An infinite class of convex polytopes with constant metric dimension*

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Abstract. A family $\mathcal{G}$ of connected graphs is a family with constant metric dimension if $dim(G)$ is finite and does not depend upon the choice of G in $\mathcal{G}$. The metric dimension of some classes of plane graphs has been determined in [3], [4], [5], [12], [14] and [18] while metric dimension of some families of convex polytopes has been studied in [8], [9], [10] and [11]and the following open problem was raised in [11].
Open Problem [11]: Let G be the graph of a convex polytope which is obtained by joining the graph of two different convex polytopes G_1 and G_2 (such that the outer cycle of G_1 is the inner cycle of G_2) both having constant metric dimension. Is it the case that G will always have the constant metric dimension?
In this paper, we extend this study to an infinite class of convex polytopes which is obtained as a combination of graph of an antiprism A_n [1] and graph of convex polytope Q_n [2] such that the outer cycle of A_n is the inner cycle of Q_n. It is natural to ask for the characterization of classes of convex polytopes with constant metric dimension. Note that the problem of determining whether $dim(G) < k$ is an NP-complete problem [7].

Keywords: *Metric dimension, basis, resolving set, plane graph, convex polytope*

1 Notation and preliminary results

Let G be a connected graph, the *distance* $d(u, v)$ between two vertices $u, v \in V(G)$ is the length of a shortest path between them. Let

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$W = \{w_1, w_2, ..., w_k\}$ be an ordered set of vertices of G and let v be a vertex of G. The *representation* $r(v|W)$ of v with respect to W is the k-tuple $(d(v,w_1), d(v,w_2), ..., d(v,w_k))$. If distinct vertices of G have distinct representations with respect to W, then W is called a *resolving set* or *locating set* for G [3]. A resolving set of minimum cardinality is called a *metric basis* for G and this cardinality is the *metric dimension* of G, denoted by $dim(G)$. The concepts of resolving set and metric basis have previously appeared in the literature (see [3-6, 8-12, 14-18]).
For a given ordered set of vertices $W = \{w_1, w_2, ..., w_k\}$ of a graph G, the *ith* component of $r(v|W)$ is 0 if and only if $v = w_i$. Thus, to show that W is a resolving set it suffices to verify that $r(x|W) \neq r(y|W)$ for each pair of distinct vertices $x, y \in V(G) \backslash W$.
A useful property in finding $dim(G)$ is the following lemma [17]:

Lemma 1. *Let W be a resolving set for a connected graph G and $u, v \in V(G)$. If $d(u,w) = d(v,w)$ for all vertices $w \in V(G) \setminus \{u,v\}$, then $\{u,v\} \cap W \neq \emptyset$.*

By denoting $G + H$ the join of G and H a *wheel* W_n is defined as $W_n = K_1 + C_n$, for $n \geq 3$, a *fan* is $f_n = K_1 + P_n$ for $n \geq 1$ and *Jahangir graph* $J_{2n}, (n \geq 2)$ (also known as *gear graph*) is obtained from the *wheel* W_{2n} by alternately deleting n spokes. Buczkowski *et al.* [3] determined the metric dimension of *wheel* W_n, Caceres *et al.* [5] the metric dimension of *fan* f_n and Tomescu and Javaid [18] the metric dimension of *Jahangir graph* J_{2n}.

Theorem 1. *([3], [5], [18]) Let W_n be a wheel of order $n \geq 3$, f_n be fan of order $n \geq 1$ and J_{2n} be a Jahangir graph. Then*
(i) For $n \geq 7$, $dim(W_n) = \lfloor \frac{2n+2}{5} \rfloor$;
(ii) For $n \geq 7$, $dim(f_n) = \lfloor \frac{2n+2}{5} \rfloor$;
(iii) For $n \geq 4$, $dim(J_{2n}) = \lfloor \frac{2n}{3} \rfloor$.

The metric dimension of all these plane graphs depends upon the number of vertices in the graph.
On the other hand, we say that a family $\mathcal{G}$ of connected graphs is a family with constant metric dimension if dim(G) is finite and does not depend upon the choice of G in $\mathcal{G}$. In [6] Chartrand et al. proved that a graph has metric dimension 1 if and only if it is a *path*, hence paths on n vertices constitute a family of graphs with constant metric dimension. They did not characterize all graphs with metric dimension 2 but they investigated few properties of graphs with metric dimension 2.

Theorem 2. *[6] A graph G with metric dimension 2 can have neither K_5 nor $K_{3,3}$ as a subgraph.*

Similarly, *cycles* with $n(\geq 3)$ vertices also constitute such a family of graphs as their metric dimension is 2 and does not depend upon on the number of vertices n. A nice property of graphs with metric dimension 2 is the following result of Khuller et al. [14].

Theorem 3. *[14] Let G be a graph with metric dimension 2 and let $\{u,v\} \subset V(G)$ be a metric basis in G. Then the following are true:*
(a) There is a unique shortest path between u and v.
(b) The degree of each u and v is at most 3.

Caceres *et al.* [4] proved that

$$dim(P_m \times C_n) = \begin{cases} 2, & \text{if } n \text{ is odd;} \\ 3, & \text{if } n \text{ is even.} \end{cases}$$

Since *prisms* D_n are the trivalent plane graphs obtained by the cross product of path P_2 with a cycle C_n, hence they constitute a family of 3-*regular graphs* with constant metric dimension. Also Javaid *et al.* proved in [12] that the plane graph *antiprism* A_n constitutes a family of regular graphs with constant metric dimension as $dim(A_n) = 3$ for every $n \geq 5$. The prism and the antiprism are *Archimedean* convex polytopes defined e.g. in [13].
The metric dimension of some classes of *convex polytopes* has been studied in in [8], [9], [10] and [11]and the following open problem was raised in [11].
Open Problem [11]: Let G be the graph of a convex polytope which is obtained by joining the graph of two different convex polytopes G_1 and G_2 (such that the outer cycle of G_1 is the inner cycle of G_2) both having constant metric dimension. Is it the case that G will always have the constant metric dimension?
In this paper, we study the metric dimension of an infinite class of convex polytopes which is obtained as a combination of graph of an antiprism A_n [1] and graph of convex polytope Q_n [2] such that the outer cycle of A_n is the inner cycle of Q_n. It is shown that this class of convex polytoes has constant metric dimension and only three vertices chosen appropriately suffice to resolve all the vertices of this class of convex polytopes.
It is natural to ask for the characterization of classes of convex polytopes with constant metric dimension.

2 The graph of convex polytope $\mathbf{T}_n$

The graph of convex polytope $\mathbf{T}_n$ (Fig. 1) can be obtained as a combination of graph of an antiprism A_n [1] and graph of convex polytope Q_n [2] such that the outer cycle of A_n is the inner cycle of Q_n.
For our purpose, we call the cycle induced by $\{a_i : 1 \leq i \leq n\}$, the inner cycle, the cycle induced by $\{b_i : 1 \leq i \leq n\}$, the interior cycle, the cycle

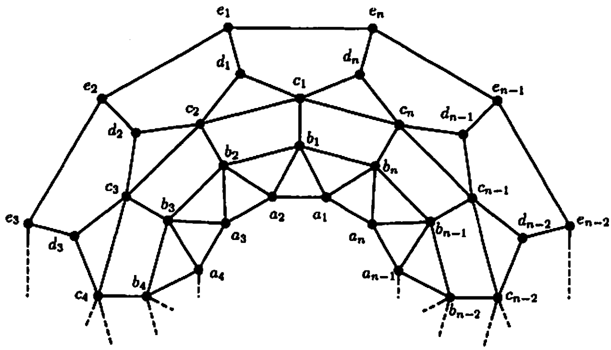

Fig. 1. The graph of convex polytope $\mathbf{T}_n$

induced by $\{c_i : 1 \le i \le n\}$, the exterior cycle, the set of vertices $\{d_i : 1 \le i \le n\}$, the interior vertices and the cycle induced by $\{e_i : 1 \le i \le n\}$, the outer cycle.

The metric dimension of graph of an antiprism A_n and graph of convex polytope Q_n has been determined in [12] and [8]where it was proved that both these graphs have constant metric dimension. In the next theorem, we prove that the metric dimension of the graph of convex polytope $\mathbf{T}_n$ is 3. Note that the choice of an appropriate basis of vertices (also referred to as landmarks in [14]) is the core of the problem.

Theorem 4. *Let the graph of convex polytopes be* $\mathbf{T}_n$*; then* $dim(\mathbf{T}_n) = 3$ *for* $n \ge 6$.

Proof. We will prove the above equality by double inequalities. We consider the two cases.

Case(i) When n is even.

In this case, we can write $n = 2k$, $k \ge 3$, $k \in \mathbf{Z}^+$. Let $W = \{a_1, a_2, a_{k+1}\} \subset V(\mathbf{T}_n)$, we show that W is a resolving set for $\mathbf{T}_n$ in this case. For this we give representations for any vertex of $V(\mathbf{T}_n)\backslash W$ with respect to W.

Representations for the vertices on inner cycle are

$$r(a_i|W) = \begin{cases} (i-1, i-2, k-i+1), & 3 \le i \le k; \\ (2k-i+1, 2k-i+2, i-k-1), & k+2 \le i \le 2k. \end{cases}$$

Representations for the vertices on interior cycle are

$$r(b_i|W) = \begin{cases} (1,1,k), & i = 1; \\ (i, i-1, k-i+1), & 2 \le i \le k; \\ (k,k,1), & i = k+1; \\ (2k-i+1, 2k-i+2, i-k), & k+2 \le i \le 2k. \end{cases}$$

Representations for the vertices on exterior cycle are

$$r(c_i|W) = (1,1,1) + r(b_i|W)$$

Representations for the interior vertices are

$$r(d_i|W) = \begin{cases} (3,3,k+1), & i=1; \\ (i+2,i+1,k-i+2), & 2 \le i \le k-1; \\ (k+2,k+1,3), & i=k; \\ (k+1,k+2,3), & i=k+1; \\ (2k-i+2,2k-i+3,i-k+2), & k+2 \le i \le 2k-1; \\ (3,3,k+2), & i=2k. \end{cases}$$

Representations for the vertices of outer cycle are $r(e_i|W) = (1,1,1) + r(d_i|W)$.
We note that there are no two vertices having the same representations implying that $dim(\mathbf{T}_n) \le 3$.
On the other hand, we show that $dim(\mathbf{T}_n) \ge 3$. Suppose on contrary that $dim(\mathbf{T}_n) = 2$, then by Theorem 2, the degree of basis vertices can be at most 3. But the graph of convex polytopes $\mathbf{T}_n$ has only vertices d_i and e_i of degree 3 and all other vertices of graph of convex polytopes $\mathbf{T}_n$ have degree 4 or 5. So we have the following possibilities to be discussed.
(1) Both vertices are in the set of interior vertices. Without loss of generality we suppose that one resolving vertex is d_1. Suppose that the second resolving vertex is d_t $(2 \le t \le k+1)$. Then for $2 \le t \le k$, we have $r(b_1|\{d_1,d_t\}) = r(e_n|\{d_1,d_t\}) = (2,t{+}1)$ and for $t = k{+}1$, $r(e_2|\{d_1,d_{k+1}\}) = r(e_n|\{d_1,d_{k+1}\}) = (2,k)$, a contradiction.
(2) Both vertices are in the outer cycle. Without loss of generality we suppose that one resolving vertex is e_1. Suppose that the second resolving vertex is e_t $(2 \le t \le k+1)$. Then for $2 \le t \le k$, we have $r(d_1|\{e_1,e_t\}) = r(e_n|\{e_1,e_t\}) = (1,t)$ and for $t = k{+}1$, $r(e_2|\{e_1,e_{k+1}\}) = r(e_n|\{e_1,e_{k+1}\}) = (1,k-1)$, a contradiction.
(3) One vertex is in the set of interior vertices and other one is in the outer cycle. Without loss of generality we suppose that one resolving vertex is d_1. Suppose that the second resolving vertex is e_t $(1 \le t \le k+1)$. Then for $1 \le t \le k$, we have $r(c_1|\{d_1,e_t\}) = r(d_n|\{d_1,e_t\}) = (1,t+1)$ and for $t = k+1$, we have $r(d_2|\{d_1,e_{k+1}\}) = r(e_1|\{d_1,e_{k+1}\}) = (1,k)$, a contradiction.
Hence, from above it follows that there is no resolving set with two vertices for $V(T_n)$ implying that $dim(T_n) = 3$ in this case.
Case(ii) When n is odd.
In this case, we can write $n = 2k{+}1$, $k \ge 3$, $k \in \mathbf{Z}^+$. Let $W = \{a_1, a_2, a_{k+1}\} \subset V(\mathbf{T}_n)$, we show that W is a resolving set for $\mathbf{T}_n$ in this case. For this we give representations for any vertex of $V(\mathbf{T}_n)\backslash W$ with respect to W.

Representations for the vertices on inner cycle are

$$r(a_i|W) = \begin{cases} (i-1, i-2, k-i+1), & 3 \leq i \leq k; \\ (k, k, 1), & i = k+2; \\ (2k-i+2, 2k-i+3, i-k-1), & k+3 \leq i \leq 2k+1. \end{cases}$$

Representations for the vertices on interior cycle are

$$r(b_i|W) = (1, 1, 1) + r(b_i|W)$$

Representations for the vertices on exterior cycle are

$$r(c_i|W) = \begin{cases} (2, 2, k+1), & i = 1; \\ (i+1, i, k-i+2), & 2 \leq i \leq k; \\ (k+2, k+1, 2), & i = k+1; \\ (2k-i+3, 2k-i+4, i-k+1), & k+2 \leq i \leq 2k+1. \end{cases}$$

Representations for the interior vertices are

$$r(d_i|W) = \begin{cases} (3, 3, k+1), & i = 1; \\ (i+2, i+1, k-i+2), & 2 \leq i \leq k-1; \\ (k+2, k+1, 3), & i = k; \\ (k+2, k+2, 3), & i = k+1; \\ (2k-i+3, 2k-i+4, i-k+2), & k+2 \leq i \leq 2k; \\ (3, 3, k+2), & i = 2k+1. \end{cases}$$

Representations for the vertices of the outer cycle are $r(e_i|W) = (1, 1, 1) + r(d_i|W)$.
Again we see that there are no two vertices having the same representations which implies that $dim(\mathbf{T}_n) \leq 3$ in this case.
On the other hand, suppose that $dim(\mathbf{T}_n) = 2$, then there are the same subcases as in case (i) and contradiction can be obtained analogously. This implies that $dim(\mathbf{T}_n) = 3$ in this case, which completes the proof.

□

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