Equitable Coloring on Corona graph of graphs

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Abstract

The notion of equitable coloring was introduced by Meyer in 1973. This paper presents exact values of equitable chromatic number of three corona graphs, which of complete graph and its complement $K_m \circ \overline{K_n}$, star graph and its complement $K_{1,m} \circ \overline{K_{1,n}}$ and complete graph and complete graph $K_m \circ K_n$.

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1 Introduction

The model of graph coloring has many applications. Everytime when we have to divide a system with binary conflicting relations into equal or almost equal conflict-free subsystems we can model such situation by means of equitable graph coloring. This subject is widely discussed in literature [1, 2, 3, 4, 5, 6, 7, 8]. In general, the problem of optimal equitable coloring, in the sense of the number of used colors, is NP-hard. So we have to look for simplified structure of graphs allowing polynomial-time algorithms. This paper gives solution for the corona graph of complete graph and its complement $K_m \circ \overline{K_n}$, corona graph of star graph and its complement $K_{1,m} \circ \overline{K_{1,n}}$ and corona graph of complete graphs $K_m \circ K_n$.

2 Preliminaries

If the set of vertices of a graph G can be partitioned into k classes $V_1, V_2, \cdots, V_k$ such that each V_i is an independent set and the condition $||V_i|-|V_j|| \leq 1$ holds for every pair (i, j), then G is said to be *equitably k-colorable.* The smallest integer k for which G is equitably k-colorable is known as the *equitable chromatic number* [1, 2, 3, 4, 5, 6, 7, 8] of G and denoted by $\chi_=(G)$.

The corona of two graphs G_1 and G_2 is the graph $G = G_1 \circ G_2$ formed from one copy of G_1 and $|V(G_1)|$ copies of G_2 where the i^{th} vertex of G_1 is adjacent to every vertex in the i^{th} copy of G_2.

3 Equitable coloring on corona graph of complete graph and its complement, star graph and its complement

Theorem 3.1. *For positive integers m and n, we have $\chi_=(K_m \circ \overline{K_n}) = m$.*

Proof. Let $V(K_m) = \{v_1, v_2, \cdots, v_m\}$ and $V(K_n) = \{u_1, u_2, \cdots, u_n\}$. By the definition of corona graph, each vertex of K_m is adjacent to every vertex of a copy of $\overline{K_n}$. Let $V(K_m \circ \overline{K_n}) = \{v_i : 1 \leq i \leq m\} \cup \{u_{ij} : 1 \leq i \leq$

$m, 1 \leq j \leq n\}$ and $E(K_m \circ \overline{K_n}) = E(K_m) \cup \bigcup_{i=1}^{m} \{v_i, u_{ij} : 1 \leq j \leq n\}$.

Now we partition the vertex set $V(K_m \circ \overline{K_n})$ as follows:

$$\begin{aligned} V_1 &= \{v_1, u_{m1}, u_{m2}, \cdots, u_{mn}\} \\ V_2 &= \{v_2, u_{11}, u_{12}, \cdots, u_{1n}\} \\ V_3 &= \{v_3, u_{21}, u_{22}, \cdots, u_{2n}\} \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ V_i &= \{v_i, u_{(i-1)1}, u_{(i-1)2}, \cdots, u_{(i-1)n}\} \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ V_m &= \{v_m, u_{(m-1)1}, u_{(m-1)2}, \cdots, u_{(m-1)n}\} \end{aligned}$$

Clearly $V_1, V_2, \cdots, V_m$ are independent sets and $|V_i| = n+1 (1 \leq i \leq n+1)$ satisfying the condition $||V_i| - |V_j|| = 0$ for every $i \neq j$. Therefore $\chi_=(K_m \circ \overline{K_n}) \leq m$. Since K_m is a subgraph contained in $K_m o \overline{K_n}$, $\chi_=(K_m \circ \overline{K_n}) \geq \chi(K_m \circ \overline{K_n}) \geq m$. Hence $\chi_=(K_m \circ \overline{K_n}) = m$. □

When $n = 1$, $K_{1,m} \circ \overline{K_{1,n}}$ is a bipartite graph with partite sets of sizes $m+2$ and $2m+1$, which is equitably 2-colorable for $m \leq 2$, but is not equitably $2(= n+1)$-colorable for $m \geq 3$, by the paper of B.L. Chen and K.W.Lih [2], let $G = G(X,Y)$ be a connected bipartite graph, then $\chi_=(G) = 2$ if and only if $||X| - |Y|| \leq 1$. By the above lemma when $n = 1$, $K_{1,m} \circ \overline{K_{1,n}}$ is not equitably 2-colorable for $m \geq 3$. But it is equitably 3-colorable by the following partition.

$$\begin{aligned} V_1 &= \{v_i : 1 \leq i \leq m\} \cup \{u_{00}\} \\ V_2 &= \{u_{i0} : 1 \leq i \leq m\} \cup \{u_{01}\} \\ V_3 &= \{u_{i1} : 1 \leq i \leq m\} \cup \{v_0\} \end{aligned}$$

Hence we find $K_{1,m} \circ \overline{K_{1,n}}$ for $n > 1$ in the following theorem.

Theorem 3.2. *For positive integers m and $n \geq 2$, we have*

$$\chi_=(K_{1,m} \circ \overline{K_{1,n}}) = n + 1.$$

Proof. Let $V(K_{1,m}) = \{v_0, v_1, v_2, \cdots, v_m\}$ and $V(K_{1,n}) = \{u_0, u_1, u_2, \cdots, u_n\}$. By the definition of corona graph, each vertex of $K_{1,m}$ is adjacent to every

vertex of a copy of $\overline{K_{1,n}}$. Let $V(K_{1,m} \circ \overline{K_{1,n}}) = \{v_i : 0 \le i \le m\} \cup \{u_{ij} : 0 \le i \le m, 0 \le j \le n\}$ and $E(K_{1,m} \circ \overline{K_{1,n}}) = E(K_{1,m}) \cup \bigcup_{i=0}^{m} \{u_{ij}, u_{ij'} : 1 \le j < j' \le n\} \cup \bigcup_{i=0}^{m} \{v_i, u_{ij} : 0 \le j \le n\}$.
It is clear that $m+1 = \left\lceil \frac{m+1}{n+1} \right\rceil + \left\lceil \frac{m+1-1}{n+1} \right\rceil + \left\lceil \frac{m+1-2}{n+1} \right\rceil + \cdots + \left\lceil \frac{m+1-n}{n+1} \right\rceil$ for $n \ge 2$. Now we define s_i as $s_i = \sum_{j=1}^{i} \left\lceil \frac{m+1-j+1}{n+1} \right\rceil$ for $1 \le i \le n+1$, then $s_{n+1} = m+1$.

Now we partition the vertex set $V(K_{1,m} \circ \overline{K_{1,n}})$ as follows:

$$
\begin{aligned}
V_1 &= \{v_j : s_1 \le j \le m\} \cup \{u_{j0}, u_{j1} : 0 \le j \le s_1 - 1\} \\
V_2 &= \{v_j : 1 \le j \le s_1 - 1\} \cup \{u_{j1} : s_1 \le j \le m\} \cup \\
&\quad \{u_{j0} : s_1 \le j \le s_2 - 1\} \cup \{u_{02}\} \\
V_3 &= \{u_{i2} : 1 \le i \le m\} \cup \{u_{j0} : s_2 \le j \le s_3 - 1\} \cup \{u_{03}\} \\
V_4 &= \{u_{i3} : 1 \le i \le m\} \cup \{u_{j0} : s_3 \le j \le s_4 - 1\} \cup \{u_{04}\} \\
&\quad \cdots\cdots\cdots\cdots\cdots\cdots \\
&\quad \cdots\cdots\cdots\cdots\cdots\cdots \\
V_n &= \{u_{i(n-1)} : 1 \le i \le m\} \cup \{u_{j0} : s_{n-1} \le j \le s_n - 1\} \cup \{u_{0n}\} \\
V_{n+1} &= \{u_{in} : 1 \le i \le m\} \cup \{u_{j0} : s_n \le j \le s_{n+1} - 1\,(= m)\} \cup \{v_0\}
\end{aligned}
$$

Clearly $V_1, V_2, \cdots, V_{n+1}$ are independent sets and $|V_i| = m+1+\left\lceil \frac{m+1-i+1}{n+1} \right\rceil$ for $1 \le i \le n+1$, satisfying the condition $||V_i| - |V_j|| \le 1$ for every $i \ne j$. Therefore $\chi_=(K_{1,m} \circ \overline{K_{1,n}}) \le n+1$. Since there exists a clique of order $n+1$ in $K_{1,m} \circ \overline{K_{1,n}}$, $\chi_=(K_{1,m} \circ \overline{K_{1,n}}) \ge \chi(K_{1,m} \circ \overline{K_{1,n}}) \ge n+1$. Hence $\chi_=(K_{1,m} \circ \overline{K_{1,n}}) = n+1$. □

4 Equitable coloring on corona graph of complete graphs

Theorem 4.1. *For positive integers m and n, we have*

$$\chi_{=}(K_m \circ K_n) = \begin{cases} n+1 & if \quad n \geq m \\ m & if \quad n < m. \end{cases}$$

Proof. Let $V(K_m) = \{v_1, v_2, \cdots, v_m\}$ and $V(K_n) = \{u_1, u_2, \cdots, u_n\}$. By the definition of corona graph, each vertex of K_m is adjacent to every vertex of a copy of K_n. Let $V(K_m \circ K_n) = \{v_i : 1 \leq i \leq m\} \cup \{u_{ij} : 1 \leq i \leq m, 1 \leq j \leq n\}$ and $E(K_m \circ K_n) = E(K_m) \cup \bigcup_{i=1}^{m} \{u_{ij}, u_{ij'} : 1 \leq j < j' \leq n\} \cup \bigcup_{i=1}^{m} \{v_i, u_{ij} : 1 \leq j \leq n\}$.

Case(i): When $n \geq m$. Now we partition the vertex set $V(K_m \circ K_n)$ as follows:

$$\begin{aligned}
V_1 &= \{v_1, u_{21}, u_{31}, \cdots, u_{m1}\} \\
V_2 &= \{v_2, u_{32}, u_{42}, \cdots, u_{m2}\} \cup \{u_{11}\} \\
V_3 &= \{v_3, u_{43}, u_{53}, \cdots, u_{m3}\} \cup \{u_{12}, u_{22}\} \\
& \ldots\ldots\ldots\ldots\ldots\ldots \\
& \ldots\ldots\ldots\ldots\ldots\ldots \\
V_m &= \{v_m\} \cup \{u_{1(m-1)}, u_{2(m-1)}, \cdots, u_{(m-1)(m-1)}\} \\
V_{m+1} &= \{u_{1m}, u_{2m}, \cdots, u_{mm}\} \\
V_{m+2} &= \{u_{1(m+1)}, u_{2(m+1)}, \cdots, u_{m(m+1)}\} \\
& \ldots\ldots\ldots\ldots\ldots \\
& \ldots\ldots\ldots\ldots\ldots \\
V_{m+(n-m)+1} &= \{u_{1n}, u_{2n}, \cdots, u_{mn}\}
\end{aligned}$$

Clearly $V_1, V_2, \cdots, V_{n+1}$ are independent sets and $|V_i| = m(1 \leq i \leq n+1)$ satisfying the condition $||V_i| - |V_j|| = 0$ for every $i \neq j$. Therefore $\chi_{=}(K_m \circ K_n) \leq n+1$. Since there exists a clique of order $n+1$ in $K_m \circ K_n$, $\chi_{=}(K_m \circ K_n) \geq \chi(K_m \circ K_n) \geq n+1$. Hence $\chi_{=}(K_m \circ K_n) = n+1$.

Case(ii): When $m > n$. Now we partition the vertex set $V(K_m \circ K_n)$ as follows:

$$
\begin{aligned}
V_1 &= \{v_1, u_{21}, u_{32}, \cdots, u_{(n+1)n}\} \\
V_2 &= \{v_2, u_{31}, u_{42}, \cdots, u_{(n+2)n}\} \\
V_3 &= \{v_3, u_{41}, u_{52}, \cdots, u_{(n+3)n}\} \\
& \cdots\cdots\cdots\cdots\cdots\cdots \\
& \cdots\cdots\cdots\cdots\cdots\cdots \\
V_{m-n} &= \{v_{m-n}, u_{(m-n+1)1}, u_{(m-n+2)2}, \cdots, u_{mn}\} \\
V_{m-n+1} &= \{v_{m-n+1}, u_{(m-n+2)1}, u_{(m-n+3)2}, \cdots, u_{m(n-1)}\} \cup \{u_{1n}\} \\
V_{m-n+2} &= \{v_{m-n+2}, u_{(m-n+3)1}, u_{(m-n+4)2}, \cdots, u_{m(n-2)}\} \cup \{u_{1(n-1)}, u_{2n}\} \\
& \cdots\cdots\cdots\cdots\cdots\cdots \\
& \cdots\cdots\cdots\cdots\cdots\cdots \\
V_m &= \{v_m\} \cup \{u_{11}, u_{22}, \cdots, u_{nn}\}
\end{aligned}
$$

Clearly $V_1, V_2, \cdots, V_m$ are independent sets and $|V_i| = n + 1 (1 \leq i \leq m)$ satisfying the condition $||V_i| - |V_j|| = 0$ for every $i \neq j$. Therefore $\chi_=(K_m \circ K_n) \leq m$. Since there exists a clique of order m in $K_m \circ K_n$, $\chi_=(K_m \circ K_n) \geq \chi(K_m \circ K_n) \geq m$. Hence $\chi_=(K_m \circ K_n) = m$. □

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