The Ramsey Numbers $r(K_5 - 2K_2, 2K_3)$, $r(K_5 - e, 2K_3)$, and $r(K_5, 2K_3)$

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Abstract

This note will complete the computation of all Ramsey numbers $r(G, H)$ for graphs G of order at most five and disconnected graphs H of order six.

In [3], the Ramsey numbers $r(G, H)$ have been determined for all graphs G of order at most five and all disconnected graphs H of order six, except the three cases $(G, H) \in \{(K_5 - 2K_2, 2K_3), (K_5 - e, 2K_3), (K_5, 2K_3)\}$, where the bounds $12 \leq r(G, 2K_3) \leq 13$ for $G \in \{K_5 - 2K_2, K_5 - e\}$ and $15 \leq r(K_5, 2K_3) \leq 16$ could be established. Here we will prove that the lower bound matches the exact value in the first two cases, whereas in the third case the upper bound gives the exact value. The following specialized notation will be used. When considering a 2-coloring χ of the edges of K_n, we refer to colors green and red. For given graphs G and H, we say that χ is a (G, H)-coloring, if it neither contains a green subgraph isomorphic to G nor a red subgraph isomorphic to H. We use V to denote the vertex set of K_n, and for $V' \subseteq V$ we define $[V']$ to be the subgraph induced by V'. In case of $V' = \{v_1, \ldots, v_k\}$ we write $[v_1, \ldots, v_k]$ instead of $[\{v_1, \ldots, v_k\}]$. Moreover, the green and red subgraphs induced by V' are denoted by $[V']_g$ and $[V']_r$, respectively.

Theorem 1.

$$r(K_5 - 2K_2, 2K_3) = r(K_5 - e, 2K_3) = 12.$$

Proof. It is already known that $r(G, 2K_3) \geq 12$ for $G \in \{K_5 - 2K_2, K_5 - e\}$. Moreover, $r(K_5 - 2K_2, 2K_3) \leq r(K_5 - e, 2K_3)$. Thus, it is sufficient to prove that $r(K_5 - e, 2K_3) \leq 12$. Suppose to the contrary that there is a $r(K_5 - e, 2K_3)$-coloring of K_{12} with vertex set V. From $r(K_5 - e, K_3) = 11$ (see [2]) we obtain $K_3 \subset [V]_r$. Let $U = \{u_1, u_2, u_3\}$ be the vertex set of a red K_3 in $[V]$ and let $W = \{w_1, w_2, \ldots, w_9\} = V \setminus U$. Then $2K_3 \not\subset [V]_r$ implies $K_3 \not\subset [W]_r$. Thus, $K_4 \subset [W]_g$ follows from $r(K_4, K_3) = 9$ (see [1]),

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and we may assume that $W' = \{w_1, w_2, w_3, w_4\}$ induces a green K_4. We distinguish the following two cases.

Case I: $2K_4 \subset [W]_g$. We may assume that $W'' = \{w_5, w_6, w_7, w_8\}$ induces a second green K_4. Since $K_5 - e \not\subset [V]_g$, at least two of the edges joining w_9 to W', say w_9w_1 and w_9w_2, have to be red. Similarly, at least two of the edges between w_9 and W'', say w_9w_5 and w_9w_6, must be red. Then $[w_1, w_2, w_5, w_6]$ has to be a green K_4 because of $K_3 \not\subset [W]_r$. Moreover, $K_5 - e \not\subset [V]_g$ forces all remaining edges in $[w_1, w_2, w_7, w_8]$ and in $[w_3, w_4, w_5, w_6]$ to be red. Furthermore, $K_3 \not\subset [W]_r$ implies only green edges between w_9 and $\{w_3, w_4, w_7, w_8\}$. Now consider the edges between U and W. From $K_5 - e \not\subset [V]_g$ we deduce that every $u \in U$ is joined by red edges to at least two vertices of the green K_4 induced by $\{w_1, w_2, w_5, w_6\}$. Thus, one vertex in $\{w_1, w_2, w_5, w_6\}$, say w_1, has to be joined by red edges to two vertices in U, say to u_1 and u_2. First assume that u_3w_1 or u_3w_2 is green. Since $K_5 - e \not\subset [V]_g$, the two green subgraphs K_4 induced by $\{w_1, w_2, w_5, w_6\}$ and $\{w_1, w_2, w_3, w_4\}$ force at least one red edge from u_3 to w_3 or w_4 and at least one red edge from u_3 to w_5 or w_6. But this produces a red $K_3 = [u_3, v, w]$ with $v \in \{w_3, w_4\}$ and $w \in \{w_5, w_6\}$ yielding a red $2K_3$ with $[w_1, u_1, u_2]$. The remaining case is that the edges u_3w_1 and u_3w_2 are both red. No green $K_5 - e$ in $[u_1, u_2, w_7, w_8, w_9]$ forces at least one red edge from u_1 or u_2, say from u_1, to $\{w_7, w_8, w_9\}$ yielding a red $K_3 = [u_1, w_1, v]$ with $v \in \{w_7, w_8, w_9\}$. Now a red $2K_3$ can only be avoided, if the edge u_2w_2 is green. But then either one of the subgraphs $[w_1, w_2, w_5, w_6]$ and $[w_1, w_2, w_3, w_4]$ together with u_2 yields a green $K_5 - e$ or we obtain a red $K_3 = [u_2, v, w]$ with $v \in \{w_3, w_4\}$ and $w \in \{w_5, w_6\}$ yielding a red $2K_3$ together with $[w_1, u_1, u_3]$, a contradiction.

Case II: $2K_4 \not\subset [W]_g$. Let $W''' = W \setminus W'$. One of the following to subcases must occur.

(i) The subgraph $[W''']_r$ contains an odd cycle. Then $K_3 \not\subset [W]_r$ forces $[W''']_r$ to be a cycle C_5. Since $K_5 - e \not\subset [V]_g$, every $w \in W'''$ must be joined by red edges to at least two vertices in W' yielding at least ten red edges between W' and W'''. Thus, one vertex in W' has to be joined red to at least three vertices of the red C_5 in W''' yielding a red K_3 in $[W]$, a contradiction.

(ii) The subgraph $[W''']_r$ is bipartite. This implies $K_1 \cup K_4 \subset [W''']_g$ or $K_2 \cup K_3 \subset [W''']_g$. Since $2K_4 \not\subset [W]_g$, only $K_2 \cup K_3 \subset [W''']_g$ is left. We may assume that $[w_5, w_6]$ and $[w_7, w_8, w_9]$ are a green K_2 and a green K_3, respectively. Then $2K_4 \not\subset [W]_g$ forces at least one red edge between every $w \in \{w_5, w_6\}$ and $\{w_7, w_8, w_9\}$. First assume that some $w \in \{w_5, w_6\}$, say w_5, is joined by red edges to two vertices in $\{w_7, w_8, w_9\}$, say to w_7 and to w_8. Moreover, since $K_5 - e \not\subset [V]_g$, we may assume that w_5w_1 and

w_5w_2 are red edges. Then $K_3 \not\subseteq [W]_r$ forces $[w_1, w_2, w_7, w_8]$ to be a green K_4. Consequently, all edges between $\{w_3, w_4\}$ and $\{w_7, w_8\}$ have to be red. Furthermore, the edges w_5w_3 and w_5w_4 must be green. To avoid a green $2K_4$ in $[W]$, one of the edges w_6w_3 and w_6w_4, say w_6w_3, has to be red. This forces w_6w_7 and w_6w_8 to be green. Then $2K_4 \not\subseteq [W]_g$ and $K_5 - e \not\subseteq [V]_g$ imply only red edges between w_6 and $\{w_1, w_2, w_9\}$. But this yields a red K_3 in $[W]$ if one of the edges between w_9 and $\{w_1, w_2, w_3\}$ is red, and otherwise we obtain a green $K_5 - e$ in $[w_9, w_1, w_2, w_3, w_4]$, a contradiction. In the remaining case, for every $w \in \{w_5, w_6\}$ there is exactly one red edge between w and $\{w_7, w_8, w_9\}$. Since $2K_4 \not\subseteq [W]_g$, a common red neighbor of w_5 and w_6 in $\{w_7, w_8, w_9\}$ is forbidden. Thus, we may assume that the edges w_5w_7 and w_6w_8 are red. Moreover, two edges between w_5 and W', say w_5w_1 and w_5w_2, have to be red because of $K_5 - e \not\subseteq [W]_g$. Consequently, w_7w_1 and w_7w_2 must be green, w_7w_3 and w_7w_4 red, w_5w_3 and w_5w_4 green. Then $K_5 - e \not\subseteq [w_1, w_2, w_6, w_7, w_8]_g$ forces at least one red edge between $\{w_6, w_8\}$ and $\{w_1, w_2\}$. We may assume that w_6w_1 is red. This implies that w_1w_8 is green. Moreover, one of the edges w_8w_3 and w_8w_4, say w_8w_3, must be red. Consequently, w_6w_3 has to be green. Now a green $2K_4$ consisting of $[w_1, w_2, w_7, w_8] \cup [w_3, w_4, w_5, w_6]$ can only be avoided if one of the edges w_8w_2 and w_6w_4, say w_8w_2, is red. Then w_6w_2 has to be green, w_6w_4 red and w_8w_4 green. Thus, all edges in $[W \setminus \{w_9\}]$ are colored. Note that $[W \setminus \{w_9\}]_r$ consists of the cycle $C_8 = (w_1, w_5, w_2, w_8, w_3, w_7, w_4, w_6)$ together with the two diagonals w_5w_7 and w_6w_8. Consider now the edges between w_9 and W'. If one of the edges w_9w_1 and w_6w_3 is colored green and also one of the edges w_9w_2 and w_9w_4, say w_9w_1 and w_9w_2, we obtain a green $K_5 - e$ in $[w_1, w_2, w_6, w_7, w_9]$, a contradiction. Thus, we may assume that the edges w_9w_1 and w_9w_3 are red. It remains to color the edges w_9w_2 and w_9w_4, and all edges between U and W. To avoid a lengthy and tedious discussion of many cases we used a simple backtracking algorithm and proved, supported by a computer, that any coloring of these edges leads to a green $K_5 - e$ or to a red $2K_3$, a contradiction. ■

Theorem 2.

$$r(K_5, 2K_3) = 16.$$

Proof. It is already known that $r(K_5, 2K_3) \leq 16$. To prove that $r(K_5, 2K_3) \geq 16$, a $(K_5, 2K_3)$-coloring of K_{15} with vertex set $V = \{v_1, v_2, \ldots, v_{15}\}$ is constructed as follows. Let $U = \{v_1, v_2, \ldots, v_{12}\}$ and $W = \{v_{13}, v_{14}, v_{15}\}$. Take the (K_5, K_3)-coloring of $[U]$ with $[U]_r$ as given in Figure 1. Note that $K_4 \not\subseteq [v_1, v_2, \ldots, v_8]_g$. Then add a red K_3 with vertex set W. Join every $v \in W$ by red edges to $v_9, v_{10}, v_{11}, v_{12}$ and by green edges to $v_1, v_2, \ldots, v_8$. The resulting coloring does not contain a red $2K_3$ since every red subgraph

K_3 contains at least two vertices belonging to W. To see that a green subgraph K_5 cannot occur, note that $K_5 \not\subset [V]_g$. Moreover, a green K_5 with at least one vertex belonging to W is impossible, because $[W]$ contains only red edges and for every $v \in W$ its green neighbors are $v_1, v_2, \ldots, v_8$ where $K_4 \not\subset [v_1, v_2, \ldots, v_8]_g$. ■

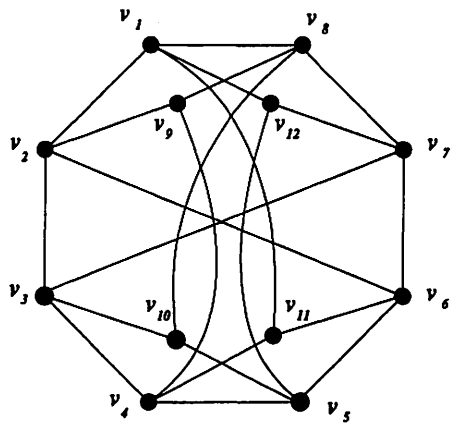

Figure 1: The graph $[U]_r$.

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