30 *There exists a decomposition of the complete 4-partite graph* $K_{24,24,24,21}$ *into 90 truncated cubes.*
Proof. Let the vertex set of $K_{24,24,24,21}$ be $\{0, 1, \ldots, 92\}$ partitioned into $\{i : i < 72,\ i \equiv 0 \pmod 3\}$, $\{i : i < 72,\ i \equiv 1 \pmod 3\}$, $\{i : i < 72,\ i \equiv 2 \pmod 3\}$ and $\{72, 73, \ldots, 92\}$. The decomposition consists of the truncated cubes

$$(7, 45, 20, 64, 12, 26, 82, 1, 54, 55, 59, 42,\ 58, 80, 66, 87, 4, 30, 9, 14, 81, 89, 8, 49),$$

$$(40, 26, 79, 9, 16, 29, 61, 56, 91, 35, 36, 86,\ 70, 45, 88, 77, 50, 33, 51, 74, 44, 90, 3, 28),$$

$$(32, 15, 82, 34, 65, 84, 27, 81, 35, 76, 61, 0,\ 80, 23, 33, 66, 55, 72, 64, 62, 30, 17, 52, 60),$$

$(0, 1, 11, 6, 2, 13, 9, 7, 23, 3, 10, 78,$
$17, 19, 63, 22, 20, 77, 26, 55, 69, 36, 46, 92),$

$(0, 4, 23, 45, 10, 76, 5, 86, 31, 71, 85, 22,$
$20, 1, 75, 83, 19, 66, 69, 35, 87, 9, 32, 78)$

under the action of the mapping $i \mapsto i + 4 \pmod{72}$ for $i < 72$, $i \mapsto 72 + (i - 72 + 7 \bmod 21)$ for $i \geq 72$. We will refer to this design as a truncated cube GDD of type $24^3 21^1$. □

Lemma 31 *There exists a truncated cube design of order* $v = 72t + 64$ *for* $t \geq 0$.

Proof. Cases $v = 64, 136$ and 208 have already been established by Lemmas 24, 25 and 26 respectively. For the rest of the proof we require 4-GDDs of types 3^4, 3^5, [5], and $6^u 9^1$ for $u \geq 4$, [12]; see also [10].

Take a 4-GDD of type 3^4, inflate points in one of the groups by a factor of 21, inflate points in the other groups by a factor of 24, and adjoin an extra point, ∞. On every inflated group of the 4-GDD, together with ∞, place the truncated cube design of order 73 from Lemma 20 or the truncated cube design of order 64 from Lemma 24, as appropriate, and replace each block of the 4-GDD by the truncated cube GDD of type $24^3 21^1$ from Lemma 30. This construction yields a truncated cube design of order 280.

Take a 4-GDD of type 3^5 and proceed as before, except that the truncated cube GDD of type 24^4 from Lemma 29 replaces any block containing only points inflated by a factor of 24. This construction yields a truncated cube design of order 352.

Take a 4-GDD of type $6^u 9^1$, $u \geq 4$, inflate points in the groups of size 6 by a factor of 12, inflate points in the group of size 9 by a factor of 15, and adjoin an extra point, ∞. On every inflated group together with ∞, place the truncated cube design of order 73 from Lemma 20 or the truncated cube design of order 136 from Lemma 25, as appropriate. Replace each block of the 4-GDD by the truncated cube GDD of type 12^4 from Lemma 27 or the truncated cube GDD of type $12^3 15^1$ from Lemma 28, as appropriate. Applying this construction yields truncated cube designs of order $72t + 64$, $t \geq 5$. □

This completes the proof of Theorem 3.

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