Further New Properties of Divisor Graphs

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Abstract

In this paper, we characterize the graphs G and H for which the Cartesian product $G \square H$ is a divisor graph. We show that divisor graphs form a proper subclass of perfect graphs. Also we prove that cycle permutation graphs of order at least 8 are divisor graphs if and only if they are perfect. Some results concerning amalgamation operations about obtaining new divisor graphs from old ones are given. Viewing block graphs as vertex amalgamations of complete graphs, we characterize those block graphs that are divisor graphs.

1 Introduction

Let S be a finite nonempty set of positive integers. The *divisor graph* $G(S)$ of S has S as its vertex set and two distinct vertices i and j are adjacent if either $i \mid j$ or $j \mid i$, that is if $\gcd(i, j) = \min\{i, j\}$. A graph G is a *divisor graph* if $G = G(S)$ for some finite nonempty set S of positive integers. For $G_n = G(S)$ where $S = \{1, 2, \cdots, n\}$, the length $f(n)$ of a longest path in G_n was studied in [8], [13], and [14]. It was shown in [5] that complete graphs, bipartite graphs, complete multipartite graphs and joins of divisor graphs are divisor graphs. Powers of paths and powers of cycles which are divisor graphs were characterized in [1] and [2], respectively. A characterization

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of nontrivial connected divisor graphs in terms of the upper orientable hull number was obtained in [4].

Divisor graphs do not contain induced odd cycles of length greater than 3, but they may contain triangles, for example complete graphs are divisor graphs, see [5]. Divisor graphs with triangles were studied in [9], where a forbidden subgraph characterization for all divisor graphs containing at most three triangles was obtained.

The following two results were proved in [5], the later one also appears in [15] with a different proof.

Lemma 1. *Every induced subgraph of a divisor graph is a divisor graph.*

Lemma 2. *No divisor graph contains an induced odd cycle of length greater than* 3.

In a digraph D, a *transmitter* is a vertex having indegree 0, a *receiver* is a vertex having outdegree 0, while a vertex v is a *transitive vertex* if it has both positive outdegree and positive indegree such that $(u, w) \in E(D)$ whenever (u, v) and (v, w) belong to $E(D)$. An orientation D of a graph G in which every vertex is a transmitter, a receiver, or a transitive vertex is called a divisor orientation of G, see [1].

A different point of view to introduce divisor graphs can be achieved by the following characterization given in [5], which will be frequently used in this paper.

Lemma 3. *A graph G is a divisor graph if and only if G has a divisor orientation.*

In [5], the graphs H for which the Cartesian product $H \square K_2$ is a divisor graph were characterized. We will determine in section 2 when the Cartesian product of two graphs is a divisor graph. In section 3, we prove that divisor graphs form a proper subclass of perfect graphs. In section 4, we determine which cycle permutation graphs are perfect, and which are divisor graphs.

In the last section, we consider amalgamations of divisor graphs that produce divisor graphs. Block graphs need not be divisor graphs, see the graph in Figure 1. This graph was given as an example of a nondivisor graph in [5]. But viewing block graphs as vertex amalgamations of complete graphs, we characterize those block graphs that are divisor graphs.

For undefined notions, the reader is referred to [3].

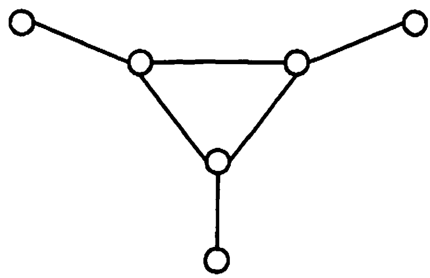

Figure 1: A block graph which is not a divisor graph.

2 Cartesian products which are divisor graphs

The Cartesian product $G_1 \Box G_2$ of two bipartite graphs G_1 and G_2 with partite sets U_1, W_1 and U_2, W_2, respectively, is also bipartite with partite sets $(U_1 \times U_2) \cup (W_1 \times W_2)$ and $(W_1 \times U_2) \cup (U_1 \times W_2)$. Therefore, since every bipartite graph is a divisor graph, see [5], we have the following result.

Lemma 4. *Let G and H be two bipartite graphs. Then $G \Box H$ is a divisor graph.*

The following two results were shown in [5] and [1], respectively.

Lemma 5. *The graph $K_2 \Box K_3$ is not a divisor graph.*

Lemma 6. *A graph G is a divisor graph if and only if each component of G is a divisor graph.*

Now we are in a position to characterize when the Cartesian product of two graphs is a divisor graph.

Theorem 7. *Let G and H be two graphs. Then $G \Box H$ is a divisor graph if and only if either both G and H are bipartite or at least one of them has size zero and the other is a divisor graph.*

Proof. Suppose the graph $G \Box H$ is a divisor graph and at least one of the two graphs G and H, say G, is not bipartite. Then G has an odd cycle. But G has no induced odd cycle of length greater than 3, for otherwise $G \Box H$ would not be a divisor graph according to Lemma 2. Thus G must have a triangle . Then H must be an edgeless graph, for if it has an edge, then the graph $K_2 \Box K_3$ (which is, by Lemma 5, not a divisor graph) would be an induced subgraph of $G \Box H$, which contradicts Lemma 1. Now since H is an edgeless graph, the graph $G\Box H$ consists of $|V(H)|$ copies of G. Therefore, by Lemma 6, G must be a divisor graph.

The converse follows by Lemma 4 and Lemma 6. □

By Theorem 7 and Lemma 2, we have the following corollary.

Corollary 7.1. *Let G and H be two divisor graphs. Then $G\Box H$ is a divisor graph if and only if either both G and H have no triangles or at least one of them has size zero.*

In [5], it was shown that the Cartesian product $G\Box K_2$ is a divisor graph if and only if G is bipartite. This result follows from Theorem 7.

3 Divisor graphs are perfect

A graph G is *perfect* if every induced subgraph of G has chromatic number equal to the size of a largest clique contained in the subgraph. An induced cycle of length at least 4 is called a *hole*, an induced subgraph that is the complement of a hole is called an *antihole*. Holes and antiholes are *odd* or *even* according to the parity of their number of vertices. A graph that does not contain any odd holes or odd antiholes is called a *Berge graph*, see [7]. Perfect graphs are Berge graphs, the converse was announced by Claude Berge in 1960 as a conjecture which is known as the *strong perfect graph conjecture*. In 2002, after more than four decades, this conjecture was proved by Maria Chudnovsky, Paul Robertson, Neil Seymour, and Robin Thomas. They published this result in 2006. Next, we state the *strong perfect graph Theorem.*

Theorem 8 (Chudnovsky, Robertson, Seymour, and Thomas [6]). *A graph is perfect if and only if it is a Berge graph.*

By Lemma 2, divisor graphs have no odd holes. Thus to show that divisor graphs are perfect, it would be enough to prove that divisor graphs also have no odd antiholes. Indeed, we will see in this section that they have neither odd nor even antiholes of order greater than 4. To this end we need the following three lemmas, the first one was shown in [1].

Lemma 9. *If D is a divisor orientation of a graph G, then the converse of D is also a divisor orientation of G.*

Lemma 10. *If D is a divisor orientation of the antihole $\overline{C_m}$, where $m \geq 5$, then every vertex of D is either a transmitter or a receiver.*

Proof. Let C_m be the cycle $12\cdots m1$, where $m \geq 5$. Assume that D is a divisor orientation of the antihole $\overline{C_m}$. In view of Lemma 9, it would be enough to show that a vertex with positive outdegree must be a transmitter. So, without loss of generality, suppose that the vertex 1 has a positive outdegree and let $(1,k) \in E(D)$ for some $k \in \{3,4,\cdots,m-1\} = N(1)$. Then for every $i_1 \in \{3,4,\cdots,m-1\}$ satisfying $|i_1 - k| = 1$ we have $i_1k \notin E(\overline{C_m})$. This implies that $(1,i_1) \in E(D)$, for otherwise we would have $(i_1,1),(1,k) \in E(D)$ while $(i_1,k) \notin E(D)$. Similarly, for each $i_2 \in \{3,4,\cdots,m-1\}$ with $|i_2 - k| = 2$, we get $(1,i_2) \in E(D)$ because $(1,i_1) \in E(D)$ for each $i_1 \in \{3,4,\cdots,m-1\}$ with $|i_1 - k| = 1$. Then, applying a similar argument repeatedly we conclude that for $t \in \{2,3,\cdots,\max\{m-1-k,k-3\}\}$, we have $(1,i_t) \in E(D)$ for each $i_t \in \{3,4,\cdots,m-1\}$ satisfying $|i_t - k| = t$. Therefore $(1,i) \in E(D)$ for each $i \in \{3,4,\cdots,m-1\} = N(1)$ and hence 1 is a transmitter. □

Lemma 11. *Every antihole of order greater than* 4 *is not a divisor graph.*

Proof. If $m = 5$, then the antihole $\overline{C_5} = C_5$ is, by Lemma 2, not a divisor graph. So, let C_m be the cycle $12\cdots m1$, where $m > 5$, and assume to the contrary that the antihole $\overline{C_m}$ is a divisor graph. Then, by Lemma 3, $\overline{C_m}$ has a divisor orientation D. According to Lemma 9, we can assume that $(1,3) \in E(D)$. Then, by Lemma 10, the vertex 1 is a transmitter in D. Therefore, again by Lemma 10, the vertices $3,4,\cdots,m-1$ are receivers in D. Since $m > 5$, we get $(3)(m-1) \in E(\overline{C_m})$ and hence either $(3,m-1) \in E(D)$ or $(m-1,3) \in E(D)$. This is a contradiction because both 3 and $m-1$ are receivers. □

The following result is a direct consequence of Lemma 11 and Lemma 1.

Theorem 12. *Let G be a divisor graph. Then G has no antihole of order greater than* 4.

Now we conclude that divisor graphs are perfect.

Theorem 13. *Every divisor graph is perfect.*

Proof. The result follows by Lemma 2, Theorem 12, and Theorem 8. □

Note that a perfect graph need not be a divisor graph, for example the perfect graph $K_2 \square K_3$ is, by Lemma 5, not a divisor graph.

The complement of a perfect graph is perfect, but the complement of a divisor graph is not necessarily a divisor graph. The path P_4 is a divisor graph whose complement is also a divisor graph, while any even cycle of length at least 6 is a divisor graph whose complement is, by Theorem 12, not a divisor graph. In fact if G is a divisor graph which has an even hole of order at least 6, then its complement has an antihole of the same order, which implies according to Theorem 12 that $\overline{G}$ is not a divisor graph. Thus we have the following result.

Theorem 14. *If G is a divisor graph which has an antihole of order at least 6, then $\overline{G}$ is not a divisor graph.*

It would be interesting to characterize those divisor graphs whose complements are also divisor graphs.

4 Which cycle permutation graphs are divisor graphs?

For $n \geq 3$, let $V(C_n) = \{1, 2, \cdots, n\}$, where $12 \cdots n1$ is the n-cycle. A *cycle permutation graph* $P_\alpha(C_n)$ consists of two copies of the n-cycle C_n such that every vertex i of one copy is adjacent to the vertex $\alpha(i)$ in the other copy, where α is a permutation on $V(C_n)$, see [12]. The Petersen graph is the cycle permutation graph $P_\alpha(C_5)$ where α is the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 1 & 3 & 5 \end{pmatrix}$.

Obviously, the cycle permutation graph $P_\alpha(C_n)$ is not a perfect graph whenever n is odd and $n \neq 3$. Even when n is even, $P_\alpha(C_n)$ is not necessarily a perfect graph. For example, $P_\alpha(C_4)$, where $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix}$, is not a divisor graph, for it has the induced 5-cycle $(1)(\alpha(1))(\alpha(3))(\alpha(2))(2)(1)$. The aim of this section is to determine precisely which cycle permutation graphs are perfect, and which are divisor graphs.

The graph $P_\alpha(C_3)$ is just $K_2 \square K_3$ for any α, thus, by Lemma 5, we have the following result.

Proposition 1. *The cycle permutation graph $P_\alpha(C_3)$ is perfect but not a divisor graph.*

Now we consider the case when $n \geq 4$.

Theorem 15. *For the cycle permutation graph $P_\alpha(C_n)$, where $n \geq 4$, the following are equivalent:*

(1) n is even, and $\alpha(2k+1)$ have the same parity for all $k \in \{0, 1, \cdots, \frac{n}{2} - 1\}$.

(2) $P_\alpha(C_n)$ is a divisor graph.

(3) $P_\alpha(C_n)$ is perfect.

Proof. (1)$\Rightarrow$(2): Suppose that n is even and $\alpha(2k+1)$ have the same parity for all $k \in \{0, 1, \cdots, \frac{n}{2}-1\}$. To show that $P_\alpha(C_n)$ is a divisor graph, we will give a divisor orientation of $P_\alpha(C_n)$. Define the orientation D of $P_\alpha(C_n)$ as follows, where C $(= C_n)$ and C' are the two copies of C_n in $P_\alpha(C_n)$:

(i) For every vertex i of C, let i be a transmitter in D if i is odd, and let i be a receiver in D if i is even.

(ii) Let the vertices of C' whose parity is similar to that of $\alpha(1)$ be receivers in the induced orientation $D_{C'}$ of D, and let those whose parity is different from that of $\alpha(1)$ be transmitters in $D_{C'}$.

Then every vertex in D is either a transmitter or a receiver and hence D is a divisor orientation of $P_\alpha(C_n)$. Therefore $P_\alpha(C_n)$ is a divisor graph.

(2)$\Rightarrow$(3): The result follows by Theorem 13.

(3)$\Rightarrow$(1): Suppose that $P_\alpha(C_n)$ is perfect. Since a perfect graph has no odd holes, n must be even. Assume to the contrary that there exist k_1 and k_2 in $\{0, 1, \cdots, \frac{n}{2} - 1\}$ such that $\alpha(2k_1 + 1)$ and $\alpha(2k_2 + 1)$ have different parities. Then we must have two successive odd integers i and j in $\{2k + 1 : k \in \{0, 1, \cdots, \frac{n}{2} - 1\}\}$ such that $\alpha(i)$ and $\alpha(j)$ have different parities. Without loss of generality, suppose that $\alpha(1)$ and $\alpha(3)$ have different parities. Then we have exactly two paths P' and P'' in C' with endvertices $\alpha(1)$ and $\alpha(3)$. Clearly, each of P' and P'' has odd length. Then the path $(\alpha(1))(1)(2)(3)(\alpha(3))$ forms two odd cycles in $P_\alpha(C_n)$ together with P' and with P''. Now since $\alpha(2)$ is an interior vertex of exactly one of the two paths P' and P'', one of these two cycles is induced in $P_\alpha(C_n)$, producing an odd hole in $P_\alpha(C_n)$, a contradiction. □

5 Which block graphs are divisor graphs?

The *block graph* of a graph G is the intersection graph whose vertex set is the family of blocks of G. A graph H is a block graph (of some graph) if and only if every block of H is complete, thus the blocks of a block graph are its cliques. Block graphs were introduced in [11]. Block graphs have neither odd holes nor odd antiholes, thus they are perfect. However, a block graph is not necessarily a divisor graph, see the graph in Figure 1. At the end of this section, we determine which block graphs are divisor graphs, but first we need to consider amalgamation operations.

A *bar amalgamation* of two disjoint graphs G and H along a vertex u of G and a vertex v of H is obtained by running a new edge e between the two vertices u and v, the resulting graph is denoted by $G_u *_e H_v$, see [10]. For example, $K_2 = G_u *_e H_v$ where each of G and H is the trivial graph.

Theorem 16. *Let G and H be two disjoint divisor graphs with divisor orientations D_G and D_H, respectively. If g and h are nontransitive vertices in D_G and D_H, respectively, then the bar amalgamation $G_g *_e H_h$ is a divisor graph.*

Proof. If one of the two vertices g and h is a transmitter while the other is a receiver, say g is a transmitter in D_G and h is a receiver in D_H, then the orientation D of $G_g *_e H_h$ defined by $E(D) = E(D_G) \cup E(D_H) \cup \{(g,h)\}$ is obviously a divisor orientation in which g is a transmitter and h is a receiver. So suppose that g is a transmitter in D_G and h is a transmitter in D_H (the case that both g and h are receivers is similar). Then the orientation D of $G_g *_e H_h$ defined by $E(D) = E(D_G) \cup E(D'_H) \cup \{(g,h)\}$, where D'_H is the converse of D_H, is obviously a divisor orientation in which g is a transmitter and h is a receiver. Therefore, by Lemma 3, $G_g *_e H_h$ is a divisor graph. □

Obviously, every vertex of a tree T is a nontransitive vertex in any divisor orientation of T. Thus, since the trivial tree K_1 is a divisor graph, and any tree of order $n > 1$ is a bar amalgamation of a tree of order $n-1$ and the trivial tree K_1, the following result follows from Theorem 16. This result was previously shown in [5].

Corollary 16.1. *Every tree is a divisor graph.*

A *vertex amalgamation* of two disjoint graphs G and H along a vertex u of G and a vertex v of H is obtained by identifying the two vertices u and v, the resulting graph is denoted by $G_u * H_v$, see [10]. For example, $P_3 = G_u * H_v$ where $G \cong H \cong K_2$.

Theorem 17. *Let G and H be two disjoint divisor graphs with divisor orientations D_G and D_H, respectively. If g and h are nontransitive vertices in D_G and D_H, respectively, then the vertex amalgamation $G_g * H_h$ is a divisor graph.*

Proof. Let $w = g = h$ be the vertex of $G_g * H_h$ obtained by identifying the two vertices g and h. If both g and h are transmitters (receivers) in D_G and D_H, respectively, then the orientation D of $G_g * H_h$ defined by $E(D) = E(D_G) \cup E(D_H)$ is clearly a divisor orientation in which w is a transmitter (receiver). So suppose that g is a transmitter in D_G and

h is a receiver in D_H (the case that g is a receiver in D_G and h is a transmitter in D_H is similar). Since, by Lemma 9, the converse D'_H of D_H is also a divisor orientation, the orientation D of $G_g * H_h$ defined by $E(D) = E(D_G) \cup E(D'_H)$ is a divisor orientation in which w is a transmitter. Therefore, by Lemma 3, $G_g * H_h$ is a divisor graph. □

Let G and H be two disjoint divisor graphs with divisor orientations D_G and D_H, respectively. Note that if $g \in V(G)$ and $h \in V(H)$ where at least one of g or h is a transitive vertex in D_G or D_H, respectively, then the vertex amalgamation $G_g * H_h$ and the bar amalgamation $G_g *_e H_h$ are not necessarily divisor graphs. Take for example the graphs G and H in Figure 2, note that h is a transitive vertex in any divisor orientation of the graph H.

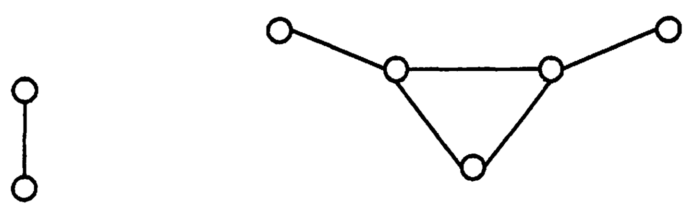

Figure 2: Two divisor graphs G and H whose amalgamations $G_g * H_h$ and $G_g *_e H_h$ are not divisor graphs.

From the proof of Theorem 17 we have a divisor orientation of $G_g * H_h$ in which the new vertex obtained by identifying the two vertices g and h is a nontransitive vertex. Therefore the following corollary follows from Theorem 17 by induction.

Corollary 17.1. *Let $G_1, G_2, \cdots, G_k$ be disjoint divisor graphs with divisor orientations $D_{G_1}, D_{G_2}, \cdots, D_{G_k}$, respectively, such that for $i = 1, 2, \cdots, k$, the set B_i of nontransitive vertices in D_{G_i} is nonempty. Then any graph obtained from $G_1, G_2, \cdots, G_k$ by iterated vertex amalgamations along vertices in $\cup\{B_i : i = 1, 2, \cdots, k\}$ is a divisor graph.*

It is known that complete graphs are divisor graphs, see [5]. Let D be a divisor orientation of the nontrivial complete graph K_n. If D has a transmitter (receiver) x, then every vertex in $V(D) - \{x\}$ has positive indegree (outdegree) and hence cannot be a transmitter (receiver) in D.

Moreover, if we label the vertices of K_n by $2, 2^2, \cdots, 2^n$, then the corresponding orientation D_{K_n} of K_n with $E(D_{K_n}) = \{(x, y) : x \text{ divides } y\}$ is a divisor orientation which has exactly one transmitter and exactly one receiver, namely 2 and 2^n, respectively. Therefore we have the following result.

Lemma 18. *If D is a divisor orientation of the nontrivial complete graph K_n, then D has at most one transmitter (receiver). Moreover K_n has a divisor orientation with exactly one transmitter and exactly one receiver.*

In particular, in the statement of Corollary 17.1, if we take the graphs $G_1, G_2, \cdots, G_k$ to be nontrivial complete graphs, then we have the following result.

Corollary 18.1. *Let $G_1, G_2, \cdots, G_k$ be nontrivial complete graphs such that for $i = 1, 2, \cdots, k$, the set B_i of nontransitive vertices in the divisor orientation D_{G_i} of G_i has exactly two elements. Then any graph obtained from $G_1, G_2, \cdots, G_k$ by iterated vertex amalgamations along vertices in $\cup\{B_i : i = 1, 2, \cdots, k\}$ is a divisor graph.*

Note that a graph obtained as described in Corollary 18.1 can be obtained from a tree by a way which is explained in the following definition.

Definition 1. *Let T be a nontrivial tree. A swollen of T is a graph G obtained from T by replacing each clique $\langle\{u, v\}\rangle$ of T induced by two adjacent vertices u and v by a complete graph $K(u, v)$ of order $n(u, v) \geq 2$ such that $\{u, v\} \subseteq V(K(u, v))$ and every vertex $x \in V(K(u, v)) - \{u, v\}$ has degree $n(u, v) - 1$ in G.*

Note that each vertex of T that is not an endvertex of T is a cutvertex of G, and that the subgraph of G induced by $V(T)$ is T itself. Now we can rewrite Corollary 18.1 using the concept of swollen of a nontrivial tree as follows.

Theorem 19. *Let T be a nontrivial tree. Then any swollen of T is a divisor graph.*

Now we can characterize the block graphs that are divisor graphs. By an *endclique* of a block graph G we mean a clique of G which contains exactly one cutvertex of G.

Theorem 20. *Let G be a nontrivial block graph. Then the following are equivalent:*

(1) G is a swollen of some nontrivial tree.

(2) G is a divisor graph.

(3) G has no induced subgraph isomorphic to the graph in Figure 1.

Proof. (1) $\Rightarrow$ (2): By Theorem 19.

(2) $\Rightarrow$ (3): Since the graph in Figure 1 is not a divisor graph, the result follows by Lemma 1.

(3) $\Rightarrow$ (1): If G is complete, then G is a swollen of the tree K_2. So, suppose that G is not complete, and let A be the set of all cutvertices of G. Let B be a set of vertices of G which consists of exactly one vertex (which is not a cutvertex of G) from each endclique of G. Clearly the sets A and B are disjoint. Let T be the induced subgraph $\langle A \cup B \rangle$ of G. Then every vertex $v \in B$ is an endvertex of T. Now, to show that T is a tree, assume not. Then T has a cycle C. Since a cycle contains neither an endvertex nor a cutvertex, all vertices of C must be from A and must belong to the same clique of G. Thus C is a 3-cycle $c_1c_2c_3c_1$ for some $c_1, c_2, c_3 \in A$. But each c_i $(i = 1, 2, 3)$ is a cutvertex of G, which implies that there exist three vertices x_i $(i = 1, 2, 3)$ in G such that for each $i = 1, 2, 3$, the vertex x_i is adjacent to c_i and the unique path joining x_i with c_j $(j \in \{1, 2, 3\} - \{i\})$ contains the vertex c_i. Then the set $\{x_1, x_2, x_3, c_1, c_2, c_3\}$ induces in G a subgraph isomorphic to that in Figure 1, which contradicts the assumption. Therefore, T is a tree. Obviously, G is a swollen of T. □

Note that the graph in Figure 1 is not induced in a block graph G if and only if every clique of G contains at most two cutvertices of G.

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