Permutation labeling of graphs *

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Abstract. A (p,q)-graph is said to be permutation graph if there exists a bijection function $f : V(G) \to \{1, 2, \cdots, p\}$ such that the induced edge function $h_f : E(G) \to \mathbb{N}$ is defined as follows

$$h_f(x_ix_j) = \begin{cases} {}^{f(x_i)}P_{f(x_j)}, & \text{if } f(x_j) < f(x_i); \\ {}^{f(x_j)}P_{f(x_i)}, & \text{if } f(x_i) < f(x_j). \end{cases}$$

In this paper, we investigate the permutation labelings of wheel related graphs.

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Subject classification: *05C78.*

1 Introduction and preliminary results

Informally, by a graph labeling we mean an assignment of integers to elements of a graph, such as vertices, or edges, or both, subject to some specified conditions. These conditions are usually expressed on the basis of the values (called weights) of some evaluating function. In our case, the evaluating function will be simply to produce permutation number of the labeled elements of the graph. One of the situations that we are particularly interested in is when all the edge weights permutation numbers are the different. In such a case we call the labeled graph has a *permutation labeling.* The study of these graphs was motivated in "A dynamic survey of graph labeling" [4]. The concept of labeling of graphs has gained a lot of popularity in the area of graph theory. This popularity is not only due to mathematical challenges of graph labelings but also to the wide range of applications that graph labelings offer to other branches of science, for instance, x-ray, crystallography, coding theory, cryptography (secret sharing schemes), astronomy, circuit design and communication

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design.
Hegde and Shetty [6] define a graph G with p vertices and q edges is said to be *permutation graph* if there exists a bijection function $f : V(G) \to \{1, 2, \cdots, p\}$ such that the induced edge function $h_f : E(G) \to \mathbb{N}$ is defined as follow

$$h_f(x_i x_j) = \begin{cases} {}^{f(x_i)}P_{f(x_j)}, & \text{if } f(x_j) < f(x_i); \\ {}^{f(x_j)}P_{f(x_i)}, & \text{if } f(x_i) < f(x_j). \end{cases}$$

In [6], Hegde and Shetty proved that the complete graph K_n is a permutation graph if and only if $n \leq 5$. The edge values in any permutation labelings are large numbers, investigations of suitable additional constraints to control edge values is a scope for further study. They strongly believe that all *trees* admit permutation labelings. In [2], Baskar and Vishnupriya proved that the following graphs are permutation graphs namely, *path* P_n, *cycle* C_n, *stars* $K_{1,n}$, graphs obtained adding a pendent edge to each edge of a star, graphs obtained by joining the centers of two identical stars with an edge or a path of length 2 and *complete binary trees* with at least three vertices. In [8] Seoud and Salim determined all permutation graphs of order ≤ 9. They also proved that every *bipartite graph* of order ≤ 50 has a permutation labeling.
In this paper, we investigate the permutation labelings of graphs namely, *k-wheel*, *k-fold wheel*, *k-fan*, *wheel*, *fan*, *gear graph*, *generalized Jahangir*, *friendship*, *sun flower*, *generalized web*, *flower* and *helm* graph.
For proving the some results of this paper, we frequently use a lemma which is given as below.

Lemma 1. *If G is a permutation graph then $G - e$(= the graph obtained from G by deleting the edge e) is also a permutation graph.*

Proof. Since there is no repeated edge labels in G then we use the same labeling of G to label $G - e$, so $G - e$ has no repeated edge labels.

□

By denoting $G + H$, the *join* of G and H, a *wheel* W_n is defined as $W_n = C_n + K_1$, a *fan* F_n is $F_n = P_n + K_1$ and *Jahangir graph* $J_{2,m}$ [9] (also known as *gear graph*) is obtained from the wheel W_{2n} by deleting alternating n spokes. A *friendship graph* f_n is obtained from the wheel W_{2n} by removing the alternating edges from the rim. A *helm graph* H_n [5] is obtained from the wheel W_n by attaching a pendent vertex to each vertex of the outer cycle of wheel.

The *k-W_n graph* is obtained from the path of order k and cycle of size n by joining edges from each vertex of the path P_k to the vertices of the cycle C_n.

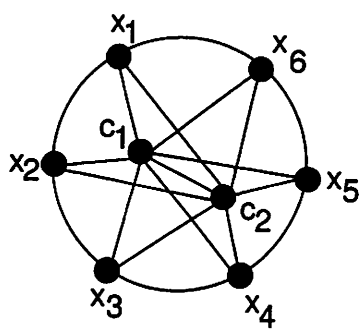

Fig. 1. 2-W_6 graph

2 Main results

Theorem 1. *The graph k-wheel admits a permutation labeling for every integer $k \geq 2$.*

Proof. Denote the vertices of k-W_n consecutively as $x_1, x_2, \ldots, x_n, c_1, \ldots, c_k$ with $|V| = n + k$ and $|E| = n(k + 1) + k - 1$.
We define the required permutation labeling in such a way, i.e. $f(x_1) = 1, f(x_n) = 2$, $f(x_i) = i + 1 : 2 \leq i \leq n - 1$ and $f(c_j) = n + j : 1 \leq j \leq k$.

Clearly, the above labeling is a permutation labeling. The induced edge labels can be listed as $h_f(E) = \{2, 3, 4!, 5! \cdots, n!, (n+1)!, \cdots, (n+k)!, n(n-1), \frac{(n+1)!}{(n+1-i)!}, \cdots, \frac{(n+1)!}{(n+1-k)!}\}$ and all these values are distinct. The edge value of $k-$wheel graph is the whole edge value labels and the pairwise intersection of any two elements from edge values labels have empty intersection i.e.,

$$A_i \cup A_j = h_f(E), A_i \cap A_j = \phi, \forall \ i \neq j.$$

So the k-W_n admits permutation labeling.

□

In [10] the graph G derived from a wheel by duplicating the hub vertex one or more times. The graph G is known as *k-fold wheel* if there are k hub vertices, each adjacent to all rim vertices and not adjacent to each other.

Corollary 1. *The k-fold wheel graph has a permutation labeling.*

□

The graph *k-fan* $F_{n,k}$ is actually obtained from the graph k-W_n by deleting one edge from the rim. Apply the same labels as in Theorem 1 on k-fan $F_{n,k}$ graph, we see by supporting of Lemma 1 the graph is again permutation. So we have the following corollary.

Corollary 2. *The graph k-fan $F_{n,k}$ admits a permutation labeling.*

Proof. By using the labeling of Theorem 1 and Lemma 1, we see that the graph has a permutation labeling.

□

Remark: If G is a permutation graph, then $G - v$(= the graph obtained from G by deleting the vertex v) have may or may not be the permutation labeling.

As in Fig. 2, we see that the graph has a permutation labeling, because there are not pair of same permutation $({}^5P_4, {}^6P_3)$ and $({}^6P_1, {}^3P_2)$ while after deleting the pendent vertex (vertices), the graph may or may not have the permutation labeling. After the deleting of one pendent vertex, we can rearrange the label to make a graph is a permutation graph but after the deleting of second pendent vertex it has not permutation labeling, because one of them above pair occur in new graph. Also, [8] Seoud and Salim showed that $\{\{K_6 - e_1\} - e_2\}$ has a permutation labeling. It means that K_6 itself has not permutation labeling. But the original graph has a permutation labeling.

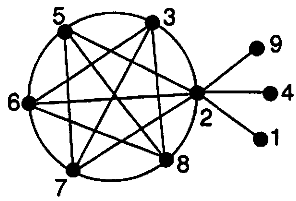

Fig. 2. Permutation graph

After removing the $k-1$ vertices of the path P_k from the graph k-W_n. We obtain the wheel W_n. Hence, we have the following theorem.

Theorem 2. *The wheel W_n has a permutation labeling for every $n \geq 3$.*

Proof. For proving this theorem, we are using the same labeling as in Theorem 1 and support of Lemma 1. Now, one can easily see that the wheel W_n admits permutation labeling.

□

With the previous lemma in hand and using the labeling of Theorem 2, we have some corollaries.

A fan $F_n (n \geq 2)$ can also be constructed from a wheel W_n by removing one rim edge. However, in the computation of the edge-weights of F_n, the label of the center is used n times, the labels of the vertices x_1 and x_n are used twice each, and the labels of all the other vertices $x_i : 2 \leq i \leq n-1$, are used three times each.

Corollary 3. *The fan F_n has a permutation labeling for every $n \geq 3$.*

□

Corollary 4. *The gear graph G_{2n} has a permutation labeling.*

□

The Jahangir graph $J_{n,m}$ [9] for $m \geq 3$ is a graph on $nm+1$ vertices, i.e., a graph consisting of a cycle C_{nm} with one additional vertex which is adjacent to m vertices of C_{nm} at distance n to each other on C_{nm}.

The following result can also be derived as an immediate consequence of Theorem 2 and Lemma 1.

Corollary 5. *The generalized graph $J_{n,m}$ admits a permutation labeling for every integer $n \geq 2, m \geq 3$.*

□

Another way of obtaining *friendship graph* is join of K_1 and n copies of K_2. The *generalized friendship graph* $f_{m,n}$ [3] is a collection of n cycles (all of order m) meeting at a common vertex.

Corollary 6. *The generalized friendship graph $f_{m,n}$ for every $n \geq 3$ admits a permutation labeling.*

□

The sun flower graph SF_n [5] is defined as follows: consider a wheel with central vertex c and an n-cycle $x_1, x_2, \cdots, x_{n-1}, x_n$ and additional n vertices $w_1, w_2, \cdots, w_{n-1}, w_n$ where w_i is joined by edges to x_i, x_{i+1} for $i = 1, 2, \cdots, n-1, n$, where $i+1$ is taken modulo n. The sun flower graph SF_n has order $2n+1$ and size $4n$.

Theorem 3. *The sun flower SF_n admits a permutation labeling for every $n \geq 7$.*

Proof. Let us define the labeling scheme in such a way, i.e., $f(x_i) = i : 1 \leq i \leq n$, $f(w_i) = n + i : 1 \leq i \leq n$ and $f(c) = 2n+1$. This labeling is a permutation labeling with induced edge labels which are listed as $h_f(E) = \{n, (i+1)!, \frac{(2n+1)!}{(2n+1-i)!}, \frac{(n+i)!}{n!}, \frac{(n+i)!}{(n-1)!}\}$ and all these values are distinct. The pairwise intersection of any two elements from edge values have empty intersection,

$$A_i \cup A_j = h_f(E), A_i \cap A_j = \phi, \forall \ i \neq j.$$

It means that sun flower graph is a permutation graph.

□

The *web graph* Wb_n [4] is obtained from helm graph by joining all pendant vertices to form a cycle. The *generalized web graph* $Wb_{n,k}$ [7] can be viewed by joining a new vertex c to vertices of the most inner cycle of generalized prism $P_m \times C_n$.

Theorem 4. *The generalized web graph $Wb_{n,k}$ has a permutation labeling for $n \geq 6$.*

Proof. Let us define the permutation labeling as follows and k denotes the last level of the generalized web graph, so we have $f(x_1^j) = 1 + n(j-1) : 1 \leq j \leq k$, $f(x_n^j) = 2 + n(j-1) : 1 \leq j \leq k$, $f(x_i^j) = i + 1 + n(j-1) : 1 \leq j \leq k, 1 \leq i \leq n$ and the center vertex c has the label $f(c) = nk + 1$.
Obviously, the above graph has a permutation labeling under the above labeling scheme with induced edge labels can be listed as $h_f(E) = \{(i+nj+2)!, \frac{(i+nj-n+1)!}{2}, \frac{(i+nj+1)!}{n!}, \frac{(nj)!}{(n-2)!}\}$ and all these are distinct. The edge values of generalized web graph $Wb_{n,k}$ is the whole edge labels and the pairwise intersection of any two elements from edge labels have empty intersection i.e.,

$$A_i \cup A_j = h_f(E), A_i \cap A_j = \phi, \forall\ i \neq j.$$

So the graph $Wb_{n,k}$ admits permutation labeling.

□

The *flower graph* F_n^* [10] can be constructed from a helm graph H_n by joining each vertex of degree 1 to the center. The flower graph F_6^* is depicted as in Fig. 3.
In the next theorem, we prove that the flower graph F_n^* is also a permutation graph.

Theorem 5. *The flower graph F_n^* is a permutation graph for every $n \geq 6$.*

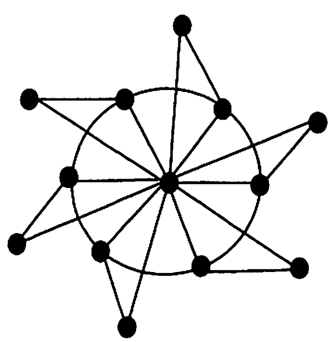

Fig. 3. F_6^*

Proof. For $n \geq 6$, let us denote the vertices of degree 4 of flower graph F_n^* for by x_i, vertices of degree 2 by y_i and the center vertex by c. The labeling scheme for the flower graph F_n^* is defined as $f(x_i) = i : 1 \leq i \leq n-2$, $f(x_{n-1}) = n$, $f(x_n) = n-1$ and $f(y_i) = f(x_i) + n : 1 \leq i \leq n$. The label assigned to the center vertex of the flower graph F_n^* is $f(c) = 2n+1$.
With this labeling scheme the flower graph F_n^* is a permutation graph with induced edge labels listed as $h_f(E) = \{n-1, n!, \frac{n!}{2}, \frac{(2n+1)!}{i!}, \frac{(n+i)!}{n!}\}$ and all these are distinct. The edge labels of flower graph F_n^* is the whole edge values and the pairwise intersection of any two elements from edge set have empty intersection. So the flower graph F_n^* admits permutation labeling.

□

By using the labeling of above Theorem and Lemma 1, we have also the following corollary.

Corollary 7. *The helm graph H_n admits a permutation labeling for every $n \geq 6$.*

□

3 Concluding remarks

In this paper, we have studied the permutation labeling for wheel related graphs. We proved that all these wheel related graphs admit permutation labeling. It seems to us that all wheel related graphs are permutation graph. So, it is natural to propose the following question as an open problem that naturally arises form the text.

Open problem: *Is it the case that every wheel related graphs is a permutation graph?*

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