On quadrilaterals in a bipartite graph *

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Abstract

Let $G = (V_1, V_2; E)$ be a bipartite graph with $|V_1|=|V_2|=2k$, where k is a positive integer. It is proved that if $d(x) + d(y) \geq 3k$ for every pair of nonadjacent vertices $x \in V_1$, $y \in V_2$, then G contains k independent quadrilaterals.

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1. Introduction

In this paper, all graphs are finite, simple, undirected and bipartite. Any undefined notation follows that of Bondy and Murty [1]. Let $G = (V_1, V_2; E)$ be a bipartite graph with $|V_1|=|V_2|$. We use $\delta(G)$ to denote the minimum degree in G and $\sigma_{1,1}(G)= \min\{d(x) + d(y) \mid x \in V_1,\ y \in V_2,\ x \neq y,\ xy \notin E(G)\}$. The order of G is $|G|$ and its size is $e(G)=|E|$. A set of graphs is said to be independent if no two of them have any common vertices. If H is a subgraph of G, then $N_H(x)=N_G(x)\bigcap V(H)$, $d(x, H)=|N_H(x)|$. Let X and Y be two independent subgraphs of G or two disjoint subsets of $V_1 \bigcup V_2$. We define $G[X]$ to be the subgraph of G induced by X, and $e(X, Y)$ to be the number of edges between X and Y. A k-cycle is a cycle of order k and a m-path is a path of order m, denoted by C^k and P^m respectively. In particular, a quadrilateral is a cycle of order 4. For a k-cycle $C=x_1x_2...x_kx_1$, x_ix_{i+1} is an edge in C. For two independent graphs G and H, $G \cup H$ is the union of G and H without adding any edge between G

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and H. Let T be a simple graph and k be a positive integer, then $G \supseteq kT$ means that G contains k independent subgraphs isomorphic to T.

One of the outstanding results on independent cycles comes from Corrádi and Hajanal [2]. It was proved that if G is a graph of order at least $3k$ with the minimum degree at least $2k$, then G contains k independent cycles. Wang [3] considered independent quadrilaterals in a bipartite graph and put forward the following conjecture which is still open.

Conjecture 1 *[3] Let $G = (V_1, V_2; E)$ be a bipartite graph with $|V_1|=|V_2|=2k$, where $k > 0$ is a positive integer. If $\delta(G) \geq k+1$, then G contains k independent quadrilaterals.*

In [3], Wang gave a result close to the Conjecture 1.

Theorem 1.1 *[3] Let $G = (V_1, V_2; E)$ be a bipartite graph with $|V_1|=|V_2|=2k$, where $k > 0$ is a positive integer. If $\delta(G) \geq k+1$, then G contains $k-1$ independent quadrilaterals and a 4-path such that they are independent.*

Yan [4] improved Theorem 1.1 and gave the following result.

Theorem 1.2 *[4] Let $G = (V_1, V_2; E)$ be a bipartite graph with $|V_1|=|V_2|=2k$, where $k > 0$ is a positive integer. If $\sigma_{1,1}(G) \geq 2k+1$, then G contains $k-1$ independent quadrilaterals and a 4-path such that they are independent.*

In this paper, we consider degree-sum conditions that ensure G contains k independent quadrilaterals. Our main result is as follows.

Theorem 1.3 *Let $G = (V_1, V_2; E)$ be a bipartite graph with $|V_1|=|V_2|=2k$, where k is a positive integer. If $\sigma_{1,1}(G) \geq 3k$, then G contains k independent quadrilaterals.*

The structure of the paper is as follows. First we will show some useful lemmas in the next section, then prove the Theorem 1.3 in Section 3.

2 Lemmas

In this section, $G = (V_1, V_2; E)$ is a bipartite graph.

Lemma 2.1 *Let $C = x_1y_1x_2y_2x_1$ be a quadrilateral and $P = a_1b_1a_2b_2$ be a path of order 4 in G with $x_1, a_1 \in V_1$. If $e(P,C) \geq 6$ and $d(a_1, C) > 0$, $d(b_2, C) > 0$, then $G[P \bigcup C]$ contains two independent quadrilaterals.*

Proof. Suppose on the contrary $G[P \bigcup C]$ doesn't contain two independent quadrilaterals. Since $e(P, C) \geq 6$, there exists a vertex $x \in V(C)$ such that $d(x, P) = 2$. Assume $d(x_1, P) = 2$. This implies $G[P - a_1 + x_1]$ contains a quadrilateral. Hence $G[C - x_1 + a_1]$ doesn't contain a quadrilateral and therefore $d(a_1, C) \leq 1$. Since $d(a_1, C) > 0$, it follows that $d(a_1, C) = 1$. Without loss of generality(denoted by w.l.o.g. for simplicity), let $a_1y_1 \in E$ and $a_1y_2 \notin E$. If $b_2x_2 \in E$, then $a_2y_1 \notin E$ and $a_2y_2 \notin E$ for otherwise $G[P \bigcup C]$ contains two independent quadrilaterals. Hence $e(P, C) \leq 5$, a contradiction. Now we have $b_2x_2 \notin E$. Since $e(P, C) \geq 6$ and $a_1y_2 \notin E$, it follows that $e(P, C) = 6$. This implies $y_2a_2 \in E$ and $x_2b_1 \in E$. Therefore, $G[P \bigcup C]$ contains two independent quadrilaterals $x_1y_2a_2b_2x_1$ and $y_1x_2b_1a_1y_1$, a contradiction. ∎

Lemma 2.2 *Let $C = x_1y_1x_2y_2x_1$ be a quadrilateral and $P = a_1b_1a_2b_2a_3b_3$ be a path of order 6 with $x_1, a_1 \in V_1$. If $e(P, C) \geq 10$, then either $G[P \bigcup C - a_1 - b_3]$ contains two independent quadrilaterals, or $G[P \bigcup C - a_1 - b_1]$ contains two independent quadrilaterals, or $G[P \bigcup C - a_3 - b_3]$ contains two independent quadrilaterals.*

Proof. Since $e(P, C) \geq 10$, $d(a_1, C) \leq 2$ and $d(b_3, C) \leq 2$, we have $e(P - a_1 - b_3, C) \geq 6$. If either $d(b_1, C) = 0$ or $d(a_3, C) = 0$, w.l.o.g., say $d(b_1, C) = 0$, then $d(a_1, C) = d(a_2, C) = d(a_3, C) = d(b_2, C) = d(b_3, C) = 2$. Hence $e(P - a_3 - b_3, C) = 6$. By Lemma 2.1, $G[(P - a_3 - b_3) \bigcup C]$ contains two independent quadrilaterals. Now assume $d(b_1, C) > 0$ and $d(a_3, C) > 0$. Since $e(P - a_1 - b_3, C) \geq 6$, it follows that $G[(P - a_1 - b_3) \bigcup C]$ contains two independent quadrilaterals from Lemma 2.1. This completes the proof. ∎

3 Proof of Theorem 1.3

In this section, we will prove the Theorem 1.3 by contradiction. Let $G = (V_1, V_2; E)$ be a bipartite graph with $|V_1| = |V_2| = 2k$ and $\sigma_{1,1}(G) \geq 3k$, where k is a positive integer. Suppose on the contrary that G doesn't contain k independent quadrilaterals. Since $\sigma_{1,1}(G) \geq 3k \geq 2k + 1$, G contains $k - 1$ independent quadrilaterals and a path of order 4 such that all of them are independent from Theorem 1.2. Let $C_1, C_2, \ldots, C_{k-1}$ be the $k - 1$ independent quadrilaterals of G and $P = a_1b_1a_2b_2$ with $a_1 \in V_1$ be the path of order 4 which is independent with the $k - 1$ quadrilaterals $C_1, C_2, \ldots, C_{k-1}$. Denote $H = \bigcup_{i=1}^{k-1} C_i$.

Since G doesn't contain k independent quadrilaterals, it follows that $G[P]$ doesn't contain a quadrilateral. Thus $a_1b_2 \notin E$ and $d(a_1, P) + d(b_2, P) = 2$. Hence,

$$e(a_1b_2, H) \geq 3k - 2 = 3(k - 1) + 1.$$

And so, there is a quadrilateral in H, say C_1, such that $e(a_1b_2, C_1) = 4$. Denote $C_1 = x_1y_1x_2y_2x_1$ with $x_1 \in V_1$ and $P' = y_1Px_1$. Then $x_1b_1 \notin E$ for otherwise $G[P \bigcup C_1]$ contains two independent quadrilaterals $y_1x_2y_2a_1y_1$ and $x_1b_2a_2b_1x_1$, a contradiction. With the same proof, $y_1a_2 \notin E$. Hence, the vertices of P' can be divided into three pairs of nonadjacent vertices. Since $d(a_1, C_1) = d(b_2, C_1) = 2$ and $G[P \bigcup C_1]$ doesn't contain two independent quadrilaterals, $e(P, C_1) \leq 5$ from Lemma 2.1. Hence

$$\sum_{x \in V(P')} d(x, H - C_1) \geq 9k - e(P, C_1) - 2e(G[P]) - e(x_1y_1, P \bigcup C_1) \geq 9(k-2)+1.$$

There is a quadrilaterals in $H - C_1$, say C_2, such that $e(P', C_2) \geq 10$. Since $G[P \bigcup C_2]$ doesn't contain two independent quadrilaterals, either $G[P' \bigcup C_2 - x_1 - b_2]$ contains two independent quadrilaterals or $G[P' \bigcup C_2 - y_1 - a_1]$ contains two independent quadrilaterals from Lemma 2.2. In the former case, $x_1y_2x_2b_2x_1$ is a quadrilateral in G; in the latter case, $y_1x_2y_2a_1y_1$ is a quadrilateral in G. Hence, $G[P \bigcup C_1 \bigcup C_2]$ contains three independent quadrilaterals and so G contains k independent quadrilaterals, a contradiction. This completes the whole proof.

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