On the sum of powers of the signless Laplacian eigenvalues of graphs

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Abstract

For a graph G and a real number $\alpha \neq 0$, the graph invariant $s_\alpha^+(G)$ is the sum of the αth power of the non-zero signless Laplacian eigenvalues of G. In this paper, Several lower and upper bounds for $s_\alpha^+(G)$ with $\alpha \neq 0, 1$ are obtained. Applying these results, we also obtain some bounds for the incidence energy of graphs, which generalize and improve on some known results.

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1. Introduction

Let G be a simple graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$. The adjacency matrix $A(G)$ of G is defined by its (i, j)-entry is 1 if vertices v_i and v_j are adjacent and 0 otherwise. Let $D(G)$ be the diagonal matrix of order n whose (i, j)-entry is the degree of the vertex v_i of the graph G. The matrices $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$ are called the Laplacian matrix and the signless Laplacian matrix of G, respectively. The eigenvalues of $L(G)$ are called the Laplacian eigenvalues and denoted by $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} \geq \mu_n = 0$. The eigenvalues of $Q(G)$ are called the signless Laplacian eigenvalues and denoted by $q_1 \geq q_2 \geq \cdots \geq q_{n-1} \geq q_n \geq 0$. For the Laplacian matrix, readers may refer to [10, 17, 19, 21, 22]

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and the references therein. For more review about the signless Laplacian matrix of G, readers may refer to [3, 6, 7, 8, 9] and the references therein.

Let G be a simple graph with n vertices. For the Laplacian matrix $L(G)$ of G, Lazić[13] studied some properties of the sum of the squares of the Laplacian eigenvalues of G where Lazić called it the Laplacian energy of the graph G. In 2008, Liu and Liu[15] introduced the so-called Laplacian-energy like invariant $LEL(G)$, as the sum of the square roots of the eigenvalues of the Laplacian matrix of G, i.e., $LEL(G) = \sum_{i=1}^{n} \sqrt{\mu_i}$. Some properties of $LEL(G)$ were established in [15, 16, 18]. Motivated by above, Zhou[22] introduced the graph invariant $s_\alpha(G)$, as the sum of the αth power of the non-zero Laplacian eigenvalues of G, i.e., $s_\alpha(G) = \sum_{i=1}^{h} \mu_i^\alpha$, where h is the number of non-zero Laplacian eigenvalues of G. Some properties of $s_\alpha(G)$ were obtained in [19, 22].

For the signless Laplacian matrix $Q(G)$ of a graph G, Jooyandeh et al.[14] introduced the concept of incidence energy $IE(G)$ of a graph G, defining it as the sum of the singular values of the incidence matrix $I(G)$ of G. In [12], Gutman et al. obtained the following basic property: $IE(G) = \sum_{i=1}^{n} \sqrt{q_i}$. Some properties of $IE(G)$ were established in [8, 11, 12, 14].

Motivated by these results, we consider the sum of the αth power of the signless Laplacian eigenvalues of a graph G. Let G be a simple graph with n vertices. For a non-zero real number α, $s_\alpha^+(G)$, called the sum of the αth power of the non-zero signless Laplacian eigenvalues of G, is defined as $s_\alpha^+(G) = \sum_{i=1}^{h} q_i^\alpha$, where h is the number of non-zero signless Laplacian eigenvalues of G. The case $\alpha = 1$ is trivial as $s_1^+(G) = 2m$, where m is the number of edges. Akbari et al.[1, 2] obtained some inequalities about $s_\alpha^+(G)$. For a non-negative integral number α, Cvetković et al.[3] established some properties of $s_\alpha^+(G)$ where it was called the αth spectral moment for the signless Laplacian eigenvalues of G (also see [5]). If $\alpha = \frac{1}{2}$, then $s_{\frac{1}{2}}^+(G) = IE(G)$, which is exactly the incidence energy of G. For more review about the incidence energy of G, readers may refer to [8, 11, 12, 14] and the references therein.

In this paper, we establish some properties for $s_\alpha^+(G)$ whenever α is real number with $\alpha \neq 0, 1$. Applying these results, we also obtain some bounds for the incidence energy of graphs, which generalize and improve on some known results.

2. Preliminaries

Lemma 1[4]. *Let G be a graph on n vertices and m edges and let e be an edge of G. Then*

$$q_1(G) \geq q_1(G-e) \geq q_2(G) \geq q_2(G-e) \geq \cdots \geq q_n(G) \geq q_n(G-e) \geq 0.$$

Lemma 2[3]. *Let G be a connected graph with diameter d. Then G has at least $d+1$ distinct signless Laplacian eigenvalues.*

Lemma 3[7]. *Let G be a connected graph without vertices of degree 1, with maximum degree Δ. Then*

$$q_1 \geq \Delta + 1 + \frac{1}{\Delta - 1}$$

with equality if and only if G is a cycle C_n.

Lemma 4[4]. *Let G be a connected graph of order $n \geq 2$ and maximum degree Δ. Then $q_1 \geq \Delta + 1$ with equality if and only if G is the star S_n.*

Proof. In [4], Cvetkovic et al. proved that the result holds whenever $n \geq 4$. It is easy to verify that the result also holds whenever $n = 2, 3$. □

Lemma 5. *Suppose that $x_1, x_2, \ldots, x_n$ are any positive numbers:*

(i) If $\alpha < 0$ or $\alpha > 1$, then

$$\sum_{i=1}^{n} x_i^{\alpha} \geq n^{1-\alpha} \left(\sum_{i=1}^{n} x_i \right)^{\alpha}$$

with equality if and only if $x_1 = x_2 = \cdots = x_n$.

(ii) If $0 < \alpha < 1$, then

$$\sum_{i=1}^{n} x_i^{\alpha} \leq n^{1-\alpha} \left(\sum_{i=1}^{n} x_i \right)^{\alpha}$$

with equality if and only if $x_1 = x_2 = \cdots = x_n$.

Proof. These are immediate consequences of the Jensen inequality.□

For a connected graph G and its vertex v_i, let d_i and t_i denote the degree of v_i and the sum of the degrees of the first neighbors of v_i in G, respectively. For convenience, we define

$$\eta(G) := \sqrt{\frac{\sum_{i=1}^{n} \left(d_i(d_i^2 + t_i) + \sum_{j \sim i} (d_j^2 + t_j) \right)^2}{\sum_{i=1}^{n} (d_i^2 + t_i)^2}}.$$

Lemma 6[19]. *Let G be a connected graph with degrees $d_1, d_2, \ldots, d_n$. Then*

$$q_1 \geq \eta(G). \tag{1}$$

The equality holds in (1) if and only if there exists a positive constant number t such that, for all $i \in \{1, 2, \ldots, n\}$,

$$\frac{d_i(d_i^2 + t_i) + \sum_{j \sim i}(d_j^2 + t_j)}{d_i^2 + t_i} = t.$$

In particular, the equality in (1) holds if G is a regular or semiregular bipartite graph.

Proof. By modifying slightly the proof of Theorem 1 in [19], we obtain the required result.□

Applying Lemma 6, we can get the following result, which is exactly Theorem 9 in [20] (also see [12]).

Corollary 1[12, 20]. *Let G be a connected graph with degrees $d_1, d_2, \ldots, d_n$. Then*

$$q_1 \geq \tau(G) := \sqrt{\frac{\sum_{i=1}^{n}(d_i^2 + t_i)^2}{\sum_{i=1}^{n} d_i^2}}, \tag{2}$$

with equality if and only if G is a regular or semiregular bipartite graph.

3. Main results

Theorem 1. *Let G be a graph with n vertices and m edges. Suppose that G has h non-zero signless Laplacian eigenvalues:*

(i) If $\alpha < 0$ or $\alpha > 1$, then

$$s_\alpha^+(G) \geq h^{1-\alpha}(2m)^\alpha \tag{3}$$

with equality if and only if $h = m$ and G consists of m copies of K_2 and possibly isolated vertices.

(ii) If $0 < \alpha < 1$, then

$$s_\alpha^+(G) \leq h^{1-\alpha}(2m)^\alpha \tag{4}$$

with equality if and only if $h = m$ and G consists of m copies of K_2 and possibly isolated vertices.

Proof. From Lemma 5, we immediately get that the inequality (3) holds and the equality holds in (3) if and only if $q_1 = q_2 = \cdots = q_h$. By Perron-Frobenius Theorem, $q_1 = q_2 = \cdots = q_h$ if and only if $h = m$ and G consists of m copies of K_2 and possibly isolated vertices, as required in (i).

By a parallel argument as (i), it can be shown that (ii) also holds. □

An alternative proof of Theorem 1. Suppose that $\alpha < 0$ or $\alpha > 1$. Let $\mathcal{L}(G)$ be the line graph of G and I_m denote the unit matrix of order m. Observe that $I(G)I(G)^T = Q(G)$ and $I(G)^T I(G) = 2I_m + A(\mathcal{L}(G))$ have same non-zero eigenvalues. From Lemma 5, we have

$$s_\alpha^+(G) = \sum_{i=1}^{h} [2 + \lambda_i(\mathcal{L}(G))]^\alpha \geq h^{1-\alpha} \left(\sum_{i=1}^{h} [2 + \lambda_i(\mathcal{L}(G))] \right)^\alpha = h^{1-\alpha}(2m)^\alpha$$

with equality if and only if $2 + \lambda_i(\mathcal{L}(G))$ is a constant for all $i = 1, 2, \ldots, h$, which implies that $\mathcal{L}(G)$ consists of h isolated vertices. By Perron-Frobenius Theorem, we get that the equality holds in (3) if and only if $h = m$ and G consists of m copies of K_2 and possibly isolated vertices.

Similarly, the result in (ii) follows. □

Theorem 2. *Let G be a graph on n vertices and e be an edge of G. Then*

(i) $s_\alpha^+(G-e) < s_\alpha^+(G)$ for $\alpha > 0$ and $s_\alpha^+(G-e) > s_\alpha^+(G)$ for $\alpha < 0$.

(ii) $s_\alpha^+(G) \leq (2n-2)^\alpha + (n-1)(n-2)^\alpha$ for $\alpha > 0$ and $s_\alpha^+(G) \geq (2n-2)^\alpha + (n-1)(n-2)^\alpha$ for $\alpha < 0$ with either equality if and only if G is the complete graph K_n.

Proof. To prove (i), observe that $\sum_{i=1}^{n} q_i(G) = 2 + \sum_{i=1}^{n} q_i(G-e)$. From Lemma 5, we get the required result.

Observe that the signless Laplacian eigenvalues of a complete graph K_n are $2n-2$ with multiplicity 1 and $n-2$ with multiplicity $n-1$. From (i), the result in (ii) holds. □

Next, we shall establish some fundamental properties of $s_\alpha^+(G)$.

Proposition 1. *(i) $s_\alpha^+(G) \geq 0$ with equality if and only if $m = 0$.*

(ii) If the graph G has components $G_1, G_2, \ldots, G_r$, then

$$s_\alpha^+(G) = \sum_{i=1}^{r} s_\alpha^+(G_i).$$

Proof. Since the signless Laplacian eigenvalues of G are nonnegative, then $s_\alpha^+(G) \geq 0$ with equality if and only if $q_1 = 0$, which implies that $m = 0$.

To see (ii), observe that the signless Laplacian spectrum of G is the union of the signless Laplacian spectrums of $G_i (i = 1, 2, \ldots, r)$, which implies that the result in (ii) follows. □

For a graph G with components $G_1, G_2, \ldots, G_r$, Proposition 1 implies that the sum of the αth power of the non-zero signless Laplacian eigenvalues of G is equal to the sum of r sums of these of $G_1, G_2, \ldots, G_r$. Without loss generality, we shall always assume that G is a connected graph throughout.

Theorem 3. *Let G be a connected graph without vertices of degree 1, with n vertices, m edges and maximum degree Δ. Suppose that G has h non-zero signless Laplacian eigenvalues:*

(i) If $\alpha < 0$ or $\alpha > 1$, then

$$s_\alpha^+(G) \geq \left(1 + \Delta + \frac{1}{\Delta - 1}\right)^\alpha + (h-1)^{1-\alpha}\left(2m - (1 + \Delta + \frac{1}{\Delta - 1})\right)^\alpha \quad (5)$$

with equality if and only if G is either K_3 or C_4.

(ii) If $0 < \alpha < 1$, then

$$s_\alpha^+(G) \leq \left(1 + \Delta + \frac{1}{\Delta - 1}\right)^\alpha + (h-1)^{1-\alpha}\left(2m - (1 + \Delta + \frac{1}{\Delta - 1})\right)^\alpha \quad (6)$$

with equality if and only if G is either K_3 or C_4.

Proof. Suppose that $\alpha < 0$ or $\alpha > 1$. From Lemma 5, we derive

$$\begin{aligned} s_\alpha^+(G) &= q_1^\alpha + \sum_{i=2}^{h} q_i^\alpha \geq q_1^\alpha + (h-1)^{1-\alpha}\left(\sum_{i=2}^{h} q_i\right)^\alpha \\ &= q_1^\alpha + (h-1)^{1-\alpha}(2m - q_1)^\alpha. \end{aligned}$$

Take $f(x) = x^\alpha + (h-1)^{1-\alpha}(2m - x)^\alpha$. By solving derivative function $f'(x) = \alpha[x^{\alpha-1} - (h-1)^{1-\alpha}(2m - x)^{\alpha-1}] \geq 0$, it is readily seen that $f(x)$ is increasing on $x \geq \frac{2m}{h}$. Since G is a connected graph, it follows from Proposition 2.1 in [5] that either $h = n$ or $h = n - 1$. Observe that $(n-1)\Delta + (n-1) \geq 2m$. From Lemma 3, we may obtain $q_1 \geq \Delta + 1 + \frac{1}{\Delta - 1} > \Delta + 1 \geq \frac{2m}{n-1} \geq \frac{2m}{h}$ and then

$$\begin{aligned} s_\alpha^+(G) &\geq f(\Delta + 1 + \frac{1}{\Delta - 1}) \\ &= \left(1 + \Delta + \frac{1}{\Delta - 1}\right)^\alpha + (h-1)^{1-\alpha}\left(2m - (1 + \Delta + \frac{1}{\Delta - 1})\right)^\alpha \end{aligned}$$

with equality if and only if $q_2 = q_3 = \cdots = q_h$ and G is a cycle C_n.

Now suppose that the equality holds in (5). We consider the following two cases: G is either bipartite or non-bipartite. If G is a connected bipartite graph, then the Proposition 2.1 in [5] implies $h = n - 1$. This show that G has exactly three distinct signless Laplacian eigenvalues. Lemma 2 implies that G has diameter at most 2. If the diameter of G equals to 1, then $G = K_2$, which contradicts with the condition that G is a cycle C_n. Hence, the diameter of G is 2, which implies that G is a complete bipartite graph $K_{r,n-r}$ with $1 \leq r \leq n-1$. Since G is also a cycle C_n, then G must is the cycle C_4. If G is a connected non-bipartite graph, then $h = n$. So, G has exactly two distinct signless Laplacian eigenvalues. By Lemma 2, the diameter of G is 1, which implies that G is a complete graph K_n. Since G is a cycle C_n, then G must is the complete graph K_3.

Conversely, suppose that $G = K_3$ or C_4. It is easy to verify that the equality holds in (5).

To prove (ii), suppose that $0 < \alpha < 1$. Lemma 5 implies that

$$s_\alpha^+(G) \leq q_1^\alpha + (h-1)^{1-\alpha}(2m - q_1)^\alpha$$

with equality if and only if $q_2 = q_3 = \cdots = q_h$. Observe that $f(x)$ is decreasing on $x \geq \frac{2m}{h}$. By similar arguments as (i), the result in (ii) also follows. □

Corollary 2. *Let G be a connected graph of order n, $n \geq 2$, with m edges and maximum degree Δ. Suppose that G has h non-zero signless Laplacian eigenvalues:*

(i) If $\alpha < 0$ or $\alpha > 1$, then

$$s_\alpha^+(G) \geq (1+\Delta)^\alpha + (h-1)^{1-\alpha}(2m - (1+\Delta))^\alpha \tag{7}$$

with equality if and only if G is a star S_n.

(ii) If $0 < \alpha < 1$, then

$$s_\alpha^+(G) \leq (1+\Delta)^\alpha + (h-1)^{1-\alpha}(2m - (1+\Delta))^\alpha \tag{8}$$

with equality if and only if G is a star S_n.

Proof. We prove only (i); the proof of (ii) is similar and is omitted. Suppose that $\alpha < 0$ or $\alpha > 1$. It is quite evident that (7) holds with equality for $n = 2$. In the following, we shall assume that $n \geq 3$. From Lemma 4 and the proof of Theorem 3, one has $q_1 \geq \Delta + 1 \geq \frac{2m}{h}$ and $f(x)$ is increasing on $x \geq \frac{2m}{h}$. Hence,

$$s_\alpha^+(G) \quad \geq \quad f(\Delta+1) = (1+\Delta)^\alpha + (h-1)^{1-\alpha}(2m - (1+\Delta))^\alpha$$

with equality if and only if $q_2 = q_3 = \cdots = q_h$ and G is a star S_n. This shows that G is bipartite. From the proof of Theorem 3, we get that

G has diameter at most 2. If the diameter of G is 1, then $G = K_2$, which contradicts with the assumption that $n \geq 3$. Hence, the diameter of G is 2, which implies that G is a complete bipartite graph $K_{r,n-r}$ with $1 \leq r \leq n-1$. Hence G must is the star S_n.

Conversely, suppose that G is the star S_n. It is easy to verify that the equality in holds (7). □

Theorem 4. *Let G be a connected graph with n vertices, m edges, maximum degree Δ and average degree $\bar{d}$ satisfying $\Delta \geq \frac{\bar{d}(n+1)}{n-1}$. Suppose that G has h non-zero signless Laplacian eigenvalues:*

(i) If $\alpha < 0$ or $\alpha > 1$, then

$$s_\alpha^+(G) > 2^{1-\alpha}\left(\Delta + \bar{d}\right)^\alpha + (h-2)^{1-\alpha}\left(2m - (\Delta + \bar{d})\right)^\alpha. \tag{9}$$

(ii) If $0 < \alpha < 1$, then

$$s_\alpha^+(G) < 2^{1-\alpha}\left(\Delta + \bar{d}\right)^\alpha + (h-2)^{1-\alpha}\left(2m - (\Delta + \bar{d})\right)^\alpha. \tag{10}$$

Proof. Suppose that $\alpha < 0$ or $\alpha > 1$. From Lemma 5, we derive

$$\begin{aligned} s_\alpha^+(G) &= (q_1^\alpha + q_2^\alpha) + \sum_{i=3}^{h} q_i^\alpha \\ &\geq 2^{1-\alpha}(q_1+q_2)^\alpha + (h-2)^{1-\alpha}\left(\sum_{i=3}^{h} q_i\right)^\alpha \\ &= 2^{1-\alpha}(q_1+q_2)^\alpha + (h-2)^{1-\alpha}(2m-(q_1+q_2))^\alpha. \end{aligned}$$

Take $g(x) = 2^{1-\alpha}x^\alpha + (h-2)^{1-\alpha}(2m-x)^\alpha$. By solving derivative function $g'(x) = \alpha[2^{1-\alpha}x^{\alpha-1} - (h-2)^{1-\alpha}(2m-x)^{\alpha-1}] \geq 0$, it is readily seen that $g(x)$ is increasing on $x \geq \frac{4m}{h}$. The condition $\Delta \geq \frac{\bar{d}(n+1)}{n-1}$ implies that $\Delta + \bar{d} \geq \frac{4m}{n-1} \geq \frac{4m}{h}$, where the last inequality holds as h is either n or $n-1$. From Lemma 4 and Theorem 5.1 in [9], we get $q_1 + q_2 \geq \Delta + \bar{d} \geq \frac{4m}{h}$. Thus, $s_\alpha^+(G) \geq f(\Delta + \bar{d})$. Now suppose that $s_\alpha^+(G) = f(\Delta + \bar{d})$. Lemma 5 implies that $q_1 = q_2$, which contradicts with the condition that G is a connected graph. Hence, the inequality (9) follows.

If $0 < \alpha < 1$, then $g(x)$ is decreasing on $x \geq \frac{4m}{h}$. By similar arguments as (i), the result in (ii) also follows. □

Corollary 3. *Let G be a connected non-bipartite graph with n vertices, m edges, maximum degree Δ and average degree $\bar{d}$.*

(i) If $\alpha < 0$ or $\alpha > 1$, then

$$s_\alpha^+(G) > 2^{1-\alpha}\left(\Delta + \bar{d}\right)^\alpha + (n-2)^{1-\alpha}\left(2m - (\Delta + \bar{d})\right)^\alpha. \tag{11}$$

(ii) *If* $0 < \alpha < 1$, *then*

$$s_\alpha^+(G) < 2^{1-\alpha}\left(\Delta+\bar{d}\right)^\alpha + (n-2)^{1-\alpha}\left(2m-(\Delta+\bar{d})\right)^\alpha. \tag{12}$$

Proof. Since G is a connected non-bipartite graph, then $h = n$. Suppose that $\alpha < 0$ or $\alpha > 1$. From the proof Theorem 4, we obtain $g(x)$ is increasing on $x \geq 2\bar{d}$. Observe that $\Delta + \bar{d} \geq 2\bar{d}$. Thus, the inequality (11) follows.

Similarly, we may get that the inequality (12) also follows. □

Theorem 5. *Let G be a connected graph with n vertices and m edges. Suppose that G has h non-zero signless Laplacian eigenvalues:*

(i) *If* $\alpha < 0$ *or* $\alpha > 1$, *then*

$$s_\alpha^+(G) \geq \eta^\alpha(G) + (h-1)^{1-\alpha}\left(2m-\eta(G)\right)^\alpha \tag{13}$$

with equality if and only if G is one of K_n, S_n or $K_{\frac{n}{2},\frac{n}{2}}$ where n is even.

(ii) *If* $0 < \alpha < 1$, *then*

$$s_\alpha^+(G) \leq \eta^\alpha(G) + (h-1)^{1-\alpha}\left(2m-\eta(G)\right)^\alpha \tag{14}$$

with equality if and only if G is one of K_n, S_n or $K_{\frac{n}{2},\frac{n}{2}}$ where n is even.

Proof. We prove only (i); the proof of (ii) is similar and is omitted.

Suppose that $\alpha < 0$ or $\alpha > 1$. From the proof of Theorem 3, we get $s_\alpha^+(G) \geq f(q_1)$, where $f(x) = x^\alpha + (h-1)^{1-\alpha}(2m-x)^\alpha$. From Lemma 6, Corollary 1 and the proof of Theorem 7 in [19], one has

$$q_1 \geq \eta(G) \geq \tau(G) \geq \frac{4m}{n} > \frac{2m}{n-1} \geq \frac{2m}{h},$$

where the last inequality holds as h is either n or $n-1$. Since $f(x)$ is increasing on $x \geq \frac{2m}{h}$, then $s_\alpha^+(G) \geq f(\eta(G))$, from which (13) follows.

Now suppose that the equality holds in (13). Then $q_1 = \eta(G)$ and $q_2 = q_3 = \cdots = q_h$. Notice that G is connected. If G is non-bipartite, then the Proposition 2.1 in [5] implies $h = n$. This show that G has exactly two distinct signless Laplacian eigenvalues. Lemma 2 implies that G has diameter 1. Thus G is the complete graph K_n. If G is bipartite, then $h = n-1$. Hence G has exactly three distinct signless Laplacian eigenvalues. From Lemma 6 in [19], G is either S_n or $K_{\frac{n}{2},\frac{n}{2}}$ where n is even.

Conversely, suppose that G is one of K_n, S_n and $K_{\frac{n}{2},\frac{n}{2}}$ where n is even. Lemma 6 implies that the equality holds in (13). □

Remark 1. For a connected graph G of order n, Proposition 2.1 in [3] implies that $h = n-1$ if and only if G is bipartite. So, it is easy to directly define the value h in above Theorems 1, 3-5 and Corollaries 2-3.

4. Some related results with $\alpha = \frac{1}{2}$

In this section, we consider the special case $\alpha = \frac{1}{2}$, which is exactly the incidence energy of a graph.

The following Corollary 4 is an immediate consequence of Theorem 1.

Corollary 4. *Let G be a graph with m edges. Suppose that G has h non-zero signless Laplacian eigenvalues. Then*

$$IE(G) \le \sqrt{2hm}$$

with equality if and only if $h = m$ and G consists of m copies of K_2 and possibly isolated vertices.

From the alternative proof of Theorem 1, we have $h \le m$. Thus, $IE(G) \le \sqrt{2hm} \le \sqrt{2}m$, which shows that Corollary 4 is an improvement on Theorem 3.2 in [12].

Corollary 5. *Let G be a connected graph without vertices of degree 1, with n vertices, m edges and maximum degree Δ. Suppose that G has h non-zero signless Laplacian eigenvalues. Then*

$$IE(G) \le \sqrt{1 + \Delta + \frac{1}{\Delta - 1}} + \sqrt{(h-1)\left[2m - (1 + \Delta + \frac{1}{\Delta - 1})\right]}$$

with equality if and only if G is either K_3 or C_4.

Proof. Taking $\alpha = \frac{1}{2}$ in Theorem 3, we immediately get the required result. □

Corollary 6. *Let G be a connected graph of order n, $n \ge 2$, with m edges and maximum degree Δ. Suppose that G has h non-zero signless Laplacian eigenvalues. Then*

$$IE(G) \le \sqrt{1 + \Delta} + \sqrt{(h-1)(2m - (1 + \Delta))}$$

with equality if and only if G is a star S_n.

Proof. This is an immediate consequence of Corollary 2. □

Observe that the function $f(x) = \sqrt{x} + \sqrt{(h-1)(2m - x)}$ (with $\alpha = \frac{1}{2}$ in the proof Theorem 3) is decreasing on $x \ge \frac{2m}{h}$. Thus, Corollary 5 improves Corollary 6. However, from Corollary 6, we immediately get

$IE(G) < \sqrt{1+\Delta} + \sqrt{(n-1)(2m-(1+\Delta))}$. This shows that Corollary 5 is an improvement on Theorem 3.6 in [12].

Taking $\alpha = \frac{1}{2}$ in Theorem 4, we may get the following Corollary 7.

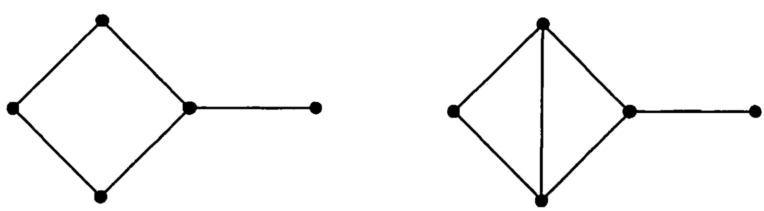
Corollary 7. *Let G be a connected graph with n vertices, m edges, maximum degree Δ and average degree $\bar{d}$ satisfying $\Delta \geq \frac{\bar{d}(n+1)}{n-1}$. Suppose that G has h non-zero signless Laplacian eigenvalues. Then*

$$IE(G) < \sqrt{2(\Delta+\bar{d})} + \sqrt{(h-2)(2m-(\Delta+\bar{d}))} \tag{15}$$

Remark 2. We firstly remark that the condition $\Delta \geq \frac{\bar{d}(n+1)}{n-1}$ can be omitted whenever G is a non-bipartite graph in Corollary 7 (see Corollary 3). Secondly, Gutman et al.[12] proved the following result:

$$IE(G) < \sqrt{1+\Delta} + \sqrt{(n-1)(2m-(1+\Delta))}. \tag{16}$$

We also remark that the inequalities (15) and (16) are incomparable. Let G_1 and G_2 be the two graphs shown in Fig. 1. For G_1 the upper bounds (15) is better than (16), whereas for G_2 the upper bounds (16) is better than (15).



G_1 G_2

Fig.1. Examples showing that inequalities (15) and (16) are incomparable.

Corollary 8. *Let G be a connected graph with n vertices and m edges. Suppose that G has h non-zero signless Laplacian eigenvalues. Then*

$$IE(G) \leq \sqrt{\eta(G)} + \sqrt{(h-1)(2m-\eta(G))}$$

with equality if and only if G is one of K_n, S_n or $K_{\frac{n}{2},\frac{n}{2}}$ where n is even.

Proof. Taking $\alpha = \frac{1}{2}$ in Theorem 5, we immediately get the required result. □

From the proof of Theorem 5, we have $f(x) = \sqrt{x} + \sqrt{(h-1)(2m-x)}$ (with $\alpha = \frac{1}{2}$ in the proof Theorem 5) is decreasing on $x \geq \frac{2m}{h}$. It follows from $q_1 \geq \eta(G) \geq \tau(G) \geq \frac{2m}{h}$ that $IE(G) \leq f(\eta(G)) \leq f(\tau(G))$. Note that h is either n or $n-1$. Hence, Corollary 8 is an improvement on Theorem 3.7 in [12].

Before this paper ends, we consider the special case $\alpha = 2$. Noting that $s_2^+(G) = s_2(G) = \sum_{i=1}^{n}(d_i^2 + d_i)$. Then $s_2^+(G)$ and $s_2(G)$ have same properties, for example,

Theorem 6[13]. *Let G be a connected graph with $n \geq 2$ vertices. Then*

$$6n - 8 \leq s_2^+(G) \leq (n-1)n^2.$$

The left equality holds if and only if G is the path P_n of order n, whereas the right equality holds if and only if G is the complete graph K_n of order n.

Proof. Let T be a spanning tree of G. From Theorem 1, we get $s_2^+(T) \leq s_2^+(G) \leq s_2^+(K_n)$, where the left equality holds if and only if G is the spanning tree T, whereas the right equality holds if and only if G is the complete graph K_n of order n. The rest proof can be seen in [13, 21]. □

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