On the geodomatic number of a graph

L. Volkmann

Lehrstuhl II für Mathematik, RWTH Aachen University,
52056 Aachen, Germany; *volkm@math2.rwth-aachen.de*

Abstract

A set S of vertices of a graph G is geodetic if every vertex in $V(G)\setminus S$ is contained in a shortest path between two vertices of S. The geodetic number $g(G)$ is the minimum cardinality of a geodetic set of G. The geodomatic number $d_g(G)$ of a graph G is the maximum number of elements in a partition of $V(G)$ into geodetic sets.

In this paper we determine $d_g(G)$ for some family of graphs, and we present different bounds on $d_g(G)$. In particular, we prove the follwing Nordhaus-Gaddum inequality, where $\overline{G}$ is the complement of the graph G. If G is a graph of order $n \geq 2$, then $d_g(G) + d_g(\overline{G}) \leq n$ with equality if and only if $n = 2$.

Keywords: Geodetic set, geodetic number, geodomatic number

2000 Mathematical subject classification: 05C12

1 Terminology

We consider finite graphs without loops and multiple edges. For any graph G the set of vertices is denoted by $V(G)$ and the edge set by $E(G)$. We define the *order* of G by $n = n(G) = |V(G)|$. For a vertex $v \in V(G)$, the *neighborhood* $N(v)$ is the set of all vertices adjacent to v, and the *degree* $d(v)$ of a vertex v is defined by $d(v) = |N(v)|$. If G is a graph, then $\overline{G}$ is its *complement.*

Let G_1 and G_2 be two disjoint graphs. The *union* $G = G_1 \cup G_2$ has $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$, and the *join* $H = G_1 + G_2$ has $V(H) = V(G_1) \cup V(G_2)$ and

$$E(H) = E(G_1) \cup E(G_2) \cup \{uv \mid u \in V(G_1) \text{ and } v \in V(G_2)\}.$$

A *complete graph* of order n is denoted by K_n and $K_{p_1,p_2,\ldots,p_r}$ is a *complete r-partite graph* such that the partite sets have cardinality $p_1, p_2, \ldots, p_r$. We denote by C_n the cycle on n vertices.

If G is a connected graph, then the *distance* $d(x, y)$ is the length of a shortest $x - y$ path in G. The *eccentricity* $e(v)$ of a vertex v is the distance to a vertex farthest from v. The *radius* $r(G)$ is the minimum eccentricity of the vertices, whereas the *diameter* $d(G)$ the maximum eccentricity. Now v is a *central vertex* if $e(v) = r(G)$. A graph is *self-centered* if every vertex is in the center. An $x - y$ path of length $d(x, y)$ is called an *$x - y$ geodesic*. A vertex v is said to *lie on* an $x - y$ geodesic P if v is a vertex of P. The closed interval $I[x, y]$ between two vertices x and y in G is the set of vertices of G which belong to an $x - y$ geodesic of G, while for $S \subseteq V(G)$,

$$I[S] = \bigcup_{x,y \in S} I[x, y].$$

If G is a connected graph, then a set S of vertices is a *geodetic set* if $I[S] = V(G)$. The minimum cardinality of a geodetic set is the *geodetic number* of G, and is denoted by $g(G)$. The geodetic number of a disconnected graph is the sum of the geodetic numbers of its components. A geodetic set of cardinality $g(G)$ is called a *$g(G)$-set.* We define the *geodomatic number* $d_g(G)$ of a graph G as the maximum number of elements in a partition of $V(G)$ into geodetic sets.

A vertex of G is *simplical* if the subgraph induced by its neighborhood is complete. It is easy to see that every simplical vertex belongs to every geodetic set. Geodetic sets and geodetic numbers have been studied in, e.g., [1, 2, 3, 4, 5, 6, 7, 8, 9].

2 Results

Since $g(G) \geq 2$ for each graph G of order $n \geq 2$, the following bounds are immediate.

Proposition 2.1. *If G is a graph of order $n \geq 2$, then*

$$1 \leq d_g(G) \leq \frac{n}{g(G)} \leq \frac{n}{2}.$$

As each simplical vertex belongs to every geodetic set, we obtain the next result.

Proposition 2.2. *If G contains a simplicial vertex, then $d_g(G) = 1$.*

Now we note the geodomatic number of each cycle.

Proposition 2.3. *If C_n is a cycle of length n, then*

(a) $d_g(C_n) = n/2$ when n is even,

(b) $d_g(C_n) = \lfloor n/3 \rfloor$ when n is odd.

Proposition 2.4. *If G is the complete r-partite graph $K_{p_1,p_2,\ldots,p_r}$ of order $n \geq 2$, then $d_g(G) = n/2$ if and only if $n = 2$ or $p_1 = p_2 = \ldots = p_r = 2$.*

Proof. If $n = 2$, then $d_g(G) = 1 = n/2$. Now let $p_1 = p_2 = \ldots = p_r = 2$, and let $\{u, v\}$ be an arbitrary partite set. Then we observe that $I[u, v] = V(G)$, and therefore we deduce that $d_g(G) = n/2$.

Coversely, assume that $d_g(G) = n/2$, and let $1 \leq p_1 \leq p_2 \leq \ldots \leq p_r$. This implies that n is even. If $n = 2$, then $d_g(G) = 1 = n/2$, and we are done. Assume now that $n \geq 4$. If $r = 1$, then $g(G) = n$ and so $d_g(G) = 1 < n/2$, a contradiction. Assume next that $r \geq 2$. If $p_1 = 1$, then there exists a partite set of G consisting of exactly one vertex u. If S is a geodetic set of G containing u, then we observe that $|S| \geq 3$, and we obtain the contradiction $d_g(G) < n/2$. This shows that $p_1 \geq 2$. If $p_r \geq 3$, then let v be a vertex of a partite set of cardinality p_r. If S is a geodetic set of G containing v, then we note that $|S| \geq 3$ and so $d_g(G) < n/2$. Consequently, $p_r \geq 3$ is impossible, and altogether we have shown that $p_1 = p_2 = \ldots = p_r = 2$. □

Propositions 2.3 and 2.4 show that $d_g(G) = n/2$ is possible, and therefore the upper bound $d_g(G) \leq n/2$ in Proposition 2.1 is sharp.

In the following theorem let $H_1 = K_{p_1,p_2,\ldots,p_r}$ with $p_1 = p_2 = \ldots = p_{r-1} = 2$, $r \geq 2$ and $p_r = 3$. If $\{u, v, w\}$ is the partite set of H_1 of cardinality 3, then let H_2 be the graph H_1 with the additional edge uv, and let H_3 be the graph H_2 with the additional edge uw.

Theorem 2.5. *Let G be a graph of order $n \geq 3$ and diameter $d(G) = 2$. Then $d_g(G) = (n-1)/2$ if and only if $G = K_{1,2}$ or $n \geq 5$ is odd and G is isomorphic to H_1, H_2 or H_3.*

Proof. If $G = K_{1,2}$ or $n \geq 5$ is odd and G is isomorphic to H_1, H_2 or H_3, then it is a simple matter to obtain $d_g(G) = (n-1)/2$.

Conversely, assume that $d_g(G) = (n-1)/2$. Then n is odd and $G = K_{1,2}$ when $n = 3$. Now let $n = 2r + 1 \geq 5$, and let

$$\{x_1, y_1\}, \{x_2, y_2\}, \ldots, \{x_{r-1}, y_{r-1}\}, \{u, v, w\}$$

be a partition of $V(G)$ into $r = (n-1)/2$ geodetic sets. The hypothesis $d(G) = 2$ implies that $d(x_i, y_i) = 2$ and $V(G) = I[x_i, y_i]$ for each $1 \leq i \leq r-1$. It follows that $d(x_i) = d(y_i) = n-2$ for each $1 \leq i \leq r-1$ and $d(u), d(v), d(w) \geq n-3$. If $\{u, v, w\}$ is an independent set, then $G = H_1$. If, without loss of generality, u is adjacent to v and $d(w) = n-3$, then $G = H_2$, and if u and v as well as u and w are adjacent and $d(v) = d(w) = n-2$, then $G = H_3$. Since $d(u) = d(v) = d(w) = n-1$ is forbidden, the proof is complete. □

Theorem 2.6. *Let G be a graph of order $n \geq 2$ such that $d_g(G) = n/2$. Then*

(a) n is even, and $G = K_2$ or $G = K_1 \cup K_1$ when $n = 2$.

(b) If $n \geq 4$, then G is connected and self-centered.

(c) If $n \geq 4$ and $d(G) = 2$, then $G = K_{p_1,p_2,\ldots,p_r}$ with $p_1 = p_2 = \ldots = p_r = 2$.

Proof. (a) The hypothesis $d_g(G) = n/2$ implies that n is even, and if $n = 2$, then $G = K_2$ or $G = K_1 \cup K_1$ is immediate.

(b) If G is not connected, then the condition $n \geq 4$ shows that $g(G) \geq 3$. Thus Propostion 2.1 leads to the contradiction $d_g(G) \leq n/3 < n/2$, and hence G is connected.

Let u be a vertex of G such that $e(u) = r(G)$. Since $d_g(G) = n/2$, there exists a vertex $v \in V(G)$ such that $V(G) = I[u, v]$. Now let x and y be two arbitrary vertices of G. Since $V(G) = I[u, v]$, the vertices x and y are contained in a $u - v$ geodesic. If there exists a $u - v$ geodesic P such that $x, y \in V(P)$, then $d(x, y) \leq d(u, v)$. Otherwise, we deduce that

$$\begin{aligned} d(x,y) &\leq \min\{d(u,x) + d(y,u), d(v,x) + d(y,v)\} \\ &\leq \frac{d(u,x) + d(y,u) + d(v,x) + d(y,v)}{2} \\ &= \frac{2 \cdot d(u,v)}{2} = d(u,v). \end{aligned}$$

In both cases we obtain $d(x, y) \leq d(u, v) \leq e(u) = r(G)$, and hence $d(G) \leq r(G)$. Consequently, $d(G) = r(G)$ and thus G is self-centered.

(c) Let $n = 2r \geq 4$, and let $\{x_1, y_1\}, \{x_2, y_2\}, \ldots, \{x_r, y_r\}$ be a partition of $V(G)$ into $r = n/2$ geodetic sets. Then $d(x_i, y_i) = 2$ and $V(G) = I[x_i, y_i]$ for each $1 \leq i \leq r$. It follows that $d(x_i) = d(y_i) = n - 2$ for each $1 \leq i \leq r$ and so we see that $G = K_{p_1,p_2,\ldots,p_r}$ with $p_1 = p_2 = \ldots = p_r = 2$. □

As an application of Theorem 2.6, we prove the following Nordhaus-Gaddum type results.

Theorem 2.7. *If G is a graph of order $n \geq 2$, then*

$$d_g(G) + d_g(\overline{G}) \leq n \tag{1}$$

with equality if and only if $n = 2$.

Proof. In view of Propsition 2.1, the Nordhaus-Gaddum inequality (1) is immediate. If $n = 2$, then $d_g(G) = 1$ and $d_g(\overline{G}) = 1$ and thus $d_g(G) + d_g(\overline{G}) = 2 = n$.

Conversely, suppose to the contrary that $d_g(G) + d_g(\overline{G}) = n$ for $n \geq 3$. Because of Proposition 2.1, it follows that $d_g(G) = n/2$

and $d_g(\overline{G}) = n/2$. Using Theorem 2.6, we deduce that $n \geq 4$ is even and G and $\overline{G}$ are both connected and self-centered. If $d(G) = 1$, then G is the complete graph, and we obtain the contradiction $d_g(G) + d_g(\overline{G}) = 2 < 4 \leq n$. If $d(G) = 2$, then we deduce from Theorem 2.6 (c) that $G = K_{p_1,p_2,\ldots,p_r}$ with $p_1 = p_2 = \ldots = p_r = 2$, and this leads to the contradiction $d_g(G) + d_g(\overline{G}) = n/2 + 1 < n$. However, if $d(G) = r(G) \geq 3$, then it is straightforward to verify that $d(\overline{G}) = r(\overline{G}) \leq 2$ (cf. also Exercise 2.2.9 in [3] on page 42). Using for $\overline{G}$ the same arguments as above, we arrive at a contradiction too, and the proof is complete. □

Theorem 2.8. *If G is a graph of even order $n \geq 4$, then*

$$d_g(G) + d_g(\overline{G}) \leq n - 1 \qquad (2)$$

with equality if and only if $G \in \{C_4, C_6\}$ or $\overline{G} \in \{C_4, C_6\}$.

Proof. Theorem 2.7 implies the Nordhaus-Gaddum inequality (2). If $G \in \{C_4, C_6\}$ or $\overline{G} \in \{C_4, C_6\}$, then it is a simple matter to verify that $d_g(G) + d_g(\overline{G}) = n - 1$.

Conversely, assume that $d_g(G) + d_g(\overline{G}) = n - 1$. Because of Proposition 2.1, it follows that $d_g(G) = n/2$ or $d_g(\overline{G}) = n/2$. We assume, without loss of generality, that $d_g(G) = n/2$ and thus $d_g(\overline{G}) = (n-2)/2$. In view of Theorem 2.6, the graph G is connected and self-centered. If $d(G) = 1$, then G is the complete graph, and we obtain the contradiction $d_g(G) = 1 < 2 \leq n/2$. If $d(G) = 2$, then Theorem 2.6 (c) shows that $G = K_{p_1,p_2,\ldots,p_r}$ with $p_1 = p_2 = \ldots = p_r = 2$. If $n = 4$, then $G = C_4$, and if $n \geq 6$, then we obtain the contradiction $d_g(G) + d_g(\overline{G}) = n/2 + 1 < n - 1$.

If $d(G) = r(G) \geq 3$, then $d(\overline{G}) = r(\overline{G}) \leq 2$. If $d(\overline{G}) = 1$, then $\overline{G}$ is complete and so $d_g(G) = 1$, a contradiction. So assume that $d(\overline{G}) = 2$. If $n = 2r \geq 4$, then the fact that $d_g(\overline{G}) = (n-2)/2 = r-1$ shows that the partition of $V(\overline{G})$ into $r - 1$ geodetic sets consists of one set of cardinality four and $r-2$ sets of cardinality two or two sets of cardinality three and $r - 3$ sets of cardinality two. If $\{x, y\}$ is a geodetic set of cardinality two in $\overline{G}$, then G contains a component H such that $V(H) = \{x, y\}$, and we obtain the contradiction $d_g(G) = 1 < n/2$. In the case that $n \geq 8$, we conclude that there exists a geodetic set of cardinality two in $\overline{G}$, and thus this is impossible. In the remaining cases that $n = 4$ or $n = 6$ it is straighforward to verify that $G = C_6$ is the only possibility. □

Buckley, Harary and Quintas [4] characterized those connected graphs G of order n for which $g(G) = n$ or $g(G) = n - 1$. As an easy consequence, we obtain the following result.

Theorem 2.9. *(a) If G is a graph of order n, then $g(G) = n$ if and only if the components of G are complete graphs.*

(b) If G is a graph of order $n \geq 3$, then $g(G) = n - 1$ if and only if there is exactly one component H of G such that $H = K_1 + (K_{n_1} \cup K_{n_2} \cup \ldots \cup K_{n_p})$ with $p \geq 2$ and the other components (if any) are complete graphs.

Theorem 2.10. *For any graph G of order n,*

$$d_g(G) + g(G) \leq n + 1 \tag{3}$$

with equality if and only if all components of G are complete graphs.

Proof. According to Proposition 2.1, we obtain

$$d_g(G) + g(G) \leq \frac{n}{g(G)} + g(G). \tag{4}$$

Using the fact that $f(x) = x + n/x$ is decreasing for $1 \leq x \leq \sqrt{n}$ and increasing for $\sqrt{n} \leq x \leq n$, this inequality leads to (3) immediately.

If the components of G are complete graphs, then $g(G) = n$ and $d_g(G) = 1$ and therefore $d_g(G) + g(G) = n + 1$.

Conversely, assume that G contains at least one non-complete component. Then Theorem 2.9 (a) implies that $g(G) \leq n - 1$. If $d_g(G) = 1$, then we deduce that $d_g(G) + g(G) \leq 1 + n - 1 = n$. If $d_g(G) \geq 2$, then $n \geq 4$ and Proposition 2.1 leads to $2 \leq g(G) \leq n/2$. Combining this with (4), we conclude that

$$d_g(G) + g(G) \leq \frac{n}{g(G)} + g(G) \leq \frac{n}{2} + 2 \leq n.$$

Hence the equality $d_g(G) + g(G) = n + 1$ is impossible in this case, and the proof of Theorem 2.10 is complete. □

Theorem 2.11. *Let G be graph of order n with at least one non-complete component. Then*

$$d_g(G) + g(G) \leq n \tag{5}$$

with equality if and only if $G = C_4$ or G contains exactly one component H such that $H = K_1 + (K_{n_1} \cup K_{n_2} \cup \ldots \cup K_{n_p})$ with $p \geq 2$ and the other components (if any) are complete graphs.

Proof. Theorem 2.10 shows $d_g(G) + g(G) \leq n$ immediately. If $G = C_4$, then $g(G) = 2$ and $d_g(G) = 2$, and thus $d_g(G) + g(G) = 4 = n$. If G contains exactly one component H such that $H = K_1 + (K_{n_1} \cup K_{n_2} \cup \ldots \cup K_{n_p})$ with $p \geq 2$ and the other components (if any) are complete graphs, then it follows from Theorem 2.9 (b) that $g(G) = n - 1$. As for such graphs $d_g(G) = 1$, we see that $d_g(G) + g(G) = 1 + n - 1 = n$.

Conversely, assume that G has not the structure described above. Then Theorem 2.9 implies that $g(G) \leq n - 2$. If $d_g(G) = 1$, then we deduce that $d_g(G) + g(G) \leq 1 + n - 2 = n - 1$. If $d_g(G) \geq 2$, then $n \geq 4$ and Proposition 2.1 leads to $2 \leq g(G) \leq n/2$. It follows that

$$d_g(G) + g(G) \leq \frac{n}{g(G)} + g(G) \leq \frac{n}{2} + 2. \tag{6}$$

If $n \geq 5$, then (6) leads to $d_g(G) + g(G) \leq n - 1$, and if $n = 4$, then $d_g(G) + g(G) \leq 3 = n - 1$ or G is of the form described in Theorem 2.11. Hence the equality $d_g(G) + g(G) = n$ is impossible in this case, and the proof is complete. □

References

[1] B. Brešar and A.T. Horvat, *On the geodetic number of median graphs*, Discrete Math. **308** (2008), 4044-4051.

[2] B. Brešar, S. Klavžar and A.T. Horvat, *On the geodetic number and related metrics sets in Cartesian product graphs*, Discrete Math. **308** (2008), 5555-5561.

[3] F. Buckley and F. Harary, *Distance in Graphs*, Addison-Wesley, 1990.

[4] F. Buckley, F. Harary, and L.V. Quintas, *Extremal results on the geodetic number of a graph*, Scientia 2A (1988), 17-26.

[5] G. Chartrand, F. Harary, H.C. Swart and P. Zhang, *Geodomination in graphs*, Bull. ICA 31 (2001), 51-59.

[6] G. Chartrand, F. Harary and P. Zhang, *Geodetic sets in graphs*, Discuss. Math. Graph Theory 20 (2000), 129-138.

[7] G. Chartrand, F. Harary and P. Zhang, *On the geodetic number of a graph*, Networks 39 (2002), 1-6.

[8] M.C. Dourado, F. Protti, D. Rautenbach and J.L. Szwarcfiter, *Some remarks on the geodetic number of a graph*, Discrete Math. 310 (2010), 832-837.

[9] R. Muntean and P. Zhang, *On geodomination in graphs*, Congr. Numer. 143 (2000), 161-174.