Decycling the Cartesian Product of a Bipartite Graph with K_2

B.L.Hartnell
Saint Mary's University, Halifax, N.S., Canada B3H 3C3
e-mail: bert.hartnell@smu.ca

C.A.Whitehead
22 Leyfield Road, Sheffield, S17 3EE, UK
e-mail: cawhitehead@talktalk.net

October 24, 2011

Abstract

A *decycling set* in a graph G is a set D of vertices such that $G - D$ is acyclic. The *decycling number* of G, $\phi(G)$, is the cardinality of a smallest decycling set in G. We obtain sharp bounds on the value of the cartesian product $\phi(G \Box K_2)$ and determine its value in the case where G is the grid graph $P_m \Box P_n$ for all $m, n \geq 2$.

1 Introduction

A *decycling set* in a graph G, also known in the literature as a *vertex feedback set*, is a set D of vertices such that $G - D$ is acyclic. The *decycling number* of G, denoted by $\phi(G)$, is the cardinality of a smallest decycling set in G. We call a decycling set of minimum size a *ϕ-set* for G.

It has been shown by Karp [7] that the decision problem of finding $\phi(G)$ for an arbitrary graph G is NP-Complete. The problem remains difficult even when restricted to some well-known families of graphs, for example, bipartite graphs or planar graphs. Since Karp's paper appeared in 1972, the problem of determining the decycling number

JCMCC **80** (2012), pp. 113-125

of specific families of graphs has attracted much interest. It has been shown to be polynomial for several classes of graphs, see for example [3], [5], [8], [9], [10], and [17]. Other results on the decycling number can be found in [1], [2], [4], [11], [14], [15] and [16]. Of particular interest to this study, hypercubes are treated in [5] and [12]; grid graphs in [5]; the cartesian product of a graph with K_r for $r \geq 3$ in [6]; the cartesian product of two cycles in [13] and the box-cross product (also known as the strong product) of two paths in [18].

The study of the decycling number of the cartesian product of two graphs was initiated by Beineke and Vandell in [5]. The *cartesian product* $G := G_1 \Box G_2$ of two graphs G_1 and G_2 has $V(G) = V(G_1) \times V(G_2)$ and two vertices (u_1, u_2) and (v_1, v_2) are adjacent in G if and only if either (i) $u_1 = v_1$ and $u_2v_2 \in E(G_2)$ or (ii) $u_2 = v_2$ and $u_1v_1 \in E(G_1)$.

We shall also refer to a *vertex cover* of G. This is a set $Q \subseteq V(G)$ such that every edge of G has at least one endvertex in Q. The minimum cardinality of a vertex cover is denoted by $\alpha(G)$. A vertex cover of minimum size is called an *α-set.*

In [5] (Theorem 1.8), sharp bounds for $\phi(G \Box K_2)$ are obtained for an arbitrary graph G in terms of $\phi(G)$ and $\alpha(G)$.

Theorem 1.1 (Beineke, Vandell) *For any graph G,*

$$2\phi(G) \leq \phi(G \Box K_2) \leq \phi(G) + \alpha(G).$$

Although the lower bound is sharp (it is achieved when G is the complete graph K_n, $n \geq 3$, for example) it is not useful in the case of graphs which have relatively few cycles. In Section 2, we show it can be improved as follows.

Theorem 1.2 *For any graph G,*

$$\phi(G \Box K_2) \geq \max\{2\phi(G), \alpha(G)\}.$$

The lower bound $\alpha(G)$ for $\phi(G \Box K_2)$ is also sharp. It is achieved when G is acyclic and indeed by a cactus containing no even cycle. The upper bound for $\phi(G \Box K_2)$ is achieved, for example, when G is any even cycle.

In [6], it is shown that when G is a member of one of several families of graphs, it is possible to find a general expression for $G\Box K_r$, when $r \geq 3$, in terms of the order of G and its decycling number.

Theorem 1.3 *Let G be a connected graph of order $n \geq 2$. When G is a cactus, a bipartite graph or a graph of maximum vertex degree 3 (other than K_4), then*

$$\phi(G\Box K_r) = \begin{cases} n + \phi(G) & \text{if } r = 3 \\ n(r-2) & \text{if } r \geq 4 \end{cases}$$

We show in Section 2 that it is not possible to obtain a general result of this kind for $G\Box K_2$ when G is bipartite or even when G is a cactus. Indeed, it is possible to construct an example of a bipartite cactus G such that $G\Box K_2 = \alpha(G) + k$, for any non-negative integer $k \leq \phi(G)$.

A further example of a family of bipartite graphs where different members achieve the upper and the lower bound of Theorem 1.1 is the family of hypercubes Q_n. We can regard Q_n as the cartesian product $Q_{n-1}\Box K_2$, $n \geq 2$, or as a graph having the set of 2^n-tuples of 0's and 1's as vertices, where two vertices are adjacent if they differ in just one position. This latter definition implies $\alpha(Q_n) = 2^{n-1}$. The values of $\phi(Q_n)$ are determined in [5] for $1 \leq n \leq 8$ and give this interesting result.

$$\phi(Q_n\Box K_2) = \begin{cases} 2\phi(Q_n) & \text{if } n = 3,5,6,7 \\ \phi(Q_n) + \alpha(Q_n) & \text{if } n = 1,2,4 \end{cases}$$

Bounds on the value of $\phi(Q_n)$ for $n \geq 9$ are given in [5] and improved in [12], but it is not clear whether the sequence $\phi(Q_n\Box K_2) = 2\phi(Q_n)$ continues beyond $n = 7$.

Although, as these comments indicate, we can say nothing in general about bipartite graphs, in Section 3 we determine $\phi(G\Box K_2)$ in the case where G is the grid graph $P_m\Box P_n$, for all $m, n \geq 2$.

All graphs considered in this paper are simple. We use the following notation. For $X \subseteq V(G)$, $\langle X\rangle$ denotes the subgraph of G induced by X. We normally regard the graph $G\Box K_2$ as K_2 in which each vertex is replaced by a copy of G. We label the copies G, G' and give label u' to the vertex of G' corresponding to the vertex u

of G. Similarly, for any set $S \subseteq V(G)$, $S' \subseteq V(G')$ denotes the set of copies of the vertices in S. Since the case when $G \cong K_1$ is trivial, we shall assume that G has order at least 2.

2 Preliminary results

We can improve the lower bound on $\phi(G\Box K_2)$ given in Theorem 1.1 by noting that corresponding to each edge uv of G, $G\Box K_2$ contains a 4-cycle $\{u, v, v', u'\}$ and hence any decycling set D for $G\Box K_2$ contains at least one of these four vertices. Thus the set $Q := \{x : x \in D \text{ or } x' \in D\}$ is a vertex cover of G so that $\phi(G\Box K_2) \geq \alpha(G)$. This gives the following result.

Theorem 2.1 *For any graph G,*

$$\max\{2\phi(G), \alpha(G)\} \leq \phi(G\Box K_2) \leq \phi(G) + \alpha(G).$$

Corollary 2.2 *For any acyclic graph G, $\phi(G\Box K_2) = \alpha(G)$.*

When $\alpha(G) \geq 2\phi(G)$, it may under some circumstances be possible to partition an α-set Q in G into two disjoint decycling sets for G, say $Q := D_1 \cup D_2$, so that $D_1 \cup D_2'$ is a ϕ-set in $G\Box K_2$. In the case when G is acyclic, D_1 can be any arbitrarily chosen subset of Q. The following lemma gives a necessary condition for such a partition to be possible when G contains cycles.

Lemma 2.3 *Let G be a graph containing a cycle C and let $D_1, D_2 \subseteq V(G)$ be such that $D := D_1 \cup D_2'$ is a ϕ-set for $G\Box K_2$. Then if $D_1 \cap D_2 \cap V(C) = \emptyset$, there exist vertices $x \in D_1$, $y \in D_2$ such that x, y are adjacent on C.*

Proof. Suppose $D_1 \cap D_2 \cap V(C) = \emptyset$ but no vertex $x \in D_1$ is adjacent on C to a vertex $y \in D_2$. Then there is a cycle C^* in $G\Box K_2 - D$ such that when the two copies of G are merged, C^* gives C, a contradiction. □

Corollary 2.4 *Let G be a graph such that no α-set for G contains a pair of adjacent vertices of each cycle in G. Then $\phi(G\Box K_2) > \alpha(G)$.*

Lemma 2.5 *Let G be a graph such that no α-set for G contains a pair of adjacent vertices of any cycle in G. Then $\phi(G\Box K_2) = \alpha(G) + \phi(G)$.*

Proof. Let $D_1, D_2 \subseteq V(G)$ be such that $D := D_1 \cup D_2'$ is a ϕ-set for $G\Box K_2$. Then $D_1 \cup D_2$ contains an α-set Q for G. If G is acyclic, the result is true by Corollary 2.2, so suppose otherwise and let C be a cycle in G. Then by Lemma 2.3, $D_1 \cup D_2$ contains a vertex of C that is not in Q. Thus $D_1 \cup D_2 = Q \cup X$, where $Q \cap X = \emptyset$ and X contains at least one vertex of each cycle in G. This implies $|D| \geq \alpha(G) + \phi(G)$ and the result follows from Theorem 2.1. □

Proposition 2.6 *Let G be an r-cycle. Then*

$$\phi(G\Box K_2) = \lfloor r/2 \rfloor + 1.$$

Proof. When r is even, the result follows immediately from Lemma 2.5. When r is odd, let Q be an α-set for G. Then Q contains a pair of adjacent vertices, say x, y. In this case, set $D_1 := Q \setminus \{x\}$ and $D_2 = \{x\}$. It is easily verified that $D_1 \cup D_2'$ is a decycling set for $G\Box K_2$. By Theorem 2.1 no smaller decycling set is possible. □

It follows from Proposition 2.6 that $\phi(G\Box K_2)$ achieves the upper bound in Theorem 2.1 when G is an even cycle and the lower bound when G is an odd cycle. We can generalise this result to a certain extent.

A *cactus* is a graph in which no two cycles have a common edge.

Proposition 2.7 *Let G be a cactus admitting an α-set Q that contains a pair of adjacent vertices of each cycle in G. Then*

$$\phi(G\Box K_2) = \alpha(G)$$

Proof. Let $C_1, \ldots, C_k$ be the cycles in G and suppose that x_i, y_i is the pair of adjacent vertices on C_i such that $x_i, y_i \in Q$, $i = 1, \ldots k$. Set $D_1 := Q \setminus \{y_1, \ldots y_k\}$ and $D_2 := \{y_1, \ldots y_k\}$. Then $D := D_1 \cup D_2'$ is a decycling set for $G\Box K_2$. The result follows from Theorem 2.1. □

In particular, $\phi(G\Box K_2)$ achieves the lower bound of Theorem 2.1 when G is a cactus in which every cycle is odd. By Lemma 2.5, it achieves the upper bound when G is a cactus in which each cycle C in G is even and no pair of adjacent vertices of C is contained in any minimum vertex cover of G.

Proposition 2.8 *Let G be the complete bipartite graph $K_{m,n}$, where $m \le n$. Then $\phi(G\Box K_2) = 2m - 1$.*

Proof. It is easily seen that $\phi(G) = m - 1$ and $\alpha(G) = m$, so that $2m - 2 \le \phi(G\Box K_2) \le 2m - 1$, by Theorem 2.1. The result is true when $m = n = 2$, from Proposition 2.6. Now consider $n \ge 3$. Let (A, B) be the bipartition of $V(G)$, where $|A| = m$ and let D be a decycling set for $G\Box K_2$. Then D contains at least $m - 1$ vertices of each of A and A'. Suppose there are vertices $a_i \in A$ and $a'_j \in A'$ such that $a_i, a'_j \notin D$, where $1 \le i, j \le m$. However, if $i = j$, then $(G\Box K_2) - D$ contains the 4-cycle $a_i b_1 b'_1 a'_i$ and if $i \ne j$, then $(G\Box K_2) - D$ contains the 6-cycle $a_i b_1 b'_1 a'_j b'_2 b_2$, a contradiction. Thus $|D| = 2m - 1$. □

3 Grid graphs

An $m \times n$ *grid* is a graph $P_m \Box P_n$, where P_k denotes a path of order k. We shall assume that $m, n \ge 2$. Denote the ith vertex in the jth copy of P_m (that is, the vertex in the ith row and jth column of the grid) by $v_{i,j}$ and let C_j denote the set of vertices in the jth copy of P_m (column of G), $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots n$.

Proposition 3.1 *Let $G := P_{2t}\Box P_n$, where $t \ge 1, n \ge 2$. Then*

$$\phi(G\Box K_2) = \begin{cases} 3tr & \text{if } n{=}2r \\ t(3r+1) & \text{if } n{=}2r{+}1 \end{cases}$$

Proof. Let D_1, D_2 be decycling sets for G such that $D := D_1 \cup D'_2$ is a ϕ-set for $G\Box K_2$. First note that the vertices of the first $2r$ columns of G can be partitioned into tr mutually disjoint sets, $X_1, \ldots, X_{tr}$, such that X_i induces a 4-cycle, $i = 1, \ldots, tr$; and when $n = 2r + 1$, the vertices of the last column can be partitioned into

t disjoint pairs, $Y_1, \ldots, Y_t$, so that Y_j induces an edge of G, $j = 1, \ldots, t$. Then in $G \Box K_2$, $\langle X_i \rangle \Box K_2$ contributes at least 3 vertices to D, $i = 1, \ldots, tr$, by Proposition 2.6; and since $\langle Y_j \rangle \Box K_2$ is a 4-cycle, it contributes at least one vertex to D, $j = 1, \ldots, t$. Thus $\phi(G \Box K_2) \geq 3tr$ when $n = 2r$ and $\phi(G \Box K_2) \geq 3tr + t$ when $n = 2r + 1$.

It remains to show that we can decycle $G \Box K_2$ with this number of vertices in either case. We define sets D_1, D_2 as follows:

$$D_1 \cap C_j = \begin{cases} \{v_{2i,j} : 1 \leq i \leq t\} & \text{if } j = 1 + 4q,\ 0 \leq q \leq (n-1)/4 \\ \{v_{2i-1,j} : 1 \leq i \leq t\} & \text{if } j = 2q,\ 1 \leq q \leq n/2 \\ \emptyset & \text{if } j = 3 + 4q,\ 0 \leq q \leq (n-3)/4 \end{cases}$$

$$D_2 \cap C_j = \begin{cases} \emptyset & \text{if } j = 1 + 4q, 0 \leq q \leq (n-1)/4 \\ \{v_{2i,j} : 1 \leq i \leq t\} & \text{if } j = 2q,\ 1 \leq q \leq n/2 \\ \{v_{2i-1,j} : 1 \leq i \leq t\} & \text{if } j = 3 + 4q,\ 1 \leq q \leq (n-3)/4 \end{cases}$$

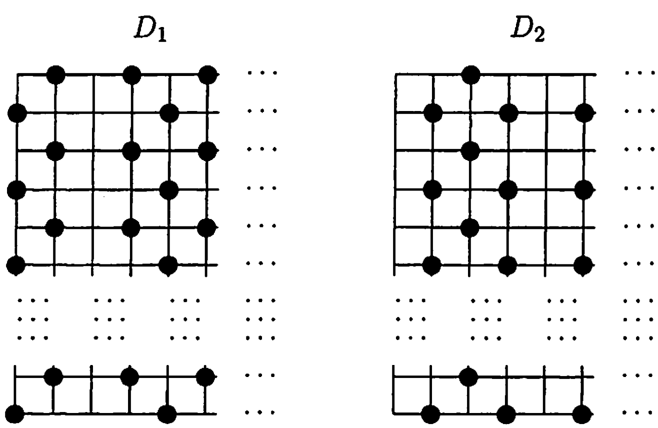

Figure 1

In Figure 1, the first six columns of G are shown, with the vertices of D_1 and D_2 in bold type (the remaining vertices of G are shown only as grid intersections to make the diagram clearer to read). It is easily seen that D_1 and D_2 are decycling sets for G. Let $D := D_1 \cup D_2'$. Then in $(G \Box K_2) - D$, no edge between G and G' has an end vertex in C_{2j}, $j = 1, 2, \ldots$. Further, the vertices of $C_{4j+1} \setminus D_1$ for $0 \leq j \leq (n-1)/4$ are isolates in $G - D_1$ and hence they are leaves in $(G \Box K_2) - D$. Similarly, the vertices of $C'_{4j+3} \setminus D'_2$ for $0 \leq j \leq (n-3)/4$ are isolates in $G' - D'_2$ and so also are leaves in $(G \Box K_2) - D$. It follows that D is a decycling set for $G \Box K_2$. The result follows by noting that $D_1 \cup D_2$ contains in total just $2t$ vertices of C_{2j} and just t vertices of C_{2j-1} for $j = 1, 2, \ldots,$. □

We remark here that if $H := P_p \Box P_q$, where $2 \leq p \leq m$ and $2 \leq q \leq n$, is a subgraph of $G := P_m \Box P_n$ and $D := D_1 \cup D'_2$ is a decycling set for $G \Box K_2$, then the vertices of $(V(H) \cap D_1) \cup (V(H) \cap D_2)'$ form a decycling set for $H \Box K_2$. We make repeated use of this observation in proving the following results.

Lemma 3.2 *Let $G := P_{2p+1} \Box P_{2q+1}$, where $p \geq q \geq 1$. Then*

$$\phi(G \Box K_2) \leq \begin{cases} (3r+2)(2p+1) & \text{if } q = 2r+1,\ r \geq 0 \\ 3r(2p+1) + p & \text{if } q = 2r,\ r \geq 1 \end{cases}$$

Proof. Let $D := D_1 \cup D'_2$ be the decycling set for $G \Box K_2$ defined in Proposition 3.1 and illustrated in Figure 1, restricted to the first $2p+1$ rows and the first $2q+1$ columns of G. There are two cases to consider.

Case (i) $q = 2r+1$, $r \geq 0$. In this case, there are exactly $3r+2$ vertices in each row of $D_1 \cup D_2$, giving the result.

Case (ii) $q = 2r$, $r \geq 1$. The total number of vertices of $D_1 \cup D_2$ in row $2i+1$ is $3r$ and that in row $2i$ is $3r+1$, $0 \leq i \leq p$. Hence in this case $\phi(G \Box K_2) \leq p(6r+1) + 3r = 3r(2p+1) + p$. □

Lemma 3.3 *Let $G := P_3 \Box P_3$. Then $\phi(G \Box K_2) = 6$. Further, if a decycling set D for $G \Box K_2$ contains two copies of $v_{1,1}$ or $v_{3,3}$, then $|D| \geq 7$ and if D contains two copies of both $v_{1,1}$ and $v_{3,3}$, then $|D| \geq 8$.*

Proof. Let $D := D_1 \cup D'_2$ be a decycling set for $G \Box K_2$. Note G contains four 4-cycles Q_1, Q_2, Q_3, Q_4, where $v_{1,1} \in V(Q_1)$, $v_{1,3} \in V(Q_2)$, $v_{3,3} \in V(Q_3)$ and $v_{3,1} \in V(Q_4)$. Let C denote the 8-cycle $G - v_{2,2}$.

Now, in $G\Box K_2$, each prism $Q_i\Box K_2$ contains three vertices of D, $1 \leq i \leq 4$, by Proposition 2.6. Let $E_1 := v_{1,3}v_{2,3}$, $E_2 := v_{3,1}v_{3,2}$. Then $D_1 \cup D_2$ contains at least three vertices of $Q_1\Box K_2$ and at least one vertex of $E_i\Box K_2$, $i = 1, 2$, giving $|D| \geq 5$. However, $|D| = 5$ only if D contains at least one copy of $v_{2,2}$ and so at most four vertices of the prism $C\Box K_2$. But again by Proposition 2.6, D contains at least 5 vertices of $C\Box K_2$ and hence $\phi(G\Box K_2) \geq 6$. It follows from case (i) of Lemma 3.2 that $\phi(G\Box K_2) = 6$.

Now suppose D contains two copies of $v_{1,1}$ or of $v_{3,3}$. However, D contains at least three further vertices of the prism $C\Box K_2$ and hence D contains at most one copy of $v_{2,2}$. But then it is not possible to choose the three remaining vertices of $D \cap V(C)$ so that D contains three vertices of $Q_i\Box K_2$, $1 \leq i \leq 4$. This contradiction shows that $|D| \geq 7$.

Lastly, suppose D contains two copies both of $v_{1,1}$ and of $v_{3,3}$. However, D also contains three vertices from each of Q_2 and Q_4 and this cannot be achieved by adding less than four additional vertices to D. Hence $|D| \geq 8$. □

Lemma 3.4 *Let $G := P_{4r+3}\Box P_{4r+3}$, where $r \geq 0$. Then*

$$\phi(G\Box K_2) = (3r + 2)(4r + 3).$$

Further, if D is a decycling set for $G\Box K_2$ containing two copies of $v_{1,1}$ or of $v_{4r+3,4r+3}$, then $|D| \geq (3r+2)(4r+3)+1$ and if D contains two copies of both $v_{1,1}$ and $v_{4r+3,4r+3}$, then $|D| \geq (3r+2)(4r+3)+2$.

Proof. We prove this result by induction. Note that it is true when $r = 0$, by Lemma 3.3. Suppose then $r \geq 1$. Let D_1, D_2 be decycling sets for G such that $D := D_1 \cup D_2'$ is a decycling set for $G\Box K_2$. Let H_1 be a 3×3 subgrid of G induced by the vertices in the first three rows and columns of G and H_2 be the $(4r - 1) \times (4r - 1)$ subgrid induced by the vertices in the last $4r - 1$ rows and columns of G. Let $X_1 := \{v_{4,3}\}$,$X_2 := \{v_{5,3}, v_{5,4}\}$, $X_3 := \{v_{3,4}, v_{4,4}, v_{4,5}, v_{3,5}\}$ and partition the remaining vertices of $V(G) - (V(H_1) \cup V(H_2))$ into $8r - 2$ subsets, $Y_1, Y_2, \ldots, Y_{8r-2}$ such that $\langle Y_j \rangle$ is a 4-cycle, $j = 1, \ldots, 8r - 2$ (the case where $r = 1$ is illustrated in Figure 2). Then D contains at least the following number of vertices: six of $H_1\Box K_2$ by Lemma 3.3, $(3r - 1)(4r - 1)$ vertices of H_2 by assumption, three of each of the $8r-2$ prisms $\langle Y_j \rangle \Box K_2$, three of the prism $\langle X_3 \rangle \Box K_2$ and one vertex of $\langle X_2 \rangle \Box K_2$, giving $|D| \geq (3r+2)(4r+3)-1$.

Suppose if possible $|D| = (3r+2)(4r+3)-1$. Then $v_{4,3} \notin D_1 \cup D_2$. Further, $(D_1 \cup D_2) \cap X_2 := \{v_{5,3}\}$ (in order to cover the edges $v_{4,3}v_{5,3}$ and $v_{5,3}v_{5,4}$) so that $v_{5,4} \notin D_1 \cup D_2$ and D contains only one copy of $v_{5,3}$. Now repeating this argument with $Z_1 := \{v_{4,5}\}$, $Z_2 := \{v_{3,4}, v_{3,5}\}$, $Z_3 := \{v_{4,3}, v_{4,4}, v_{5,4}, v_{5,3}\}$ in place of X_1, X_2, X_3 respectively, implies that the vertices $v_{4,5}$ and $v_{3,4}$ are not in $D_1 \cup D_2$ and D contains just one copy of $v_{3,5}$. However D contains three vertices of each of $X_3 \Box K_2$ and $Z_3 \Box K_2$ and so we may assume D contains two copies of $v_{4,4}$. Further, D contains at least 5 vertices of the prism $C \Box K_2$, where $C := v_{3,3}v_{3,4}v_{3,5}v_{4,5}v_{5,5}v_{5,4}v_{5,3}v_{4,3}$, and hence two copies of $v_{3,3}$ or $v_{5,5}$. The first alternative is ruled out by Lemma 3.3 and the second by hypothesis. Hence $\phi(G \Box K_2) \geq (3r+2)(4r+3)$ and equality follows from Lemma 3.2.

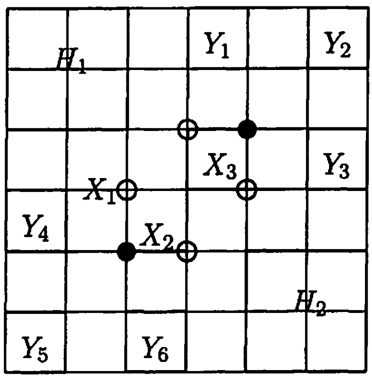

Figure 2

Let D be a ϕ-set for $G \Box K_2$. Then $|D| = (3r+2)(4r+3)$ and as noted above, D contains at least 5 vertices of the prism $C \Box K_2$. Suppose first this includes no more than one copy of $v_{3,3}$ or of $v_{5,5}$. If D contains two copies of $v_{1,1}$ or $v_{4r+3,4r+3}$, then D restricted to $H_i \Box K_2$ contains at least $\phi(H_i \Box K_2)+1$ vertices, $i=1$ or $i=2$, by Lemma 3.3 or the induction hypothesis. Hence if D contains just one of $v_{1,1}$ and $v_{4r+3,4r+3}$, then $|D| \geq (3r+2)(4r+3)+1$ and if D contains two copies of both $v_{1,1}$ and $v_{4r+3,4r+3}$, then $|D| \geq (3r+2)(4r+3)+2$.

Now suppose that D contains two copies of $v_{3,3}$ or $v_{5,5}$, say of $v_{3,3}$. Then again if D contains two copies of just one of $v_{1,1}$ and $v_{4r+3,4r+3}$, we have $|D| \geq (3r+2)(4r+3)+1$ and if two copies of both, then $|D| \geq (3r+2)(4r+3)+2$. □

Proposition 3.5 *Let $G := P_{2p+1} \square P_{2q+1}$, where $p \geq q \geq 1$. Then*

$$\phi(G \square K_2) = \begin{cases} (3r+2)(2p+1) & \textit{if } q = 2r+1,\ r \geq 0 \\ 3r(2p+1)+p & \textit{if } q = 2r,\ r \geq 1 \end{cases}$$

Proof. Let D be a decycling set for G.

Case (i) $q = 2r+1$, $r \geq 0$. Let H_1, H_2 be the subgraphs of G induced by the vertices in the first $4r+3$ rows and in the remaining $2(p-2r-1)$ rows of G respectively. Then D contains at least $(3r+2)(4r+3)$ vertices of $H_1 \square K_2$, by Lemma 3.4, and at least $(p-2r-1)(3q+1))$ vertices of $H_2 \square K_2$, by Proposition 3.1, giving $|D| \geq (2p+1)(3r+2)$. Equality follows from Lemma 3.2.

Case (ii) $q = 2r$, $r \geq 1$. Let H be the subgraph of G induced by the vertices in the first $4r-1$ columns of G. Partition the vertices in the remaining two columns of G into p sets X_j such that $\langle X_j \rangle$ is a 4-cycle in G, $1 \leq j \leq p$, and a set Y such that $\langle Y \rangle$ is an edge of G. Then D contains at least $(3r-1)(2p+1)$ vertices of $H \square K_2$, by case (i), at least three vertices from each of the prisms $\langle X_j \rangle \square K_2$ and at least one vertex of $\langle Y \rangle \square K_2$, giving $|D| \geq 3r(2p+1)+p$. Again, equality follows from Lemma 3.2. □

References

[1] N.Alon, D. Mubayi and R. Thomas, *Large induced forests in sparse graphs*, J. Graph Theory **38** (2001), 3, pp.113-123.

[2] Sheng Bau and Lowell W. Beineke, *The decycling number of graphs*, Australas. J. Combin. **25** (2002), 285-298.

[3] Sheng Bau, Lowell W. Beineke and Robert C. Vandell, *Decycling snakes*, Cong. Num. **134** (1998), 79-87.

[4] Sheng Bau, Nicholas C. Wormald, and Sanming Zhou, *Decycling numbers of random regular graphs*, Random Structures and Algorithms **21** (2002), 3-4, 397-413.

[5] Lowell W. Beineke and Robert C. Vandell, *Decycling graphs*, J. Graph Theory **25** (1997), 1, 59-77.

[6] B.L. Hartnell and C.A. Whitehead, *Decycling sets in certain cartesian product graphs with one factor complete*, Australas. J. Combin **40** (2008), 305-315.

[7] R.M. Karp, *Reducibility among combinatorial problems, Complexity of Computer Computations*, R.E. Miller and J.W. Thatcher, ed., Plenum Press, New York-London (1972), 85-103.

[8] Y.D. Liang, *On the feedback vertex set problem in permutation graphs*, Inf Process Lett **52** (1994), 123-129.

[9] Y.D. Liang and M.-S. Chang, *Minimum feedback vertex sets in cocomparability graphs and convex bipartite graphs*, Acta Inf **34** (1997), 337-346.

[10] D-M Li and Y-P Liu, *A polynomial algorithm for finding the mimimum feedback vertex set of a 3-regular simple graph*, Acta Math. Sci. **19** (1999), 4, 375-381.

[11] J.-P. Liu and C. Zhao, *A new bound on the feedback vertex sets in cubic graphs*, Discrete Math **148** (1996), 119-131.

[12] David A. Pike, *Decycling hypercubes*, Graphs Combin. **19** (2003), 4, 547-550.

[13] David A. Pike and Yubo Zou, *Decycling cartesian products of two cycles*, SIAM J. Discrete Math **19** (2005), 3, 551-663.

[14] Narong Punnim, *The decycling number of cubic graphs*, Combinatorial Geometry and Graph Theory, pp. 141- 145, Lecture Notes in Computing Sci., **3330**, Springer, Berlin, 2005.

[15] Narong Punnim *Decycling regular graphs*, Australas. J. Combin. **32** (2005), 147-162.

[16] Narong Punnim *Decycling connected regular graphs*, Australas. J. Combin. **35** (2006), 115-169.

[17] S. Ueno, Y. Kajitani and S. Gotoh, *On the non-separating independent set problem and feedback set problem for graphs with no vertex degree exceeding three*, Discrete Math **72** (1988), 355-360.

[18] Jin-shan Xie, *Decycling number of box-cross product of two paths*, J.Fuzhou Univ. Nat. Sci. Ed. **35** (2007), no 1, 16-19.