R-total domination on convex bipartite graphs

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Abstract

Let $\mathcal{P} = \{I_1, I_1 + d, I_1 + 2d, \ldots, I_1 + (\ell - 1) \cdot d\}$, where ℓ, d, I_1 are fixed integers and $\ell, d > 0$. Suppose that $G = (V, E)$ is a graph and R is a labeling function which assigns an integer $R(v)$ to each $v \in V$. An *R-total dominating function* of G is a function $f : V \to \mathcal{P}$ such that $\sum_{u \in N_G(v)} f(u) \geq R(v)$ for all vertices $v \in V$, where $N_G(v) = \{u \mid (u, v) \in E\}$. The *$R$-total domination problem* is to find an R-total dominating function f of G such that $\sum_{v \in V} f(v)$ is minimum. In this paper we present a linear-time algorithm to solve the R-total domination problem on convex bipartite graphs. Our algorithm gives a unified approach to the k-total, signed total, and minus total domination problems for convex bipartite graphs.

Keywords: Graph algorithms; Minus total dominating functions; Signed total dominating functions; Biconvex bipartite graphs; Planar bipartite graphs

1 Introduction

Let $G = (V, E)$ be a finite, simple, undirected graph with *vertex set* V and *edge set* E. It is understood that $|V| = n$ and $|E| = m$ if nothing else is stated. We also use $V(G)$ and $E(G)$ to denote the vertex and edge sets of G, respectively. We denote by $G[W]$ the subgraph of G induced by the vertex set $W \subseteq V$. For any vertex $v \in V$, the *neighborhood* of v in G is $N_G(v) = \{u \in V | (u, v) \in E\}$ and the *closed neighborhood* of v in G is $N_G[v] = N_G(v) \cup \{v\}$. The *degree* of a vertex v in G is $deg_G(v) = |N_G(v)|$.

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A *clique* is a subset of pairwise adjacent vertices of V. A *maximal clique* is a clique that is not a proper subset of any other clique.

A *total dominating set* of a graph $G = (V, E)$ is a subset D of V such that $|D \cap N_G(v)| \geq 1$ for every vertex $v \in V$. The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set of G. The total domination problem is to find a total dominating set of G of minimum cardinality. For a fixed positive integer k, a *k-total dominating set* of G is a subset D of V such that $|D \cap N_G(v)| \geq k$ for every vertex $v \in V$. The k-total domination problem is to find a k-total dominating set of G of minimum cardinality.

Definition 1. Suppose that $G = (V, E)$ is a finite, simple, undirected graph. Let $\mathcal{P}$ be a subset of real numbers. Let $f : V \to \mathcal{P}$ be a function which assigns to each $v \in V$ a value in $\mathcal{P}$. The set $\mathcal{P}$ is called the *weight set* of f. Let $f(S) = \sum_{u \in S} f(u)$ for any subset S of V. Then $f(V)$ is called the *weight* of f.

A function $f : V \to \{0, 1\}$ is a *total dominating function* of a graph $G = (V, E)$ if $f(N_G(v)) \geq 1$ for every vertex $v \in V$. A total dominating set can be viewed as a total dominating function f and thus $\gamma_t(G) = \min\{f(V) \mid f \text{ is a total dominating function of } G\}$. A function $f : V \to \mathcal{P}$ is a *signed* (respectively, *minus*) *total dominating function* of G if $\mathcal{P}$ is $\{-1, 1\}$ (respectively, $\{-1, 0, 1\}$). The *signed* (respectively, *minus*) *total domination number* of G, denoted by $\gamma_t^s(G)$ (respectively, $\gamma_t^-(G)$), is the minimum weight of a signed (respectively, minus) total dominating function of G. The signed (respectively, minus) total domination problem is to find a signed (respectively, minus) total dominating function of G of minimum weight.

Total domination is a well-known subject in graph theory. It has been thoroughly studied in the literature, and surveyed in [8, 9, 11]. Recently its variations, *signed total domination* and *minus total domination*, have been widely studied in [7, 10, 12, 13, 14, 19, 20, 21, 22, 23]. From the algorithmic point of view, the signed and minus total domination problems are linear-time solvable for trees [7, 13] and biconvex bipartite graphs [14], and these two problems can be solved in $O(n^2)$ time for chordal bipartite graphs [13]. However, the decision problems corresponding to these two problems are NP-complete for bipartite graphs, planar bipartite graphs, and doubly chordal graphs [7, 13, 14].

In [13], Lee introduced the concept of *R-total domination* to develop a unified approach to the signed and minus total domination problems for trees and chordal bipartite graphs.

Definition 2. Let ℓ, d, I_1 be fixed integers and $\ell, d > 0$. Let $\mathcal{P}$ be the weight set $\{I_1, I_1 + d, I_1 + 2d, \ldots, I_1 + (\ell - 1) \cdot d\}$. Suppose that $G = (V, E)$

is a graph and R is a labeling function which assigns an integer $R(v)$ to each $v \in V$. An *R-total dominating function* of $G = (V, E)$ is a function $f : V \to \mathcal{P}$ such that $f(N_G(v)) \geq R(v)$ for all vertices $v \in V$. The *R-total domination number* $\gamma_{t,R}(G)$ is the minimum weight of an R-total dominating function of G. The *R-total domination problem* is to find an R-total dominating function of G of minimum weight.

The R-total domination problem includes the k-total, signed total, and minus total domination problems as special cases. For example, if $\mathcal{P} = \{0, 1\}$ and $R(v) = k$ (where k is an integer) for every vertex v of a graph G, then we obtain the k-total domination problem. If $\mathcal{P} = \{-1, 1\}$ and $R(v) = 1$ for every vertex v of G, then we obtain the signed total domination problem. Finally, if $\mathcal{P} = \{-1, 0, 1\}$ and $R(v) = 1$ for every vertex v of G, then we obtain the minus total domination problem.

Lee showed that the R-total domination problem can be solved in $O(n+m)$ time for trees [13] and biconvex bipartite graphs [14], and that it can be solved in $O(n^2)$ time for chordal bipartite graphs [13].

Convex bipartite graphs are a subclass of chordal bipartite graphs and a superclass of biconvex bipartite graphs. They were introduced by Glover [5], motivated by several industrial and scheduling applications. In this paper, we present a linear-time algorithm for the R-total domination problem on convex bipartite graphs. Our algorithm gives a unified approach to the k-total, signed total, and minus total domination problems for convex bipartite graphs.

2 Preliminaries

Given a graph $G = (V, E)$, a vertex v is *simplicial* if all vertices of $N_G[v]$ form a clique. The ordering $v_1, v_2, \ldots, v_n$ of the vertices of V is a *perfect elimination ordering* of G if for all $i \in \{1, \ldots, n\}$, v_i is a simplicial vertex of the subgraph G_i of G induced by $\{v_i, v_{i+1}, \ldots, v_n\}$. A *chord* of a cycle is an edge between two vertices of the cycle that is not an edge of the cycle. A graph G is called a *chordal* graph if each cycle in G of length at least 4 has at least one chord. Rose [17] showed the characterization that a graph is *chordal* if and only if it has a perfect elimination ordering. Let $N_i[v]$ denote the closed neighborhood of v in G_i. A perfect elimination ordering is called a *strong elimination ordering* if it has the following property:

For $i < j < k$ if v_j and v_k belong to $N_i[v_i]$ in G_i, then $N_i[v_j] \subseteq N_i[v_k]$. Farber [4] showed that a graph is *strongly chordal* if and only if it admits a strong elimination ordering. Currently, the fastest algorithms for recognizing a strongly chordal graph and giving a strong elimination ordering run in $O(m \log n)$ [16] or $O(n^2)$ time [18].

A graph $G = (V, E)$ is a *bipartite* graph if V can be partitioned into two disjoint sets A and B such that every edge has its ends in different sets. We call the sets, A and B, the *bipartition* of V and use $G = (A, B, E)$ to denote a bipartite graph. An ordering of the vertices in B (respectively, A) has the *adjacency property* if for each vertex $a \in A$ (respectively, $b \in B$) $N_G(a)$ (respectively, $N_G(b)$) consists of vertices which are consecutive in the ordering of the vertices in B (respectively, A). We say that G is *convex* on B (respectively, A) if an ordering of the vertices in B (respectively, A) fulfills the adjacency property.

A bipartite graph is a *chordal* bipartite graph if every cycle of length at least 6 has a chord. A bipartite graph $G = (A, B, E)$ is a *convex* bipartite graph if G is convex on A or B, and G is a *biconvex* bipartite graph if it is convex on *both* A and B.

Convex bipartite graphs are a subclass of chordal bipartite graphs, and a superclass of biconvex bipartite graphs and bipartite permutation graphs [3]. For convex bipartite, biconvex bipartite, and bipartite permutations graphs, there are linear-time recognition algorithms that produce the corresponding orderings on the vertex sets in linear time [1, 6, 15].

Definition 3. Let $G = (A, B, E)$ be a convex bipartite graph. An ordering of the vertices in $A \cup B$ is a *convex* ordering of G if the corresponding ordering of the vertices in B fulfills the adjacency property.

Definition 4. Suppose that $G = (A, B, E)$ is a bipartite graph. Let G_A (respectively, G_B) be the graph obtained by adding all possible edges between vertices of A (respectively, B) such that the set A (respectively, B) is a clique of G_A (respectively, G_B).

Lemma 1 shows a connection between chordal bipartite graphs and strongly chordal graphs.

Lemma 1 ([2, 4]). *The graphs G_A and G_B obtained from a chordal bipartite graph $G = (A, B, E)$ are strongly chordal graphs.*

Throughout this paper, we assume that a convex bipartite graph $G = (A, B, E)$ is convex on B, and that the corresponding orderings of vertices in A and B are $a_1, a_2, \ldots, a_{|A|}$ and $b_1, b_2, \ldots, b_{|B|}$. Figure 1 shows a convex bipartite graph.

Note that an R-total dominating function of a graph does not exist if the graph contains an isolated vertex. We assume that all graphs considered in this paper do not contain isolated vertices.

3 R-total domination on convex bipartite graphs

In this section, we develop a linear-time algorithm to solve the R-total domination problem on convex bipartite graphs. Suppose that $G = (A, B, E)$

is a convex bipartite graph with $|A \cup B| = n$ and $|E| = m$. By Lemma 1, the graphs G_A and G_B are strongly chordal graphs. We compute a strong elimination ordering of G_A (respectively, G_B) from a convex ordering of G in Section 3.1. Then, using strong elimination orderings of G_A and G_B, we give a linear-time algorithm to solve the R-total domination problem for G in Section 3.2.

Note that the fastest algorithms for recognizing a strongly chordal graph and giving a strong elimination ordering run in $O(m \log n)$ [16] or $O(n^2)$ time [18]. We can use one of these two recognition algorithms to compute strong elimination orderings of G_A and G_B, but the running time is not linear. Furthermore, to our knowledge, there is no algorithm for computing a strong elimination ordering of G_A (respectively, G_B) from a convex ordering of G. Therefore, we give a linear-time algorithm in Section 3.1 to compute a strong elimination ordering of G_A (respectively, G_B). The idea of the method for constructing a strong elimination ordering of G_A (respectively, G_B) is as follows.

By our assumption, a convex bipartite graph $G = (A, B, E)$ is convex on B and the corresponding orderings of vertices in A and B are $a_1, a_2, \ldots, a_{|A|}$ and $b_1, b_2, \ldots, b_{|B|}$. The ordering of the vertices in B has the adjacency property. However, the ordering of the vertices in A does not necessarily have the adjacency property. Therefore, we partition the set A into $A_1, A_2, \ldots, A_{|B|}$ such that the maximum index of the vertices in $N_G(a)$ is i for each vertex $a \in A_i$ if A_i is nonempty, where $1 \le i \le |B|$. For each nonempty set A_i, we arrange the vertices of A_i in non-decreasing order of their degrees. For $1 \le i \le |B|$, let $|A_i| = n_i$ and let $a_{i_1}, a_{i_2}, \ldots, a_{i_{n_i}}$ be the vertices of A_i if A_i is nonempty. Then, we visit the sets A_1, A_2, ..., $A_{|B|}$ one by one and output the vertices $a_{i_1}, a_{i_2}, \ldots, a_{i_{n_i}}$ while a visited set A_i is nonempty. The sequence of all vertices output from the visited sets results in a new ordering of the vertices in A. The new ordering of the vertices in A can lead to a new convex ordering of the vertices in $A \cup B$. Let $a'_1, a'_2, \ldots, a'_{|A|}, b_1, b_2, \ldots, b_{|B|}$ be the new convex ordering. This ordering is a strong elimination ordering of G_B and the ordering $b_1, b_2, \ldots, b_{|B|}, a'_1, a'_2, \ldots, a'_{|A|}$ is a strong elimination ordering of G_A. See Section 3.1 for further details.

3.1 Strong convex orderings and strong elimination orderings

Let $G = (A, B, E)$ be a convex bipartite graph. Define $\ell(a)$ and $r(a)$ such that $N_G(a) = \{b_{\ell(a)}, b_{\ell(a)+1}, \ldots, b_{r(a)}\}$ for all $a \in A$. For $1 \le i \le |B|$, we use A_i to denote a maximum subset of A such that $b_{r(a)} = b_i$ for all $a \in A_i$. In other words, the maximum index of the vertices in $N_G(a)$ is i for each vertex $a \in A_i$ if A_i is nonempty. Then, $A = A_1 \cup A_2 \cup \cdots \cup A_{|B|}$. Let

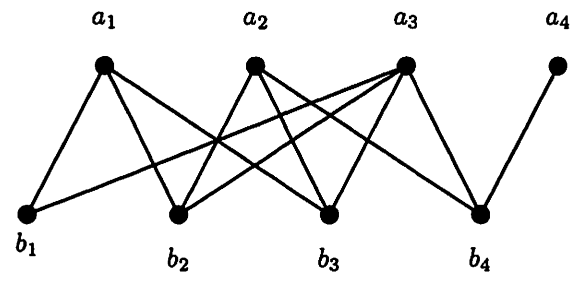

Figure 1: A convex bipartite graph.

$|A_i| = n_i$. If $n_i \neq 0$, let $\ell(A_i) = \min\{\ell(a) | a \in A_i\}$ and let the vertices in A_i be ordered as $a_{i_1}, a_{i_2}, \ldots, a_{i_{n_i}}$ such that $\ell(a_{i_{n_i}}) \leq \ell(a_{i_{n_i-1}}) \leq \cdots \leq \ell(a_{i_1})$. Clearly, $deg_G(a_{i_1}) \leq deg_G(a_{i_2}) \leq \cdots \leq deg_G(a_{i_{n_i}})$. Suppose that W is a collection of all nonempty sets of $A_1, A_2, \ldots, A_{|B|}$. Obviously, any two nonempty sets of W are disjoint if $|W| \geq 2$.

We give the function ConvexSets(G, A, B) to compute a list $\mathcal{M}$ of sets $\hat{A}_1, \ldots, \hat{A}_{|B|}$ for a convex bipartite graph $G = (A, B, E)$. For $1 \leq j \leq |B|$, $\hat{A}_j$ is a subset of A and implemented as a list of vertices. To illustrate, we let $H = (A, B, E)$ be the convex bipartite graph with $A = \{a_1, a_2, a_3, a_4\}$ and $B = \{b_1, b_2, b_3, b_4\}$ as shown in Figure 1. In this function, the set $\mathcal{M}$ is implemented as a list of sets, and S_i and $\hat{A}_i$ are implemented as lists of vertices for $1 \leq i \leq 4$. At the end of Step 9, $S_1 = \{a_1, a_3\}$, $S_2 = \{a_2\}$, $S_3 = \emptyset$, and $S_4 = \{a_4\}$. Note that $\ell(a_4) = r(a_4) = 4$. In Steps 10–16, the function visits the sets S_4, S_3, S_2, S_1 one by one in decreasing order of their indices. At the end of Step 16, $\hat{A}_1 = \emptyset$, $\hat{A}_2 = \emptyset$, $\hat{A}_3 = \{a_1\}$, and $\hat{A}_4 = \{a_4, a_2, a_3\}$ with $\ell(a_3) \leq \ell(a_2) \leq \ell(a_4)$. The function ConvexSets(H, A, B) returns a list of sets $\mathcal{M} = \emptyset, \emptyset, \{a_1\}, \{a_4, a_2, a_3\}$.

Lemma 2. *Let $G = (A, B, E)$ be a convex bipartite graph. Suppose that a list $\mathcal{M} = \hat{A}_1, \hat{A}_2, \ldots, \hat{A}_{|B|}$ is returned from the function* ConvexSets(G, A, B). *For each nonempty set $\hat{A}_j$ in the list $\mathcal{M}$, where $1 \leq j \leq |B|$, let $a_{j_1}, a_{j_2}, \ldots, a_{j_{n_j}}$ be the vertices in $\hat{A}_j$ listed from the beginning to the end of $\hat{A}_j$. Then, $\ell(a_{j_{n_j}}) \leq \ell(a_{j_{n_j-1}}) \leq \cdots \leq \ell(a_{j_1})$.*

Proof. In this function, the set $\mathcal{M}$ is implemented as a list of sets, and S_i and $\hat{A}_i$ are implemented as lists of vertices for $1 \leq i \leq |B|$. At the end of Step 9, S_j consists of all vertices a in A with $\ell(a) = j$ for $1 \leq j \leq |B|$. Then $A = S_1 \cup S_2 \cup \cdots \cup S_{|B|}$.

In Steps 10–16, the function visits the sets $S_{|B|}, S_{|B|-1}, \ldots, S_1$ one by

Function ConvexSets(G, A, B)

1: $\mathcal{M} \leftarrow$ empty list of sets;
2: **for** $i \leftarrow 1$ **to** $|B|$ **do**
3:     $S_i \leftarrow$ empty list of vertices;
4:     $\hat{A}_i \leftarrow$ empty list of vertices;
5: **end for**
6: **for** $i \leftarrow 1$ **to** $|A|$ **do**
7:     $j \leftarrow \ell(a_i)$;
8:     Append a_i to S_j;
9: **end for**
10: **for** $i \leftarrow |B|$ **to** 1 **do**
11:     **if** S_i is not empty **then**
12:         **for** each vertex $a \in S_i$ **do**
13:             $j \leftarrow r(a)$;
14:             Append a to $\hat{A}_j$;
15:         **end for**
16: **end for**
17: **for** $i \leftarrow 1$ **to** $|B|$ **do**
18:     Append $\hat{A}_i$ to $\mathcal{M}$;
19: **end for**
20: **return** $\mathcal{M}$;

one in decreasing order of their indices. At the end of Step 16, $\hat{A}_j$ consists of all vertices a in A with $r(a) = j$ for $1 \leq j \leq |B|$. If $\hat{A}_j \neq \emptyset$, then the vertices in $\hat{A}_j$ are $a_{j_1}, a_{j_2}, \ldots, a_{j_{n_j}}$. The lemma is trivial if $|\hat{A}_j| = 1$. Suppose that $|\hat{A}_j| \geq 2$. For any two distinct vertices $a_{j_x}, a_{j_y} \in \hat{A}_j$ with $x < y$, let $p, q \in \{1, 2, \ldots, |B|\}$ such that $a_{j_x} \in S_p$ and $a_{j_y} \in S_q$. If $p = q$, then $\ell(a_{j_y}) = \ell(a_{j_x})$. Suppose that $p \neq q$. Since $x < y$, the function appends a_{j_x} to $\hat{A}_j$ before appending a_{j_y} to $\hat{A}_j$. It implies that the function visits S_p before visiting S_q. Therefore, $q < p$ and $\ell(a_{j_y}) < \ell(a_{j_x})$. Following the discussion above, we know that $\ell(a_{j_y}) \leq \ell(a_{j_x})$ for any two distinct vertices $a_{j_x}, a_{j_y} \in \hat{A}_j$ with $x < y$. Hence, $\ell(a_{j_{n_j}}) \leq \ell(a_{j_{n_j-1}}) \leq \cdots \leq \ell(a_{j_1})$ □

Lemma 3. *The function* ConvexSets(G, A, B) *can be implemented in* $O(n+m)$ *time.*

Proof. Let $G = (A, B, E)$ be a convex bipartite graph. Note that G is convex on B. For every $a \in A$, $\ell(a)$ and $r(a)$ can be computed in $O(deg_G(a))$ time. Hence, the running time of the function is $O\left(\sum_{a \in A}(deg_G(a) + 1)\right) = O(n + m)$ time. □

Following Lemmas 2 and 3, we can compute A_1, A_2, A_3, $\ldots, A_{|B|}$

by the function ConvexSets(G, A, B) in $O(n+m)$ time for a convex bipartite graph $G = (A, B, E)$. We can visit the sets A_1, A_2, ..., $A_{|B|}$ one by one in increasing order of their indices, and output the vertices $a_{i_1}, a_{i_2}, \ldots, a_{i_{n_i}}$ while a visited set A_i is nonempty. The sequence of all vertices output from the visited sets results in a new ordering of the vertices in A. The new ordering of the vertices in A can lead to a new convex ordering of the vertices in $A \cup B$. We call the new convex ordering a *strong convex ordering* of G. To illustrate, we let $H = (A, B, E)$ be the convex bipartite graph with $A = \{a_1, a_2, a_3, a_4\}$ and $B = \{b_1, b_2, b_3, b_4\}$ as shown in Figure 1. The function ConvexSets(H, A, B) returns a list of sets $\mathcal{M} = \emptyset, \emptyset, \{a_1\}, \{a_4, a_2, a_3\}$. Let $A_1 = \emptyset$, $A_2 = \emptyset$, $A_3 = \{a_1\}$, and $A_4 = \{a_4, a_2, a_3\}$. Clearly, $\ell(a_3) \leq \ell(a_2) \leq \ell(a_4)$. The ordering $a_1, a_4, a_2, a_3, b_1, b_2, b_3, b_4$ (respectively, $b_1, b_2, b_3, b_4, a_1, a_4, a_2, a_3$) is a strong convex ordering of H.

The following theorem can be easily verified according to the discussion above.

Theorem 1. *Let $G = (A, B, E)$ be a convex bipartite graph. A strong convex ordering of G can be computed in $O(n+m)$ time.*

Let $G = (A, B, E)$ be a convex bipartite graph. We use $\langle A, B\rangle$ (respectively, $\langle B, A\rangle$) to denote a strong convex ordering $v_1, v_2, \ldots, v_n$ of G such that $A = \{v_1, \ldots, v_{|A|}\}$ and $B = \{v_{|A|+1}, \ldots, v_n\}$ (respectively, $B = \{v_1, \ldots, v_{|B|}\}$ and $A = \{v_{|B|+1}, \ldots, v_n\}$). For any strong convex ordering $\langle A, B\rangle = v_1, v_2, \ldots, v_n$ (respectively, $\langle B, A\rangle = v_1, v_2, \ldots, v_n$), $v_{|A|+i} = b_i$ (respectively, $v_i = b_i$) for $1 \leq i \leq |B|$.

Suppose that $A_i \neq \emptyset$ for some integer $i \in \{1, 2, \ldots, |B|\}$. By the construction of a strong convex ordering of G as mentioned above, the vertices of A_i in the ordering $\langle A, B\rangle$ are consecutive. Let $v_{k_i+j-1} = a_{i_j}$ for $1 \leq j \leq n_i$. Then $A_i = \{v_{k_i}, v_{k_i+1}, \ldots, v_{k_i+n_i-1}\}$, where $\ell(v_{k_i+n_i-1}) \leq \ell(v_{k_i+n_i-2}) \leq \cdots \leq \ell(v_{k_i})$.

We use $\hat{H}_i$ (respectively, H_i) to represent the subgraph of G induced by $A_i \cup B$ (respectively, $A_i \cup A_{i+1} \cup \cdots \cup A_{|B|} \cup B$) for $1 \leq i \leq |B|$. For $1 \leq j \leq n_i$, we use $\hat{H}_{i_j}$ (respectively, H_{i_j}) be the subgraph of $\hat{H}_i$ (respectively, H_i) induced by $\{v_{k_i+j-1}, v_{k_i+j}, \ldots, v_{k_i+n_i-1}\} \cup B$ (respectively, $\{v_{k_i+j-1}, v_{k_i+j}, \ldots, v_{k_i+n_i-1}\} \cup A_{i+1} \cup \cdots \cup A_{|B|} \cup B$).

Lemma 4. *Let $G = (A, B, E)$ be a convex bipartite graph with a strong convex ordering $\langle A, B\rangle = v_1, v_2, \ldots, v_n$. Suppose that $A_i \neq \emptyset$ for some integer $i \in \{1, 2, \ldots, |B|\}$. In $\hat{H}_i$, the ordering $v_{k_i}, v_{k_i+1}, \ldots, v_{k_i+n_i-1}$ for the set A_i fulfills the adjacency property.*

Proof. The lemma is trivial if $|A_i| = 1$ or 2. Suppose that $|A_i| \geq 3$. Note that $\ell(v_{k_i+n_i-1}) \leq \ell(v_{k_i+n_i-2}) \leq \cdots \leq \ell(v_{k_i})$. Clearly, $\ell(A_i) =$

$\ell(v_{k_i+n_i-1})$. Let $h = \ell(v_{k_i+n_i-1})$. Then $\bigcup_{v\in A_i} N_{\hat{H}_i}(v) = \{b_h, b_{h+1}, \ldots, b_i\}$ $\subseteq B$. We assume for contrary that the ordering $v_{k_i}, v_{k_i+1}, \ldots, v_{k_i+n_i-1}$ does not have the adjacency property. There exists a vertex b_j, where $h \leq j \leq i$, such that the vertices in $N_{\hat{H}_i}(b_j)$ are not consecutive. Let $v_{k_i+x}, v_{k_i+x+1}, \ldots, v_{k_i+x+s} \in A_i$ be consecutive vertices such that v_{k_i+x}, $v_{k_i+x+s} \in N_{\hat{H}_i}(b_j)$, $v_{k_i+x+p} \notin N_{\hat{H}_i}(b_j)$, where $1 \leq p \leq s-1$. Therefore $\ell(v_{k_i+x}) \leq b_j$ and $b_j < \ell(v_{k_i+x+p})$. Then $\ell(v_{k_i+x}) \leq b_j < \ell(v_{k_i+x+p})$, which contradicts that $\ell(v_{k_i+x+p}) \leq \ell(v_{k_i+x})$. Hence, the ordering v_{k_i}, $v_{k_i+1}, \ldots, v_{k_i+n_i-1}$ fulfills the adjacency property. □

Lemma 5. *Let $G = (A, B, E)$ be a convex bipartite graph with a strong convex ordering $\langle A, B\rangle = v_1, v_2, \ldots, v_n$. Suppose that $A_i \neq \emptyset$ for some integer $i \in \{1, 2, \ldots, |B|\}$. Let j be an integer such that $1 \leq j \leq n_i$. For any two vertices $b_x, b_y \in N_{\hat{H}_{i_j}}(v_{k_i+j-1})$ with $x < y$, $N_{\hat{H}_{i_j}}(b_x) = N_{\hat{H}_{i_j}}(b_y)$.*

Proof. Notice that $\ell(v_{k_i+n_i-1}) \leq \ell(v_{k_i+n_i-2}) \leq \cdots \leq \ell(v_{k_i})$. Then $\ell(v_{k_i+n_i-1}) \leq \ell(v_{k_i+j-1})$. Since $b_x, b_y \in N_{\hat{H}_{i_j}}(v_{k_i+j-1})$ and $x < y$, $\ell(v_{k_i+j-1})$ $\leq x < y \leq i$. We have $\ell(v_{k_i+n_i-1}) \leq x < y \leq i$. The vertex $v_{k_i+n_i-1}$ is therefore adjacent to b_x and b_y. Following Lemma 4, the ordering $v_{k_i}, v_{k_i+1}, \ldots, v_{k_i+n_i-1}$ for the set A_i fulfills the adjacency property in $\hat{H}_i$. Since both b_x and b_y are adjacent to v_{k_i+j-1} and $v_{k_i+n_i-1}$, we have $N_{\hat{H}_{i_j}}(b_x) = N_{\hat{H}_{i_j}}(b_y) = \{v_{k_i+j-1}, v_{k_i+j}, \ldots, v_{k_i+n_i-1}\}$. □

Theorem 2. *Let $G = (A, B, E)$ be a convex bipartite graph with a strong convex ordering $\langle A, B\rangle = v_1, v_2, \ldots, v_n$. Suppose that $A_i \neq \emptyset$ for some integer $i \in \{1, 2, \ldots, |B|\}$. Let j be an integer such that $1 \leq j \leq n_i$. Then $N_{H_{i_j}}(b_x) \subseteq N_{H_{i_j}}(b_y)$ for any two vertices $b_x, b_y \in N_{H_{i_j}}(v_{k_i+j-1})$ with $x < y$.*

Proof. The theorem can be proved by induction as follows. Let p be a positive integer. Suppose that $p = |B|$. The graph H_p is the subgraph of G induced by $A_p \cup B$, and thus $H_p = \hat{H}_p$. By Lemma 5, we have $N_{H_{p_j}}(b_x) = N_{\hat{H}_{p_j}}(b_x) = N_{\hat{H}_{p_j}}(b_y) = N_{H_{p_j}}(b_y)$. The theorem therefore holds. We assume that the theorem holds for H_{p_j} with $i+1 \leq p$ (the inductive hypothesis). Note that H_i is the subgraph of G induced by $A_i \cup V(H_{i+1})$. Let $b \in B$ be a vertex in $N_{H_{i_j}}(v_{k_i+j-1})$. It can be easily verified that $N_{H_{i_j}}(b) = N_{\hat{H}_{i_j}}(b) \cup N_{H_{i+1}}(b)$. We consider the following cases.

Case 1: b_x is not adjacent to any vertex in $A_{i+1} \cup A_{i+2} \cup \cdots \cup A_{|B|}$. Then $N_{H_{i_j}}(b_x) = N_{\hat{H}_{i_j}}(b_x)$. By Lemma 5, we have $N_{H_{i_j}}(b_x) = N_{\hat{H}_{i_j}}(b_x) = N_{\hat{H}_{i_j}}(b_y)$. Hence $N_{H_{i_j}}(b_x) \subseteq N_{H_{i_j}}(b_y)$.

Case 2: b_x is adjacent to a vertex in $A_{i+1} \cup A_{i+2} \cup \cdots \cup A_{|B|}$. Let $W = \{A_t | i+1 \leq t \leq |B| \text{ and } b_x \text{ is adjacent to a vertex in } A_t\}$. Suppose

ℓ is the smallest index of the sets of W. Let s be the minimum number of $1, 2, \ldots, |A_\ell|$ such that $v_{k_\ell+s-1} \in A_\ell$ is adjacent to b_x. It is clear that $N_{H_{i+1}}(b_x) = N_{H_{\ell,s}}(b_x)$. Since b_y is adjacent to v_{k_i+j-1}, we have $x < y \leq i < \ell$. Therefore, $v_{k_\ell+s-1}$ is adjacent to b_y. By the inductive hypothesis, $N_{H_{\ell,s}}(b_x) \subseteq N_{H_{\ell,s}}(b_y)$. Following Lemma 5, we have $N_{\hat{H}_{i_j}}(b_x) = N_{\hat{H}_{i_j}}(b_y)$.

Hence $N_{H_{i_j}}(b_x) = N_{\hat{H}_{i_j}}(b_x) \cup N_{H_{i+1}}(b_x) = N_{\hat{H}_{i_j}}(b_x) \cup N_{H_{\ell,s}}(b_x) \subseteq N_{\hat{H}_{i_j}}(b_y) \cup N_{H_{\ell,s}}(b_y) \subseteq N_{\hat{H}_{i_j}}(b_y) \cup N_{H_{i+1}}(b_y) = N_{H_{i_j}}(b_y)$. Following the discussion above, the theorem holds. □

Theorem 3. *Suppose that $G = (A, B, E)$ is a convex bipartite graph with a strong convex ordering $\langle A, B\rangle$. Then, the strong convex ordering $\langle A, B\rangle = v_1, v_2, \ldots, v_n$ is a strong elimination ordering of G_B.*

Proof. Following Lemma 1, the graph G_B obtained from G is a strongly chordal graph. Let G_p be the subgraph of G_B induced by $\{v_p, v_{p+1}, \ldots, v_n\}$, where $1 \leq p \leq n$. Let $N_p(v)$ denote the neighborhood of v in G_p and let $N_p[v] = N_p(v) \cup \{v\}$ denote the closed neighborhood of v in G_p. It can be easily verified that $N_p[v_p]$ is a clique of G_p for $1 \leq p \leq n$. Therefore, the ordering $v_1, v_2, \ldots, v_n$ is a perfect elimination ordering of G_B. Suppose that there exist three positive integers p, s, and t such that $1 \leq p < s < t \leq n$ and $v_s, v_t \in N_p[v_p]$. We prove that the strong convex ordering $v_1, v_2, \ldots, v_n$ is a strong elimination ordering of G_B by showing that $N_p[v_s] \subseteq N_p[v_t]$. We consider the following cases:

Case 1: $|A| + 1 \leq p \leq n$. Then $V(G_p) \subseteq B$. Note that B is a clique of G_B. Hence, $N_p[v_s] = N_p[v_t]$.

Case 2: $1 \leq p \leq |A|$. Then $v_p \in A$, $B \subset V(G_p)$, and $v_s, v_t \in B$. Clearly, $N_p[v_s] = (N_p[v_s] \cap A) \cup (N_p[v_s] \cap B)$ and $N_p[v_t] = (N_p[v_t] \cap A) \cup (N_p[v_t] \cap B)$. Note that B is a clique of G_B. Then $(N_p[v_s] \cap B) = (N_p[v_t] \cap B)$. In the following, we consider the inclusion relationship between $N_p[v_s] \cap A$ and $N_p[v_t] \cap A$.

Since $v_s, v_t \in B$, $N_p[v_s] \cap A = N_p(v_s) \cap A$ and $N_p[v_t] \cap A = N_p(v_t) \cap A$. Note that G_p is the subgraph of G_B induced by $\{v_p, v_{p+1}, \ldots, v_{|A|}, v_{|A|+1} \ldots, v_n\}$. Let $b_x, b_y \in B$ with $x < y$ such that $b_x = v_s$ and $b_y = v_t$. Suppose that $v_p \in A_i$ and $v_p = v_{k_i+j-1}$, where $1 \leq i \leq |B|$ and $1 \leq j \leq n_i$. We have $\{v_p, v_{p+1}, \ldots, v_{|A|}, v_{|A|+1} \ldots, v_n\} = \{v_{k_i+j-1}, v_{k_i+j}, \ldots, v_{k_i+n_i-1}\} \cup A_{i+1} \cup \cdots \cup A_{|B|} \cup B$. Therefore, $N_p(v_s) \cap A = N_p(b_x) \cap A = N_{H_{i_j}}(b_x)$ and $N_p(v_t) \cap A = N_p(b_y) \cap A = N_{H_{i_j}}(b_y)$. By Theorem 2, $N_{H_{i_j}}(b_x) \subseteq N_{H_{i_j}}(b_y)$. Then, $(N_p[v_s] \cap A) \subseteq (N_p[v_t] \cap A)$.

Following the discussion above, $N_p[v_s] \subseteq N_p[v_t]$. Hence, the strong convex ordering $\langle A, B\rangle = v_1, v_2, \ldots, v_n$ is a strong elimination ordering of G_B. □

Theorem 4. *Suppose that $G = (A, B, E)$ is a convex bipartite graph with a strong convex ordering $\langle B, A\rangle = v_1, v_2, \ldots, v_n$. Then, $v_1, v_2, \ldots, v_n$ is a strong elimination ordering of G_A.*

Proof. Following Lemma 1, the graph G_A obtained from G is a strongly chordal graph. Let G_i be the subgraph of G_A induced by $\{v_i, v_{i+1}, \ldots, v_n\}$, where $1 \leq i \leq n$. Let $N_i[v]$ denote the closed neighborhood of v in G_i. It can be easily verified that $N_i[v_i]$ is a clique of G_i for $1 \leq i \leq n$. Therefore, the ordering $v_1, v_2, \ldots, v_n$ is a perfect elimination ordering of G_A. Suppose that there exist three positive integers i, j, and k such that $1 \leq i < j < k \leq n$ and $v_j, v_k \in N_i[v_i]$. We prove the strong convex ordering $v_1, v_2, \ldots, v_n$ is a strong elimination ordering of G_A by showing that $N_i[v_j] \subseteq N_i[v_k]$. We consider the following cases:

Case 1: $|B| + 1 \leq i \leq n$. Then $V(G_i) \subseteq A$. Note that A is a clique of G_A. Hence, $N_i[v_j] = N_i[v_k]$.

Case 2: $1 \leq i \leq |B|$. Then $v_i \in B$, $A \subset V(G_i)$, and $v_j, v_k \in A$. Clearly, $N_i[v_j] = (N_i[v_j] \cap A) \cup (N_i[v_j] \cap B)$ and $N_i[v_k] = (N_i[v_k] \cap A) \cup (N_i[v_k] \cap B)$. Since A is a clique of G_A, $(N_i[v_j] \cap A) = (N_i[v_k] \cap A)$. In the following, we consider the inclusion relationship between $N_i[v_j] \cap B$ and $N_i[v_k] \cap B$.

Note that G is convex on B. The vertices in $N_i[v_j] \cap B$ (respectively, $N_i[v_k] \cap B$) are consecutive in the strong convex ordering. Therefore, $N_i[v_j] \cap B = \{v_i, v_{i+1}, \ldots, v_{r(v_j)}\}$ and $N_i[v_k] \cap B = \{v_i, v_{i+1}, \ldots, v_{r(v_k)}\}$. If $r(v_j) = r(v_k)$, then $(N_i[v_j] \cap B) = (N_i[v_k] \cap B)$. We have $N_i[v_j] = N_i[v_k]$. Suppose that $r(v_j) \neq r(v_k)$. Let $v_j \in A_p$ and $v_k \in A_q$, where $1 \leq p, q \leq |B|$. Note that $j < k$. By the construction of a strong convex ordering of G, we know that $p < q$ and thus $r(v_j) < r(v_k)$. Then, $(N_i[v_j] \cap B) \subseteq (N_i[v_k] \cap B)$. We have $N_i[v_j] \subseteq N_i[v_k]$.

Following the discussion above, the strong convex ordering $v_1, v_2, \ldots, v_n$ is a strong elimination ordering of G_A. □

Theorem 5. *Let $G = (A, B, E)$ be a convex bipartite graph with $|A \cup B| = n$ and $|E| = m$. A strong elimination ordering of G_A (respectively, G_B) can be computed from G in $O(n + m)$ time.*

Proof. It follows from Theorems 1, 3 and 4. □

3.2 Algorithm

Let ℓ, d, I_1 be fixed integers and $\ell, d > 0$. Let $\mathcal{P}$ be the weight set $\{I_1, I_1 + d, I_1 + 2d, \ldots, I_1 + (\ell - 1) \cdot d\}$. Suppose that $G = (A, B, E)$ is a bipartite graph with a labeling function R which assigns an integer $R(v)$ to each vertex $v \in V(G)$. Let R_A (respectively, R_B) be a labeling function of G which assigns an integer $R_A(v)$ (respectively, $R_B(v)$) to each vertex in G such that $R_A(v) = I_1 \cdot deg_G(v)$ (respectively, $R_B(v) = I_1 \cdot deg_G(v)$)

for every $v \in A$ (respectively, $v \in B$), and $R_A(v) = R(v)$ (respectively, $R_B(v) = R(v)$) for every $v \in B$ (respectively, $v \in A$).

Definition 5. An R_A-total dominating function f of a bipartite graph $G = (A, B, E)$ is called an *R_A^*-total dominating function* of G if $f(v) = I_1 + (\ell - 1) \cdot d$ for every $v \in B$. An R_B-total dominating function g of G is called an *R_B^*-total dominating function* of G if $g(v) = I_1 + (\ell - 1) \cdot d$ for every $v \in A$.

Lemma 6 shows that a minimum R-total dominating function of a chordal bipartite graph G can be obtained from a minimum R_A^*-total dominating function and a minimum R_B^*-total dominating function of G.

Lemma 6 ([13]). *Suppose that $G = (A, B, E)$ is a bipartite graph with a labeling function R as mentioned above. Let f_A (respectively, f_B) be a minimum R_A^*-total (respectively, R_B^*-total) dominating function of G. Let f be a function of G defined by $f(v) = f_A(v)$ for every $v \in A$ and $f(v) = f_B(v)$ for every $v \in B$. Then f is a minimum R-total dominating function of G.*

In [14], Lee proposed a linear-time algorithm for computing a minimum R_A^*-total (respectively, R_B^*-total) dominating function of a biconvex bipartite graph. Based upon Lee's algorithm, we give the function $\mathsf{RTD}(G, \langle X, Y\rangle, R, I_1, \ell, d)$ for computing a minimum R_Y^*-dominating function of a convex bipartite graph G. The function $\mathsf{RTD}(G, \langle X, Y\rangle, R, I_1, \ell, d)$ takes G, $\langle X, Y\rangle$, R, I_1, ℓ, and d as inputs. Input G represents a convex bipartite graph, and X and Y are the bipartition of $V(G)$. Input $\langle X, Y\rangle$ is a strong convex ordering of G. Input R is a labeling function assigning an integer $R(v)$ to each vertex $v \in X \cup Y$. Inputs ℓ, d, I_1 are integers and $\ell, d > 0$. The weight set $\mathcal{P}$ is assumed to be the set $\{I_1, I_1 + d, I_1 + 2d, \ldots, I_1 + (\ell - 1) \cdot d\}$.

If $\langle X, Y\rangle = \langle B, A\rangle$, the function $\mathsf{RTD}(G, \langle B, A\rangle, R, I_1, \ell, d)$ computes a minimum R_A^*-total dominating function of a convex bipartite graph $G = (A, B, E)$. If $\langle X, Y\rangle = \langle A, B\rangle$, the function $\mathsf{RTD}(G, \langle A, B\rangle, R, I_1, \ell, d)$ computes a minimum R_B^*-total dominating function of G.

To illustrate $\mathsf{RTD}(G, \langle X, Y\rangle, R, I_1, \ell, d)$, we let $H = (A, B, E)$ be the convex bipartite graph with $A = \{a_1, a_2, a_3, a_4\}$ and $B = \{b_1, b_2, b_3, b_4\}$ as shown in Figure 1. Let $X = A$, $Y = B$, $I_1 = -1$, $\ell = 3$, and $d = 1$. Then the weight set $\mathcal{P} = \{-1, 0, 1\}$. Let $R(v) = 1$ for every vertex $v \in A \cup B$. By the function $\mathsf{ConvexSets}(H, A, B)$, we know that $\langle A, B\rangle = a_1, a_4, a_2, a_3, b_1, b_2, b_3, b_4$ is a strong convex ordering of G.

At the end of Step 4 of $\mathsf{RTD}(G, \langle A, B\rangle, R, -1, 3, 1)$, $R_B(b_1) = -2$, $R_B(b_2) = R_B(b_3) = R_B(b_4) = -3$, and $R_B(v) = R(v) = 1$ for every vertex $v \in A$. At the end of Step 5, $v_1 = a_1$, $v_2 = a_4$, $v_3 = a_2$, $v_4 = a_3$, and

Function RTD$(G, \langle X, Y\rangle, R, I_1, \ell, d)$

1: **for** every vertex $v \in X \cup Y$ **do**
2:     **if** $v \in Y$ **then** $R_Y(v) = I_1 \cdot deg_G(v)$;
3:     **else** $R_Y(v) = R(v)$;
4: **end for**
5: $v_1, \ldots, v_n \leftarrow \langle X, Y\rangle$;
6: **for** $i \leftarrow 1$ **to** n **do**
7:     $f(v_i) \leftarrow I_1 + (\ell - 1) \cdot d$;
8: **end for**
9: **for** $i \leftarrow 1$ **to** n **do**
10:     **if** $R_Y(v_i) > f(N_G(v_i))$
11:     **then** stop and return the infeasibility of the problem;
12: **end for**
13: **for** $i \leftarrow 1$ **to** n **do**
14:     **if** $v_i \in Y$ **then**
15:         $M \leftarrow \min\{f(N_G(v)) - R_Y(v) | v \in N_G(v_i)\}$;
16:         $f(v_i) \leftarrow \max\{I_1, I_1 + (\lceil \ell - \frac{M}{d} \rceil - 1) \cdot d\}$;
17: **end for**
18: return the function f;

$v_{i+4} = b_i$ for $1 \le i \le 4$. In Steps 6–8, $f(v_i)$ is initialized with the value 1 for $1 \le i \le n$. For $1 \le i \le n$, it can be easily verified that $R_B(v_i) \ge f(N_G(v_i))$ for $1 \le i \le n$. Therefore, RTD$(G, \langle A, B\rangle, R, -1, 3, 1)$ does not stop in Step 11. For $5 \le i \le 8$, v_i is a vertex of B. RTD$(G, \langle A, B\rangle, R, -1, 3, 1)$ assigns the values -1, 1, 1, 1 to $f(v_5)$, $f(v_6)$, $f(v_7)$, $f(v_8)$, respectively. Then the function f is an R_B^*-total dominating set of G.

In the following, Lemmas 7–9 show the correctness of RTD$(G, \langle X, Y\rangle, R, I_1, \ell, d)$. Lemma 10 shows that the running time of RTD$(G, \langle X, Y\rangle, R, I_1, \ell, d)$ is $O(n + m)$ time. They can be proved by the arguments similar to those for Lemmas 6–9 in [14].

Lemma 7. *If the function f initialized by* RTD$(G, \langle X, Y\rangle, R, I_1, \ell, d)$ *in Steps 6–8 is not an R_Y^*-total dominating function of G, then G has no R-total dominating functions.*

Lemma 8. *The function f returned from Step 18 of* RTD$(G, \langle X, Y\rangle, R, I_1, \ell, d)$ *is an R_Y^*-total dominating function of G.*

Lemma 9. *The function f found by* RTD$(G, \langle X, Y\rangle, R, I_1, \ell, d)$ *is a minimum R_Y^*-total dominating function of G.*

Lemma 10. *The function* RTD$(G, \langle X, Y\rangle, R, I_1, \ell, d)$ *computes a minimum R_Y^*-total dominating function of a convex bipartite graph $G = (X, Y, E)$ in $O(n + m)$ time.*

Theorem 6. *Given a convex bipartite graph $G = (A, B, E)$ with $|A \cup B| = n$ and $|E| = m$, the R-total domination problem can be solved in $O(n + m)$ time.*

Proof. By Lemmas 7–9, a function f_A (respectively, f_B) obtained by $\mathsf{RTD}(G, \langle B, A \rangle, R, I_1, \ell, d)$ (respectively, $\mathsf{RTD}(G, \langle A, B \rangle, R, I_1, \ell, d)$) is a minimum R_A^*-total (respectively, R_B^*-total) dominating function of G. Following Lemmas 6 and 10, the R-total domination problem is linear-time solvable for a convex bipartite graph G. □

4 Conclusions

In this paper, we have presented a linear-time algorithm for the R-total domination problem on convex bipartite graphs. Since the R-total domination problem includes the k-total, signed total, and minus total domination problems as special cases, our algorithm can also solve these problems in linear time. In [13], the author solved the R-total domination problem in $O(n^2)$ time for chordal bipartite graphs. Suppose that we are given a chordal bipartite graph G. For further study, it is a great challenge to design an algorithm to solve this problem on G in $o(n^2)$ time.

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