The Edge Spectrum of K_4-Saturated Graphs

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Abstract

Any H-free graph G is called H-saturated if the addition of any edge $e \notin E(G)$ results in H as a subgraph of G. The minimum size of an H-saturated graph on n vertices is denoted by $sat(n, H)$. The edge spectrum for the family of graphs with property P is the set of all sizes of graphs with property P. In this paper, we find the edge spectrum of K_4-saturated graphs. We also show that if G is a K_4-saturated graph, then either $G \cong K_{1,1,n-2}$ or $\delta(G) \geq 3$, and we show the exact structure of a K_4-saturated graph with $\kappa(G) = 2$ and $\kappa(G) = 3$.

1 Introduction

All graphs in this paper are simple graphs, namely, finite graphs without loops or multiple edges. Notation will be standard, and generally follow the notation of [2]. Let G be a graph, and let $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. Let $d(x)$ denote the degree of the vertex x, $|V(G)|$ denote the order of the graph G, $|E(G)|$ denote the size of the graph G, and $d(u, v)$ is the distance between u and v. If W is a nonempty subset of the vertex set $V(G)$, then the subgraph $\langle W \rangle$ of G induced by W is the graph having vertex set W and whose edge set consists of those edges of G incident with two vertices of W. Furthermore,

$|V(G)| = n$, unless otherwise specified. Also, K_p denotes the complete graph on p vertices.

A graph G is called an H-saturated graph if G does not contain H as a subgraph but the addition of any edge $e \notin E(G)$ produces H as a subgraph of G. The saturation number of a graph is the minimum number of edges in an H-saturated graph of order n and it is denoted by $sat(n, H)$. This parameter was introduced by Erdős, Hajnal, and Moon in [3]. The maximum number of edges in a H-saturated graph of order n is the well known Turán extremal number and is usually denoted by $ex(n, H)$.

It is known that any K_3-saturated graph has at least $n-1$ edges [3] and at most $\lfloor n^2/4 \rfloor$ edges [5] and [6]. Furthermore, these bounds are sharp as shown in [3] and [6]. Also, any K_4-saturated graph has at least $2n-3$ edges and at most $\lfloor n^2/3 \rfloor$ edges and these bounds are sharp. The emphasis of this paper will be on determining the sizes of K_4-saturated graphs of order n, that is the edge spectrum of K_4-saturated graphs.

2 Results on K_4-saturated Graphs

Barefoot et. al [1] studied the edge spectrum of K_3-saturated graphs and proved the following result.

Theorem 2.1 *Let $n \geq 5$ and m be nonnegative integers. There is an (n, m) K_3-saturated graph if and only if $2n - 5 \leq m \leq \lfloor (n-1)^2/4 \rfloor + 1$ or $m = k(n-k)$ for some positive integer k.*

This result says that a K_3-saturated graph is either a complete bipartite graph or its size falls in the given range and all values in this range are possible.

In this section, we will show a similar result about K_4-saturated graphs. First we make two simple observations shown in the following two Propositions.

Proposition 2.2 *Let G be a K_4-saturated graph. Then $diam(G) = 2$.*

Proof. Let G be a K_4-saturated graph. Suppose $diam(G) \neq 2$, that is, suppose there exists vertices $u, v \in V(G)$ such that $d(u, v) = 3$. Say u, x, y, v is a $u - v$ distance 3 path. Then the addition of the edge uv must produce a K_4. So there must exist vertices a, b such that the induced subgraph $\langle u, a, b, v\rangle \cong K_4$. But now we have $d(u, v) = 2$, as u, a, v is such a path, a contradiction. □

Proposition 2.3 *Let G be a K_4-saturated graph. Then G is 2-connected.*

Proof. Let G be a K_4-saturated graph. Suppose G is not 2-connected. Let u be the cut vertex and let A and B be components of $G - u$ with $x \in V(A)$ and $y \in V(B)$ (see Figure 1).

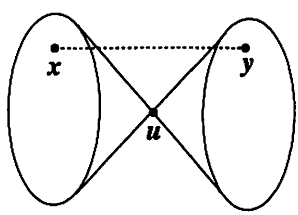

Figure 1: G

Now the addition of the edge xy creates a K_4. So there exists a vertex $w \neq u$ such that $wx \in E(G)$ and $wy \in E(G)$. Then $G - u$ is not disconnected, a contradiction. □

Next we show that there is only one K_4-saturated graph with minimum degree two.

Theorem 2.4 *Let G be a K_4-saturated graph. Then either $G \cong K_{1,1,n-2}$ or $\delta(G) \geq 3$.*

Proof. Let G be a K_4-saturated graph. Let $v \in V(G)$ such that $N(v) = \{x, y\}$. Then for all the vertices $z \in V(G) - \{x, y, v\}$, the addition of the edge vz must produce a K_4 with vertex set $\{v, x, y, z\}$. Thus, $xy, xz, yz \in E(G)$. Since z was chosen arbitrarily, in fact, all vertices in $V(G) - \{x, y, v\}$ are adjacent to both x and y. So $K_{1,1,n-2} \subseteq G$. But $K_{1,1,n-2}$ is K_4-saturated, so $G \cong K_{1,1,n-2}$. □

Next we prove three lemmas that lead to a short proof of a result that all K_4-saturated graph on n vertices other than $K_{1,1,n-2}$ have at least $3n-8$ edges. Before we prove these lemmas, we make the following two observations about K_4-saturated graphs G:

1. The neighborhood of every vertex contains an edge.
2. For all vertices $u, v \in V(G)$, $uv \notin E(G)$ if and only if there exists an edge in the common neighborhood of u and v.

Lemma 2.5 *If G is a K_4-saturated graph of order n and $\delta(G) = 3$, then $|E(G)| \geq 3n - 8$.*

Proof. Let $v \in V(G)$ such that $N(v) = \{x, y, z\}$. Let $W = V(G) - \{v, x, y, z\}$. Note that the induced graph $\langle x, y, z \rangle$ is not isomorphic to K_3.

If $\langle x, y, z \rangle$ contains precisely one edge, say xy, then by observation 2, every vertex $w \in W$ must also be adjacent to x, y and W is therefore independent. But this would force $N(z)$ to be independent, contradicting observation 1.

If $\langle x, y, z \rangle$ contains precisely two edges, say xy, yz, then by observation 2, W can be partitioned into three sets W_{xy}, W_{yz}, W_{xyz}, where $W_{xy} = \{w \in W | N(w) \cap \{x, y, z\} = \{x, y\}\}$ with W_{yz} and W_{xyz} defined similarly (note that $W_{xz} = \emptyset$). Each of these are independent sets of vertices. Furthermore, if $w \in W_{xyz}$, then it has no adjacencies to W_{xy} or W_{yz}. Let $w_1 \in W_{xy}$ and $w_2 \in W_{yz}$. Then $N(w_1) \cap N(w_2) = \{y\}$. Thus, by observation 2, they must be adjacent. Hence, W_{xy} and W_{yz} must induce a complete bipartite graph. Let $|W_{xyz}| = h, |W_{yz}| = l, |W_{xy}| = n - h - l - 4$. Then,

$$
\begin{aligned}
|E(G)| &= 5 + 2l + 2(n - h - l - 4) + l(n - h - l - 4) + 3h \\
&= (2 + l)n + (1 - l)h - l^2 - 4l - 3.
\end{aligned}
$$

Let $f(l) = (2 + l)n + (1 - l)h - l^2 - 4l - 3$. Note if $l = 0$, then we are done by our previous argument. Now $f(1) = 3n - 8$. Furthermore, f is maximized at $l = \frac{n-h-4}{2}$ and is increasing between $l = 1$ and $l = \frac{n-h-4}{2}$. Note this is all of the relevant interval for l, since we can assume, without loss of generality, $|W_{yz}| \leq |W_{xy}|$. □

Lemma 2.6 *If G is a K_4-saturated graph of order n and $\delta(G) = 4$, then* $|E(G)| \geq 3n - 8$.

Proof. Let $v \in V(G)$ such that $d(v) = 4$ and $|V(G)| = n$. Let $N(v) = A = \{a_1, a_2, a_3, a_4\}$. Let $W = V(G) - A - \{v\}$. Then there must exist an edge $a_i a_j$ for some $i, j, 1 \leq i, j \leq 4$ as for any vertex $w \in W$, addition of the edge vw must create a K_4.

If $\langle A \rangle$ contains precisely one edge, say $a_1 a_2$, then every vertex $w \in W$ must also be adjacent to a_1, a_2 by observation 2 above and W is therefore independent. But this would force $N(a_3)$ and $N(a_4)$ to be independent, contradicting observation 1. So $\langle A \rangle$ must contain at least two edges. So now assume that there are precisely two edges among the vertices of A. Then we have two possibilities as shown in the Figure 2.

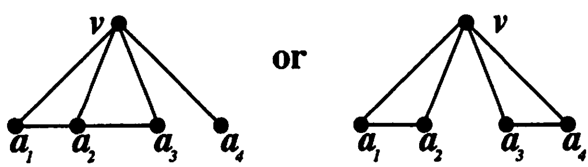

Figure 2: G

Pick the vertices a_i and a_j in different components of $\langle N(v) \rangle$. The addition of the edge $a_i a_j$must produce a K_4 with an edge in W, say $w_1 w_2$. But each vertex w_i, for $i = 1, 2$, must be adjacent to an edge in A, or else the addition of the edge vw_i would not produce a K_4. Hence, w_1 and w_2 are adjacent to at least three vertices of A.

The count below comes from counting the degree sum in the following parts: (1) the degree sum in $v \cup A$, (2) the degree sum from A to W, (3) the degree sum of vertices in W. In the last instance we use the assumption that $\delta(G) = 4$.

$$\sum_{x\in V(G)} d(x) \geq (4+4+4)+(2(n-5)+2)+(4(n-5))$$
$$= 6n-16.$$

Observe that if there are three edges among the vertices of A, we also obtain the same count. Hence, $|E(G)| \geq 3n-8$. □

Lemma 2.7 *If G is a K_4-saturated graph and $\delta(G) = 5$, then $|E(G)| \geq 3n-8$.*

Proof. Note that $|E(G)| \geq \frac{5n}{2}$ and $\frac{5n}{2} \geq 3n-8$ for $n \leq 16$.

Let $v \in V(G)$ with $d(v) = 5$ and $|V(G)| = n \geq 17$. Let $N(v) = A = \{a_1, a_2, a_3, a_4, a_5\}$. Let $W = V(G) - A - \{v\}$. We know there must exist an edge a_ia_j for some i, j. In fact, as in Lemma 2.6, there must exist at least two edges. Furthermore, for any vertex $w \in W$, wa_i and wa_j must exist for some $i, j, 1 \leq i, j \leq 5$. Also, two of the vertices in W must be adjacent to at least 3 vertices in A, as shown in Lemma 2.6. Since $\delta(G) = 5$ and counting as we did in the previous lemma, we have

$$\sum_{x\in V(G)} d(x) \geq (5+5+4)+(2(n-6)+2)+(5(n-6))$$
$$= 7n-26$$

So, $|E(G)| \geq \frac{7n-26}{2} \geq 3n-8$ for $n \geq 10$. □

Theorem 2.8 *Every 2-connected K_4-saturated graph of order n with $\delta(G) \geq 3$ has at least $3n-8$ edges.*

Proof. From Lemmas 2.5 - 2.7, the result holds for graphs G with $3 \leq \delta(G) \leq 5$. For graphs G with $\delta(G) \geq 6$, $|E(G)| \geq \frac{6n}{2} = 3n$. □

In the following two theorems, we classify all K_4-saturated graphs G with $\kappa(G) = 2$ or $\kappa(G) = 3$.

Theorem 2.9 *If G is a K_4-saturated graph of order n with $\kappa(G) = 2$, then $G \cong K_{1,1,n-2}$.*

Proof. Let G be a K_4-saturated with $\kappa(G) = 2$. Let K be a minimal cut set of G and let C_1, C_2 be distinct components of $G - K$. Let $x_i \in V(C_i)$ for $i = 1, 2$. Then the addition of the edge x_1x_2 produces a K_4. So if $u, v \in K$, then $\langle x_1, x_2, u, v\rangle \cong K_4$. As x_1 (x_2) was an arbitrary vertex of C_1 (C_2), each vertex in C_1 (C_2) is also adjacent to u and v. So $G - K$ is an independent set and the result is shown. □

Now we will classify all K_4-saturated graphs with $\kappa(G) = 3$. But first, we define the graph $W(a, b, c, d, e)$ to be a wheel on 5 sets totaling n vertices such that $a + b + c + d + e = n - 1$ and each of the 5 sets of the wheel are independent sets of sizes a, b, c, d, e, respectively, and two consecutive independent sets on the wheel form a complete bipartite subgraph. For example, $W(1, 3, 2, 1, 2)$ on 10 vertices is shown in Figure 3.

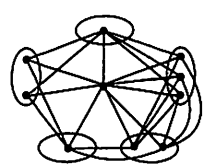

Figure 3: $W(1, 2, 1, 3, 2)$

Theorem 2.10 *If G is a K_4-saturated graph of order n with $\kappa(G) = 3$, then $G \cong K_{1,2,n-3}$ or $G \cong W(1, t, 1, r, s)$.*

Proof. Let K be a minimal cut set of G, say $\{v_1, v_2, v_3\}$. Let C_1, C_2 be distinct components of $G - K$. Let $x_i \in V(C_i)$ for $i = 1, 2$. Then addition of the edge x_1x_2 produces a K_4. Without loss of generality, suppose $\langle x_1, x_2, v_1, v_2\rangle \cong K_4$. If all the components of $G - K$ are trivial, then all are adjacent to all three vertices of K by the connectivity assumption. Now one of (but not both) v_1 or v_2 is adjacent to v_3 (say v_2) or else inserting the edge v_2v_3 would not produce a K_4. But now G is $K_{1,2,n-3}$.

We now assume C_1 is nontrivial. Let x_3 be a neighbor of x_1 in C_1. Inserting the edge x_2x_3, we know (without loss of generality) $\langle x_2, x_3, v_2, v_3\rangle \cong$

K_4, since x_3 cannot be adjacent to both v_1 and v_2. If there exists a vertex $x_4 \in V(C_2)$ such that $x_2x_4 \in E(G)$, then by a similar argument $x_4v_1, x_4v_2, x_4v_3 \in E(G)$. Hence, $\langle x_2, x_4, v_1, v_2\rangle \cong K_4$. Hence, $V(C_2) = \{x_2\}$. In fact $G - \{K \cup C_1\}$ is an independent set. For every $w \in V(C_1), N(w) \cap K = \{v_1, v_2\}$ or $N(w) \cap K = \{v_2, v_3\}$ since w must be adjacent to an edge in K but cannot be in a K_4. Now partition C_1 into two classes A and B, where vertices of A are adjacent to v_1 and v_2, while vertices of B are adjacent to v_2 and v_3. Clearly, A is an independent set. Similarly, B is an independent set.

We claim $\langle A \cup B\rangle$ is a complete bipartite graph. Let $a \in A, b \in B$ such that $ab \notin E(G)$. Then addition of the edge ab must produce a K_4. Since A and B are independent sets, the edge they have in common has to be in K, a contradiction. Hence, $\langle A \cup B\rangle$ is complete bipartite. Thus, if $|A| = s$ and $|B| = r$, using v_2 as the center of the wheel, we have $G \cong W(1, t, 1, r, s)$, where $3 + t + r + s = n$. □

In [4], Hanson and Toft gave the following construction. The graphs in the family $\mathcal{T}_n^k$ are on n vertices consisting of a complete $(k-1)$-partite graph on $n-2$ vertices, with classes of independent points $C_1, C_2, ..., C_{k-1}$, together with two adjacent vertices x and y and where each vertex of C_1 is joined to precisely one of x or y, x and y are each adjacent to at least one vertex of C_1, no vertex of C_2 is adjacent to either x or y and all vertices of C_i, $i > 2$ are adjacent to both x and y. For $k \geq 3$, define $T'_{k-1,n}$ to be graphs in $\mathcal{T}_n^k$ for which $|C_1| + 1, |C_2| + 2, |C_3|, ..., |C_{k-1}|$ are equal or as equal as possible. For $n \geq 3k - 4$ we can describe $T'_{k-1,n}$ as follows: let $n + 1 = t(k-1) + r$, $0 \leq r < k - 1$ and let G denote a member of $\mathcal{T}_{n_0}^k$ on $n_0 = n - r$ vertices and $e_0 = e(T_{k-1,n-r}) - (t-2)$ edges where the classes C_i satisfy $|C_1| = t - 1, |C_2| = t - 2$ and $|C_i| = t$, $i > 2$ (G is unique up to adjacencies of x and y to class C_1). Define $T'_{k-1,n}$ to be a graph G with one vertex added to precisely r of the classes $C_1, ..., C_{k-1}$. Note that the graphs $T'_{k-1,n}$ are maximal, with respect to the number of edges, in the family $\mathcal{T}_n^k$. Then Hanson and Toft [4] showed the following result.

Theorem 2.11 *Let G be a maximal K_k-saturated graph on $n \geq k + 2 \geq 5$ vertices with $\chi(G) \geq k$, then G is a $T'_{k-1,n}$ graph.*

Theorem 2.12 *If G is a K_4-saturated graph of order n and G is not complete tripartite, then $|E(G)| \leq \frac{n^2-n+4}{3}$.*

Proof. Let G be a K_4-saturated graph of order n. Suppose G is not a complete tripartite graph. Since G is not tripartite, $\chi(G) \geq 4 = k$. Hence, by Theorem 2.11, $|E(G)| \leq |E(T'_{3,n})|$. For $n+1 = 3t+r$, a straight forward computation shows, $|E(T'_{3,n})| \leq \frac{n^2-n+4}{3}$. In fact, when $r = 0$, $|E(T'_{3,n})| = \frac{n^2-n+4}{3}$. Hence, $|E(G)| \leq \frac{n^2-n+4}{3}$. □

Theorem 2.13 *Let $n \geq 5$ and m be nonnegative integers. There is an (n, m) K_4-saturated graph G if and only if $3n-8 \leq m \leq \frac{n^2-n+4}{3}$ or $m = rs+st+rt$ for some positive integers r, s, t where $n = r+s+t$.*

Proof. Let $n \geq 5$ and m be nonnegative integers. Let G be an (n, m) K_4-saturated graph. If G is a tripartite graph, then G must be a complete tripartite graph, otherwise an edge may be added without creating a K_4. Now if $G \cong K_{1,1,n-2}$, then $m = 2n-3$ and clearly $r = s = 1$ while $t = n-2$, otherwise $m = rs+st+rt$ for some positive integers r, s, t such that $n = r+s+t$.

Now let G be a nontripartite graph. Then from Theorem 2.8, Theorem 2.10, and Theorem 2.12, we have that $3n-8 \leq m \leq \frac{n^2-n+4}{3}$.

It is sufficient to construct an (n, m) K_4-saturated graph for each value of m. If $m = 2n-3$, then $G \cong K_{1,1,n-2}$. If $m = rs+st+rt$ for some positive integers r, s, t where $n = r+s+t$, then $G \cong K_{r,s,t}$ with m edges.

Now if $3n-8 \leq m \leq \frac{n^2-n+4}{3}$, then consider an (n, m) K_4-saturated graph $G \cong H + \overline{K_q}$, where H is a K_3-saturated graph of order $n-q$. From Theorem 2.1, $2(n-q)-5 \leq |E(H)| \leq \lfloor \frac{(n-q-1)^2}{4} \rfloor + 1$. Hence, $2(n-q)-5+(n-q)q \leq m \leq \lfloor \frac{(n-q-1)^2}{4} \rfloor + 1 + (n-q)q$. When $q = 1$, we obtain the lower bound on $m = 3n-8$. Now let $f(q) = \lfloor \frac{(n-q-1)^2}{4} \rfloor + 1 + (n-q)q$. Then $f(q)$ is maximum when $q = \frac{n+1}{3}$ and $f(\frac{n+1}{3}) = \frac{n^2-n+4}{3}$. □

In the above Theorem, the lower and upper bounds are achieved. For example, for the following graphs G_1 (Figure 4) and G_2 (Figure 5), lower and upper bounds, respectively, are achieved.

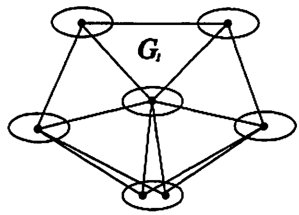

Figure 4: G_1

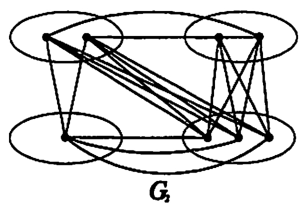

Figure 5: G_2

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