The Hosoya index order of three type of special graphs

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Abstract

The Hosoya index of a graph is defined as the summation of the coefficients of the matching polynomial of a graph. In this paper, we give explicitly expression of the graph $C(n, v_1v_i), Q(n, v_1v_s)$ and $D(s,t)$, and also characterized the extremal graphs with respect to the upper and lower bounds of the Hosoya index of these graphs. In particular, we give the Hosoya index order for graph $C(n, v_1v_i)$ and $Q(n, v_1v_s)$, respectively.

1 Introduction

Let G be a *simple graph*, $V(G)$ and $E(G)$ be its vertex set and edge set, respectively. Let $m(G, k)$ denote the number of $k-$element matchings in G. For convenience, we set $m(G, 0) = 1$. Furthermore, $m(G, 1) = |E(G)|$ is the number of edges. In [7], E. J. Farrell defines the *matching polynomial* as

$$u(G,x) = \sum_{k\geq 0}^{n/2} (-1)^k m(G,k) x^{n-2k} \tag{1}$$

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The Hosoya index of a graph originated from the work of Hosoya [2], in 1971. A topological parameter to study the relationship between the molecular structure and the physical and chemical properties of the certain hydrocarbon compounds, the Hosoya index is denoted as

$$Z(G) = \sum_{k=0} m(G, k). \tag{2}$$

Let P_n, C_n be the path and cycle of length n. Among all n vertex trees, the path P_n has the greatest Hosoya index $Z(P_n)$, and the star S_n has the smallest Hosoya index $Z(S_n) = n$. Among the trees we have following inequality:

$$n = Z(S_n) \leq Z(T) \leq Z(P_n) = f_{n+1}, \tag{3}$$

where f_{n+1} is the $(n+1)th$ *Fibonacci number.* This fact was established a long time ago. Let l_n be the Lucas number, $l_{n+1} = l_n + l_{n-1}$, $l_0 = 2, l_1 = 1$. $C(n, v_1v_i)$ is obtained by joining vertices of v_1 and v_i of C_n with an edge (see in Fig.1.), where $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$. $Q(n, v_1v_s)$ is the graph obtained by identifying a vertex of C_s and a one degree vertex of P_{n-s+1} (see in Fig.2.). $D(s,t), s \leq t$ is the graph obtained by identifying two vertices C_s and C_t on vertex u, (see in Fig 3.). In this paper, we give explicitly expressions of the graph $C(n, v_1v_i), Q(n, v_1v_s)$ and $D(s,t)$, and also characterized the extremal graphs with respect to the upper and lower bounds of the Hosoya index of these graphs. In particular, we give the Hosoya index order for graph $C(n, v_1v_i)$ and $Q(n, v_1v_s)$, respectively. All other graph terminologies are not introduced here, please refer to [1].

2 Basic lemmas

The following basic results will be used, we cited as lemmas.

Lemma 2.1. *[3] Let G be a graph, then*

1. *If* $uv \in E(G)$, *then* $Z(G) = Z(G - uv) + Z(G - \{uv\})$,
2. *If* $u \in V(G)$, *then* $Z(G) = Z(G - v) + \sum_{v \in N_G(u)} Z(G - \{u, v\})$,
3. *If* $G_1, G_2, \ldots, G_k$ *are k components of a graph G, then* $Z(G) = \prod_{i=1}^{i=k} Z(G_i)$.

Meanwhile, we need some results about Fibonacci numbers:

Lemma 2.2. *Let f_n, l_n be the Fibonacci and Lucas number. Then*

1. $f_{m+n} = f_m f_{n+1} + f_n f_{m-1}$;
2. $f_n = f_k f_{n-k+1} + f_{k-1} f_{n-k}$, *for* $1 \leq k \leq n$;
3. $f_{n-1} f_n + f_n f_{n+1} = f_{2n}$;
4. *(Lucas 1680)* $f_{n+1} f_{n-1} - f_n^2 = (-1)^n$;
5. $f_{n-k} f_{m+k} - f_n f_m = (-1)^n f_{m-n-k} f_k$,

where ω is golden rational.

Lemma 2.3.

$$Z(P_n) = f(n+1) = \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^{n+1} - \frac{1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})^{n+1};$$

$$Z(C_n) = f(n-1) + f(n+1).$$

We need another two important relations between Fibonacci numbers and Lucas number which are used to simplify our equations, we give as lemmas 2.4 and 2.5.

Lemma 2.4.

$$f_k f_{n-k} = \frac{1}{5}[l_n - (-1)^k l_{n-2k}], \tag{4}$$

where f_n and l_n be the Fibonacci and Lucas numbers, respectively.

Proof. By the Binet's formula $f_n = \frac{1}{\sqrt{5}}[\phi^n - (-\phi)^{-n}]$, we have

$$f_{n-k} f_k = \frac{1}{5}[\phi^k - (-\phi)^{-k}][\phi^{n-k} - (-\phi)^{-(n-k)}],$$

expand this equation, and with $(-\phi\phi) = 1$ we have

$$= \frac{1}{5}[\phi^n + (-\phi)^{-n} - (-1)^k[\phi^{n-2k} + (-\phi)^{-(n-2k)}],$$

meanwhile $l_n = \phi^n + (-\phi)^{-n}$, therefore

$$f_{n-k} f_k = \frac{1}{5}[l_n - (-1)^k l_{n-2k}].$$

□

Lemma 2.5.

$$f_s f_{t+1} + f_{s-1} f_t = \frac{1}{5}(l_{s+t+1} + l_{s+t-1}), \tag{5}$$

where f_n and l_n be the nth Fibonacci number and nth Lucas number respectively.

By the Binet's formula, we can have Lemma 2.5 easily.

3 Main results

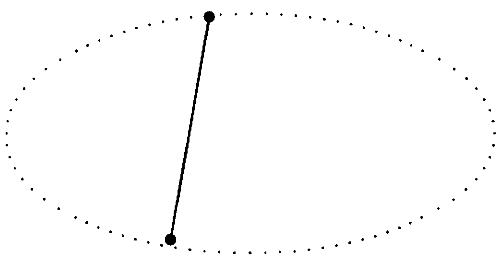

Fig.1. Graph $C(n, v_1 v_i)$

Theorem 3.1. *Let $n = 4m + i, i \in \{1, 2, 3, 4\}, m \geq 0$ then the Hosoya index of $C(n, v_1 v_i)$ is*

$$Z(C(n, v_1 v_i)) = f_{n-1} + f_{n+1} + f_{i-1} f_{n-i+1},$$

and

$$Z(C(n, v_1 v_4)) > Z(C(n, v_1 v_6)) > \ldots > Z(C(n, v_1 v_{2m+2l})) >$$

$$Z(C(n, v_1 v_{2m+1})) > Z(C(n, v_1 v_{2m-1})) > \ldots > Z(C(n, v_1 v_5)) > Z(C(n, v_1 v_3)),$$

where $l = \lfloor \frac{i-1}{2} \rfloor$.

Proof. By Lemma 2.1, take the chord $v_1 v_i = e$ as the edge we delete, then easily have:

$$Z(C(n, v_1 v_i)) = f_{n-1} + f_{n+1} + f_{i-1} f_{n-i+1} \tag{6}$$

by the Lemma 2.4 E.q(4),

$$f_{n-i+1} f_{i-1} = \frac{1}{5}[l_n - (-1)^{i-1} l_{n-2i+2}]$$

hence

$$Z(C(n, v_1v_i)) = f(n-1) + f(n+1) + \frac{1}{5}l_n - (-1)^{i-1}\frac{1}{5}l_{n-2i+2}$$

the only term depending on i is $(-1)^{i-1}l_{n-2i+2}$. Since l_{n-2i+2} is always positive and monotonically decrease respect to i and $C(n, v_1v_i) \cong C(n, v_1v_{n-i+1})$, then we may the order of the Hosoya index is:

$$Z(C(n, v_1v_4)) > Z(C(n, v_1v_6)) > \ldots > Z(C(n, v_1v_{2m+2l})) >$$
$$Z(C(n, v_1v_{2m+1})) > Z(C(n, v_1v_{2m-1}))$$
$$> \ldots > Z(C(n, v_1v_5)) > Z(C(n, v_1v_3)),$$

where $l = \lfloor \frac{i-1}{2} \rfloor$, or in the sequence

$$Z(C(n, v_1v_4)) > Z(C(n, v_1v_6) > \ldots > Z(C(n, v_1, v_{\lfloor n/2 \rfloor}))$$
$$> Z(C(n, v_1v_{n/2})) > \ldots > Z(C(n, v_1v_3))$$

□

Corollary 3.2. *For graph $C(n, v_1v_4)$ has the largest Hosoya index $f_{n-1} + f_{n+1} + 2f_{n-3}$ and $C(n, v_1v_3)$ has the smallest Hosoya index $f_{n-1} + f_{n+1} + f_{n-2}$ the bound is*

$$f_{n-1} + f_{n+1} + f_{n-2} < Z(C(v_1v_i)) < f_{n-1} + f_{n+1} + 2f_{n-3}.$$

REMARK:
when n is even, then

$$Z(C(n, v_1v_4)) > Z(C(n, v_1v_6)) > \ldots > Z(C(n, v_1v_{\lfloor n/2 \rfloor}))$$
$$= f_{n-1} + f_{n+1} + \frac{1}{5}l_n + \frac{2}{5} > Z(C(1, n/2)) = f_{n-1} + f_{n+1} + \frac{1}{5}l_n - \frac{3}{5}$$
$$> \ldots > Z(C(n, v_1v_3))$$

For graph $Q(n, v_1v_s)$, we have the following theorem.

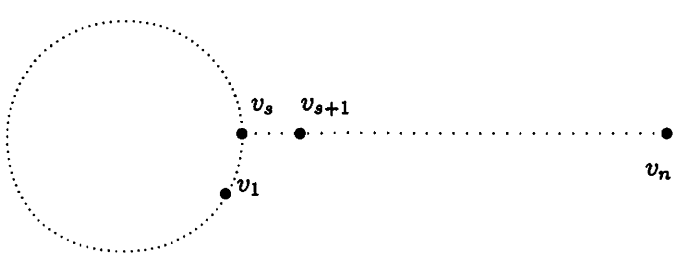

Fig.2. Graph $Q(n, v_1v_s)$

For the sake of simplicity, we write $Q(i,j)$ (where i,j is the length of the cycle and the length of the path) instead of $Q(n, v_1v_s)$ in theorem 3.3.

Theorem 3.3. *The Hosoya index of graph $Q(s, n-s)$ respect to s is:*

$$Z(Q(4,n-4)) > Z(Q(6,n-6)) > \ldots > Z(Q(2m,n-2m))$$
$$> Z(Q(2m-1,n-2m+1)) > \ldots > Z(Q(3,n-3)).$$

Proof. Take an edge from C_s which adjacent to the degree of vertex 3, by Lemma 2.1,we have

$$Z(Q(s,n-s)) = f_{n+1} + f_{s-1}f_{n-s+1},$$

by lemma 2.4 E.q (4),

$$Z(Q(s,n-s)) = f_{n+1} + \frac{1}{5}l_n - (-1)^{s-1}\frac{1}{5}l_{n-2s+2},$$

similar discuss as theorem 3.1, we have

$$Z(Q(4,n-4)) > Z(Q(6,n-6)) > \ldots > Z(Q(2m,n-2m))$$
$$> Z(Q(2m-1,n-2m+1)) > \ldots > Z(Q(3,n-3)).$$

□

For graph $Q(s,t)$, $Q(4,n-4)$ has the largest Hosoya index and $Q(3,n-3)$ has least Hosoya index, the bound is

$$f_{n-2} + 4f_{n+1} < Z(Q(s,t)) < f_{n+1} + 2f_{n-3}.$$

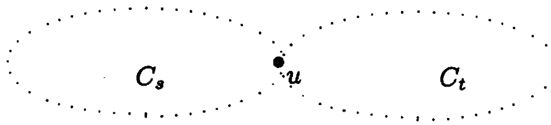

Fig. 3. Infinity graph $D(s,t)$

Theorem 3.4. *The Hosoya index of infinity graph $D(s,t), n = s+t-1, s \leq t$ is*

$$Z(D(s,t)) = 2f_{s+t} - f_sf_t,$$

furthermore, when $s+t \geq 8$, $D(3, s+t-3)$ has the minimum Hosoya index which is $4f_{n-1}$. $D(\lfloor(n+1)/2\lfloor, \lceil(n+1)/2\rceil)$ has the largest Hosoya index.

Proof. By Lemma 2.1, we have

$$Z(D(s,t)) = Z(P_{s-1})Z(P_{t-1}) + 2Z(P_{s-2})Z(P_{t-1}) + 2Z(P_{t-2})Z(P_{s-1}).$$

Since $Z(P_n) = f_{n+1}$, hence

$$Z(D(s,t)) = f_s f_t + 2f_{s-1}f_t + 2f_{t-1}f_s$$

$$= f_s f_t + 2f_{s-1}f_t + 2f_{t-1}f_s + 2f_s f_{t+1} - 2f_s f_{t+1}$$

$$= f_s f_t + 2(f_{s-1}f_t + f_s f_{t+1}) + 2(f_{t-1}f_s - f_s f_{t+1}),$$

by Lemma 2.2 $f_{s-1}f_t + f_s f_{t+1} = f_{s+t}$, and $f_{t+1} = f_t + f_{t-1}$, we have

$$Z(D(s,t)) = 2f_{s+t} - f_s f_t$$

By Lemma 2.4, $f_s f_t = f_{n-1-s} f_s = \frac{1}{5}[l_{n-1} - (-1)^s l_{n-1-2s}]$ has largest value when $s = 3$. Then $D(3, s+t-3)$ has the minimum Hosoya index. $D(s,t)$ when $|s-t| \leq 1$ has the largest Hosoya index. □

REMARK:
For the infinity graph of order $5 \leq n \leq 8$, it is to calculate the Hosoya index of it.

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