Computation of the Folkman Number $F_v(3,5;6)$†

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Abstract We prove that $F_v(3,5;6) = 16$, which solves the smallest open case of vertex Folkman numbers of the form $F_v(3,k;k+1)$. The proof uses computer algorithms.

1 Introduction

We shall only consider simple graphs without multiple edges or loops. If G is a graph, then the set of vertices of G is denoted by $V(G)$, the set of

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edges by $E(G)$ and the complementary graph of G by $\overline{G}$. The subgraph of G induced by $S \subseteq V(G)$ will be written as $G[S]$.

Graph G is an (s,t)-graph if G contains neither clique of order s nor independent set of order t. We denote by $\mathcal{R}(s,t)$ the set of all (s,t)-graphs, and an (s,t)-graph of order n is called an $(s,t;n)$-graph. Let $\mathcal{R}(s,t;n)$ be the set of all $(s,t;n)$-graphs.

For a graph G, complete graph K and a vertex set $S \subseteq V(G)$, we say that S is $(G,+v,K)$-maximal if and only if $K \not\subseteq G[S]$ and $K \subseteq G[S \cup \{v\}]$, for every vertex $v \in V(G) - S$.

In this note, we study vertex Folkman numbers and graphs, which form a branch of Ramsey theory. For a graph G and positive integers $a_1, a_2, \cdots, a_r$, we write $G \rightarrow (a_1, a_2, \cdots, v_r)^v$ if every r-coloring of the vertices of G results in a monochromatic a_i-clique of color i, for some $i \in \{1, 2, \cdots, r\}$. Let

$$\mathcal{F}_v(a_1, a_2, \cdots, a_r; k) = \{G : G \rightarrow (a_1, a_2, \cdots, a_r)^v \text{ and } K_k \not\subseteq G\}.$$

The graphs in $\mathcal{F}_v(a_1, a_2, \cdots, a_r; k)$ are called (Folkman) $(a_1,\ a_2,\ \cdots,\ a_r;k)^v$-graphs. An $(a_1, a_2, \cdots, a_r; k)^v$-graph of order n will be called an $(a_1, a_2, \cdots, a_r; k; n)^v$-graph. The set of all $(a_1, a_2, \cdots, a_r; k; n)^v$-graphs will be denoted by $\mathcal{F}_v(a_1, a_2, \cdots, a_r; k; n)$. Then, the vertex Folkman numbers are defined by

$$F_v(a_1, a_2, \cdots, a_r; k) = \min\{|V(G)| : G \in \mathcal{F}_v(a_1, a_2, \cdots, a_r; k)\}.$$

One can easily see that $F_v(a_1, a_2, \cdots, a_r; k)$ does not depend on the order of $a_1, \ldots, a_r$ and thus without loss of generality we will assume that $a_1 \leq a_2 \leq \ldots \leq a_r$. Folkman [1] proved that $\mathcal{F}_v(a_1, \ldots, a_r; k)$ is nonempty if and only if $k > \max\{a_1, \ldots, a_r\}$. By the pigeonhole principle, we observe that $K_m \rightarrow (a_1, \ldots, a_r)^v$, if

$$m = 1 + \sum_{i=1}^{r}(a_i - 1).$$

This easily leads to the solution in the case of $k = m + 1$, namely the equality $F_v(a_1, \ldots, a_r; m + 1) = m$. As k becomes smaller the problem of computing Folkman numbers of this form is getting harder. Łuczak, Ruciński and Urbański [2] proved that $F_v(a_1, \ldots, a_r; m) = a_r + m$. For $k \leq m - 1$ only partial results are known. Interestingly, one of the smallest nontrivial cases, $F_v(3,3;4) = 14$ ($k = m - 1 = 4$), was quite difficult to prove [7, 10]. It is also the first case in the family of vertex Folkman numbers of the form $F_v(3, k-1; k)$, which attracted significant attention in previous studies. In particular, Nenov, in 2001 [9], proved that $F_v(3,4;5) = 13$ and later [8] he also established a general bound as in the following theorem.

Theorem 1 *For $k \geq 3$, $2k+4 \leq F_v(3,k;k+1) \leq 4k+2$.*

In this note, we obtain the exact value for the first open case in this family, $F_v(3,5;6)$. The proof is computational. The exact values of $F_v(3,k;k+1)$ remain open for $k \geq 6$.

By Theorem 1, we have $F_v(3,5;6) \geq 14$. Some general bounds for numbers of this form were obtained in [11], in particular the bound $F_v(3,5;6) \leq 16$. Hence we know that $14 \leq F_v(3,5;6) \leq 16$. The computations described in the next sections show that there is no graph of order 15 which is K_6-free and satisfies $G \rightarrow (3,5)^v$. This will imply that $F_v(3,5;6) \geq 16$ and thus $F_v(3,5;6) = 16$.

2 Proof of $F_v(3,5;6) \geq 15$

Luczak, Ruciński and S. Urbański [2] proved the following theorem.

Theorem 2 *Let $m = 1 + \sum_{i=1}^{r}(a_i - 1)$. The graph $K_{a_r+m} - C_{2a_r+1}$ is the unique (a_r+m)-vertex graph G with properties $G \rightarrow (a_1, a_2, \cdots, a_r)^v$ and $K_m \not\subseteq G$.*

If $r = 2, a_1 = 2anda_2 = 5$, then for $m = 6$ by Theorem 2 we have that $\overline{C_{11}}$ is the unique 11-vertex K_6-free graph with the property $\overline{C_{11}} \rightarrow (2,5)^v$. All 263520 $(6,3;14)$-graphs are given in [5]. With the help of a computer, we found that none of these graphs is a Folkman $(3,5;6)^v$-graph. Hence, if there exists a Folkman $(3,5;6;14)^v$-graph G, then G contains a 3-independent set. Let the graph H be obtained by removing a 3-independent set from G. Then, clearly, $H \rightarrow (2,5)^v$. By Theorem 2, $\overline{C_{11}}$ is the unique graph H of order 11 satisfying $H \rightarrow (2,5)^v$ and thus $H \cong \overline{C_{11}}$. The graph $\overline{C_{11}}$ was extended to potential Folkman graps of order 14 by the following procedure FindF_v356Graph.

We implemented the algorithm FindF_v356Graph. Multiple independent checks were done to ensure the correctness of the intermediate steps. The computations using FindF_v356Graph and starting with $H \cong \overline{C_{11}}$ produced an empty set $\mathcal{T}$. Hence we have the following lemma.

Lemma 1 $F_v(3,5;6) \geq 15$.

Procedure 1 FindF_v356Graph

$\mathcal{T} \leftarrow \emptyset$;
find the family $\mathcal{M} = \{S \subseteq V(H) : S \text{ is } (H, +v, K_5)\text{-maximal}\}$
let $|\mathcal{M}| = n$ and $\mathcal{M} = \{S_1, S_2, \cdots, S_n\}$;
for all S_1, S_2, $S_3 \in \mathcal{M}$ **do**
  construct in all possible ways graph F by adding three vertices v_1, v_2, v_3 to H, such that $N_F(v_i) = S_i$ for $i = 1, 2, 3$;
  if $F \in \mathcal{F}_v(3,5;6)$ **then**
    add F to the set $\mathcal{T}$;
  end if
end for

3 Proof of $F_v(3,5;6) \geq 16$

We will say that the graph G has *Property A* if

$$G \in \mathcal{F}_v(3,4;6;12) \cap \mathcal{F}_v(2,5;6;12).$$

All the 116792 $(6,3;12)$-graphs are known and they are available from [4]. Processing them with straightforward computer algorithms and using some simple reasonings, we found that the facts stated in the next three observations hold.

Observation 1 *There are exactly* 283 *graphs which have Property A and have no* 3*-independent sets.*

Observation 2 *If $G \in \mathcal{F}_v(3,5;6;15)$, then G has a 3-independent set.*

Proof. Suppose that there is a graph $G \in \mathcal{F}_v(3,5;6;15)$ containing no 3-independent sets. Then G is a $(6,3;15)$-graph. All 64732 $(6,3;15)$-graphs are given in [6]. By a simple computer search, it is easy to verify that there is no $(6,3;15)$-graph G satisfying $G \rightarrow (3,5)^v$. □

Observation 3 *If $G \in \mathcal{F}_v(3,5;6;15)$, then G has no* 4*-independent sets.*

Proof. Suppose that there exists a graph $G \in \mathcal{F}_v(3,5;6;15)$ containing a 4-independent set. Let H be a graph obtained from G by removing a 4-independent set, so clearly $H \rightarrow (2,5)^v$. By Theorem 2, $\overline{C_{11}}$ is the unique graph H of order 11 satisfying $H \rightarrow (2,5)^v$. Thus we have $H \cong \overline{C_{11}}$. The graph $\overline{C_{11}}$ was extended to all potential Folkman graphs of order 15 by a slightly modified procedure FindF_v356Graph, which was adding 4 new vertices instead of 3. Similarly as before no graphs were produced. □

Observations 2 and 3 imply that the largest independent set in every graph in $\mathcal{F}_v(3,5;6;15)$ has exactly 3 vertices. Hence, if H is a graph obtained by removing a 3-independent set from any such graph, then we have $H \in \mathcal{R}(6,4;12)$, $H \to (2,5)^v$ and $H \to (3,4)^v$. We will consider two cases:

Case 1. H contains no 3-independent set.
By Observation 1, there are 283 graphs possible graphs H. All of them were processed by the algorithm FindF_v356Graph and no Folkman graphs of the type $(3,5;6;15)$ were found.

Case 2. H contains 3-independent set but no 4-independent set.
First, we claim that $\Delta(H) \leq 9$. Let v be any vertex in $V(H)$. Since H is K_6-free we know the subgraph of H induced by the neighbors of v in H is K_5-free. Because $H \to (2,5)^v$, we can see that there must be a K_2 in the subgraph of H induced by the non-neighbors of v in H. This implies that the degree of v in H is at most 9. Next, we will consider the following two subcases:
Subcase 2.1. $\delta(H) \geq 5$. Let

$$\begin{aligned}
\mathcal{A}_1 &= \{G | G \in \mathcal{R}(6,4;12), \delta(G) \geq 5, \Delta(G) \leq 9\}, \\
\mathcal{A}_2 &= \{G | G \in \mathcal{A}_1, G \text{ contains } K_5\}, \\
\mathcal{A}_3 &= \{G | G \in \mathcal{A}_2, G \in \mathcal{F}_v(2,5;6)\}, \\
\mathcal{A}_4 &= \{G | G \in \mathcal{A}_3, G \in \mathcal{F}_v(3,4;6)\}.
\end{aligned}$$

We generated all $(6,4;12)$-graphs with vertex degrees ranging from 5 to 9 (set $\mathcal{A}_1$) using program *geng* [3]. Next, the sets $\mathcal{A}_2, \mathcal{A}_3$ and $\mathcal{A}_4$ were obtained from $\mathcal{A}_1$ by direct computations. In Table 1, we give the statistics of the number of graphs in $\mathcal{A}_1, \mathcal{A}_2$, $\mathcal{A}_3$ and $\mathcal{A}_4$, broken by the number of edges ranging between 30 and 54. All graphs in $\mathcal{A}_4$ were extended by the algorithm FindF_v356Graph and no target Folkman graphs were found. We note that because of large size of $\mathcal{A}_1$, direct computations from the definitions were not feasible.

Table 1: Statistics of $|\mathcal{A}_i|$ by the number of edges

edges	$\lvert\mathcal{A}_1\rvert$	$\lvert\mathcal{A}_2\rvert$	$\lvert\mathcal{A}_3\rvert$	$\lvert\mathcal{A}_4\rvert$
54	5	4	1	0
53	37	36	1	0
52	384	378	22	11
51	3609	3583	119	72
50	28772	28626	440	225
49	185632	184794	785	332

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Table 1 – continued from previous page

edges	$\|\mathcal{A}_1\|$	$\|\mathcal{A}_2\|$	$\|\mathcal{A}_3\|$	$\|\mathcal{A}_4\|$
48	962357	957268	872	241
47	4044300	4014482	635	120
46	13939178	13773581	331	36
45	39783385	38955541	112	9
44	94687495	91128734	26	1
43	188829276	176174913	3	0
42	316441087	280162007	0	0
41	446190267	363551169	0	0
40	529053391	380734861	0	0
39	525790315	317400595	0	0
38	434789038	207235423	0	0
37	295260018	104024127	0	0
36	161235786	39297469	0	0
35	68573182	10882328	0	0
34	21640455	2127510	0	0
33	4697021	276658	0	0
32	616123	21663	0	0
31	37832	871	0	0
30	590	15	0	0

Subcase 2.2. $\delta(H) \leq 4$.
Consider $v \in V(H)$ with $d(v) \leq 4$. Since $H \rightarrow (2,5)^v$, we have $H \setminus \{v\} \rightarrow (2,5)^v$. By Theorem 2, $H \setminus \{v\} \cong \overline{C_{11}}$. An initial graph family $\mathcal{B}_1$ was constructed by adding a vertex v to $\overline{C_{11}}$ in all possible ways, so that $d(v) \leq 4$. Then by processing $\mathcal{B}_2$, $\mathcal{B}_3$ and $\mathcal{B}_4$ defined similarly to $\mathcal{A}_i$'s from the subcase 2.1, we obtained no final Folkman graphs. The computations for the subcase 2.2 were much faster.

Thus, the above analysis implies that $F_v(3,5;6) \geq 16$. Since $F_v(3,5;6) \leq 16$ [11], the following theorem holds.

Theorem 3 $F_v(3,5;6) = 16$.

3.1 Next Challenges

The next open case of the form $F_v(3,k;k+1)$ is for $k = 6$. Theorem 1 implies that $16 \leq F_v(3,6;7)$ and the upper bound of 18 was obtained in [11]. Computing the exact value of $F_v(3,6;7)$ is likely difficult, but might be doable.

For slightly different type of parameters, now avoiding K_{m-2}, it is known that $17 \leq F_v(4,4;5) \leq 23$ [12]. This is a very elegant case, yet despite significant effort the gap between lower and upper bounds indicates that we don't understand it very well. Try to make this gap smaller!

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