Some Results On Felicitous Labeling of Graphs

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Abstract

Figueroa-centeno, Ichishima and Muntaner-Batle [3, 4] proved some results on felicitous graph and raised the following conjectures:

(i) The one point union of m copies of C_n is felicitous if and only if $mn \equiv 2(mod\ 4)$.

(ii) mC_n is felicitous if and only if $mn \not\equiv 2(mod\ 4)$.

In this paper, the conjectures are partially settled by proving the following results.

1. For any odd positive integers m and n the one-point union of m copies of C_n is felicitous if $mn \equiv 1, 3$.
2. For any positive integer m, the one-point union of m copies of C_4 is felicitous.
3. For any two odd positive integers m and n, mC_n is felicitous if $mn = 1, 3(mod\ 4)$.
4. For any positive integer m, mC_4 is felicitous.

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1 Introduction

In 1967, Rosa [5] introduced the concept of labeling in graph theory. Thereafter, many authors introduced families of graphs such as magic, graceful, harmonious, felicitous etc. Labelled graphs serve as useful models for broad range of applications such as coding theory, X-ray, crystallography, radar, astronomy, circuit design and communication addressing refer [1, 2] for details.

By $G(p,q)$, we mean a graph G having p vertices and q edges. Let $V(G)$ and $E(G)$ denote respectively the vertex set and the edge set of G.

In this paper, aided by the use of certain number theoretic properties, we have partially settled the conjecture raised by Figueroa-centeno et al.[5]. We need the following definition.

Definition 1.1. Lee, Schmeichel and Shee [6] *An injective function f from the vertices of a graph G with q edges to the set $\{0,1,2,\ldots,q\}$ is called felicitous if the edge labels induced by $(f(x)+f(y))(mod\ q)$ for each edge xy, are distinct.*

2 Main Results

In this section, we partially prove the conjecture raised by Figueroa-centeno [3, 4].

Result 2.1. *For any odd positive integers m and n, the one point union of m copies of C_n is felicitous if $mn \equiv 1,3(mod\ 4)$.*

Proof. Let G be the one-point union of m copies of cycles of length n where m and n are both odd. Let $V(G) = \{V_0^{(j)} = V_0 \text{ for all } j\} \cup \{V_i^{(j)} : 1 \leq i \leq n-1; 1 \leq j \leq m\}$ be the vertices of G and $E(G) = \{V_0V_1^{(j)} : 1 \leq j \leq m\} \cup \{V_i^{(j)}V_{i+1}^{(j)} : 2 \leq i \leq n-2, 1 \leq j \leq m\} \cup \{V_{n-1}^{(j)}V_0 : 1 \leq j \leq m\}$ be the edges of G.

Define an injection $f : V(G) \rightarrow \{0,1,2,3,\ldots,mn\}$ as follows:

$$\begin{aligned}
f(V_0) &= 1 \\
f(V_i^{(j)}) &= i+1+n(j-1) : 1 \leq i \leq n-1, 1 \leq j \leq m \\
f(V_0) + f(V_1^{(j)}) &= 3+n(j-1)(mod\ mn) : 1 \leq j \leq m \\
f(V_i^{(j)}) + f(V_{i+1}^{(j)}) &= 2i+3+2n(j-1)(mod\ mn) : 1 \leq i \leq n-2, 1 \leq j \leq m \\
f(V_{n-1}^{(j)}) + f(V_0) &= nj+1(mod\ mn) : 1 \leq j \leq m.
\end{aligned}$$

We verify that the edge labels are distinct and $(f(x)+f(y)) \in \{0,1,2,3,\ldots,(mn-1)\}$ for every edge $xy \in E(G)$ and hence f is a felicitous labeling. □

Result 2.2. *For any positive integer m, the one point union of m copies of C_4 is felicitous.*

Proof. Let G be the one point union of m copies of cycles of length 4. Let $V(G) = \{V_0^{(j)} : V_0 : \text{for all } j\} \cup \{V_i^{(j)} : 1 \leq i \leq 3, 1 \leq j \leq m\}$ be the vertices of G and let $E(G) = \{V_0V_1^{(j)} : 1 \leq j \leq m\} \cup \{V_i^{(j)}V_{i+1}^{(j)} : 1 \leq i \leq 2, 1 \leq j \leq m\} \cup \{V_3^{(j)}V_0 : 1 \leq j \leq m\}$ be the edges of G.

Define an injection $f : V(G) \to \{0, 1, 2, \ldots, 4m\}$ as follows:

$$\begin{aligned}
f(V_0) &= 0 \\
f(V_1^{(j)}) &= j : 1 \leq j \leq m \\
f(V_2^{(j)}) &= 3m + 1 - 2j : 1 \leq j \leq m \\
f(V_3^{(j)}) &= 3m + j : 1 \leq j \leq m \\
f(V_0) + f(V_1^{(j)}) &= j : 1 \leq j \leq m \\
f(V_1^{(j)}) + f(V_2^{(j)}) &= 3m + 1 - j (mod\ 4m) : 1 \leq j \leq m \\
f(V_2^{(j)}) + f(V_3^{(j)}) &= 6m + 1 - j (mod\ 4m) : 1 \leq j \leq m \\
f(V_3^{(j)}) + f(V_0) &= 3m + j (mod\ 4m) : 1 \leq j \leq m.
\end{aligned}$$

We verify that the edge labels are distinct and $(f(x)+f(y)) \in \{0, 1, 2, 3 \ldots, (4m+1)\}$ for every edge $xy \in E(G)$ and hence f is a felicitous labeling. □

Remark 2.3. *Result 2.2 proves a particular case of the conjecture given in [3] by R. Figueroa-Centeno et al., for any value of m with n = 4.*

Result 2.4. *For any two odd positive integers m and n, mC_n is felicitous if $mn \equiv 1, 3 (mod\ 4)$.*

Proof. Let $G = mC_n$ be the m disjoint copies of C_n.

Let $V(G) = \{v_i^{(j)} : 1 \leq i \leq n, 1 \leq j \leq m\}$ be the vertices of G and let $E(G) = \{v_i^{(j)}v_{i+1}^{(j)} : 1 \leq i \leq n-1, 1 \leq j \leq m\} \cup \{v_n^{(j)}v_1^{(j)} : 1 \leq j \leq m\}$ be the edges of G. Let $G = mC_n$ where m and n are odd.

Define an injection $f : V(G) \to \{0, 1, 2, 3, \ldots, mn\}$ as follows.

$$f(v_i^{(j)}) = \begin{cases} j & \text{if } i = 1 \text{ and } 1 \leq j \leq m. \\ \frac{5m-j+2}{2} & \text{if } i = 2, n = 3, 1 \leq j \leq m \text{ and } j \text{ is odd.} \\ \frac{6m-j+2}{2} & \text{if } i = 2, n = 3, 1 < j < m \text{ and } j \text{ is even.} \\ \frac{4m-j+1}{2} & \text{if } i = 3, n = 3, 1 \leq j \leq m \text{ and } j \text{ is odd.} \\ \frac{3m-j+1}{2} & \text{if } i = 3, n = 3, 1 < j < m \text{ and } j \text{ is even.} \\ \frac{m(n+i)-j+2}{2} & \text{if } 2 \leq i \leq n-3, n > 3, \ i \text{ is even,} \\ & 1 \leq j \leq m \text{ and } j \text{ is odd.} \\ \frac{m(n+i+1)-j+2}{2} & \text{if } 2 \leq i \leq n-3, n > 3, \ i \text{ is even,} \\ & 1 < j < m \text{ and } j \text{ is even.} \\ \frac{m(i+1)-j+1}{2} & \text{if } 3 \leq i \leq n-2, n > 3, \ i \text{ is odd,} \\ & 1 \leq j \leq m \text{ and } j \text{ is odd.} \\ \frac{mi-j+1}{2} & \text{if } 3 \leq i \leq n-2, n > 3, \ i \text{ is odd,} \\ & 1 < j < m \text{ and } j \text{ is even.} \\ mi + j & \text{if } i = n-1, n > 3 \text{ and } 1 \leq j \leq m. \\ \frac{mi-j+2}{2} & \text{if } i = n, n > 3, 1 \leq j \leq m \text{ and } j \text{ is odd.} \\ \frac{m(i+1)-j+2}{2} & \text{if } i = n, n > 3, 1 < j < m \text{ and } j \text{ is even.} \end{cases}$$

$$
f(v_i^{(j)})+f(v_{i+1}^{(j)}) = \begin{cases}
\frac{5m+j+2}{2} & \text{if } i=1, n=3, m\geq 5, \\ & 1\leq j\leq m-4 \\ & \text{and } j \text{ is odd} \\
\frac{-m+j+2}{2} & \begin{cases} \text{if } i=1, n=3, m=1 \\ \text{and } j=1 \\ \text{if } i=1, n=3, m\geq 3, \\ m-2\leq j\leq m \text{and } j \textit{ is odd.} \end{cases} \\
\frac{j+2}{2} & \text{if } i=1, n=3, m\geq 3, 1<j<m \\ & \text{and } j \text{ is even.} \\
\frac{m(n+2)+j+2}{2} & \text{if } i=1, n>3, 1\leq j\leq m \\ & \text{and } j \text{ is odd.} \\
\frac{m(n+3)+j+2}{2} & \text{if } i=1, n>3, 1<j<m \\ & \text{and } j \text{ is even.} \\
\frac{3m-2j+3}{2} & \text{if } i=2, n=3, 1\leq j\leq m. \\
\frac{m(n+2i+2)-2j+3}{2} & \begin{cases} if\ 2\leq i\leq \frac{n-5}{2}, & n\geq 9, j=1 \\ & \text{and } m=1 \\ if\ 2\leq i\leq \frac{n-3}{2}, & n>3 \text{ and} \\ & 1\leq j\leq \frac{m+3}{2} \\ if\ 2\leq i\leq \frac{n-1}{2}, & n>3 \text{ and} \\ & \frac{m+5}{2}\leq j\leq m \end{cases} \\
\frac{m(-n+2i+2)-2j+3}{2} & \begin{cases} if\ \frac{n-5}{2}<i\leq n-3, & n\geq 5, j=1 \\ & \text{and } m=1 \\ if\ \frac{n-3}{2}<i\leq n-3, & n\geq 5 \text{ and} \\ & 1\leq j\leq \frac{m+3}{2} \\ if\ \frac{n-1}{2}<i\leq n-3, & n\geq 5 \text{ and} \\ & \frac{m+5}{2}\leq j\leq m \end{cases} \\
\frac{m(i-1)+j+1}{2} & \text{if } i=n-2, n>3, i \text{ is odd,} \\ & 1\leq j\leq m \text{ and } j \text{ is odd.} \\
\frac{m(i-2)+j+1}{2} & \text{if } i=n-2, n>3, i \text{ is odd,} \\ & 1<j<m \text{ and } j \text{ is even.} \\
\frac{m(i-1)+j+2}{2} & \text{if } i=n-1, n>3, 1\leq j\leq m \\ & \text{and } j \text{ is odd.} \\
\frac{mi+j+2}{2} & \text{if } i=n-1, n>3, 1<j<m \\ & \text{and } j \text{ is even.}
\end{cases}
$$

$$f(v_n^{(j)}) + f(v_1^{(j)}) = \begin{cases} \frac{-2m+j+1}{2} & \text{if } n = 3, m = 1 \text{ and } j = 1 \\ \frac{4m+j+1}{2} & \text{if } n = 3, m \neq 1, 1 \leq i \leq m \\ & \text{and } j \text{ is odd.} \\ \frac{3m+j+1}{2} & \text{if } n = 3, 1 < j < m \text{ and } j \text{ is even.} \\ \frac{mn+j+2}{2} & \text{if } n > 3, 1 \leq j \leq m \text{ and } j \text{ is odd.} \\ \frac{m(n+1)+j+2}{2} & \text{if } n > 3, 1 < j < m \text{ and } j \text{ is even.} \end{cases}$$

It is easy to verify that the edge labels are distinct and $f(x) + f(y) \in \{0, 1, 2, 3, \ldots, (mn-1)\}$ for every edge $xy \in E(G)$ and hence f is a felicitous labeling. □

Result 2.5. *For any positive integer m, mC_4 is felicitous.*

Proof. Let $G = mC_4$ be the m disjoint copies of cycles of length 4.

Let $V(G) = \{v_i^{(j)} : 1 \leq i \leq 4, 1 \leq j \leq m\}$ be the vertices of G and let $E(G) = \{v_i^{(j)} v_{i+1}^{(j)} : 1 \leq i \leq 3, 1 \leq j \leq m\} \cup \{v_4^{(j)} v_1^{(j)}\}$ be the edges of G. Let $G = mC_4$ where m is any positive integer.

Define an injection $f : V(G) \to \{0, 1, 2, \ldots, 4m\}$ as follows.

$$f(v_i^{(j)}) = \begin{cases} m(i-1) + j - 1 & \text{if } 1 \leq i \leq 3 \text{ and } 1 \leq j \leq m \\ 3m + j & \text{if } i = 4 \text{ and } 1 \leq j \leq m. \end{cases}$$

$$f(v_i^{(j)}) + f(v_{i+1}^{(j)}) = \begin{cases} m + 2j - 2 & \text{if } 1 \leq j \leq m \text{ and } i = 1 \\ 3m + 2j - 2 & \begin{cases} if\ 1 \leq j \leq \frac{m+1}{2} & m \text{ is odd and } i = 2 \\ if\ 1 \leq j \leq \frac{m}{2} & m \text{ is even and } i = 2 \end{cases} \\ -m + 2j - 2 & \begin{cases} if\ \frac{m+1}{2} < j \leq m & m \text{ is odd and } i = 2 \\ if\ \frac{m}{2} < j \leq m & m \text{ is even and } i = 2 \end{cases} \\ m + 2j - 1 & \text{if } 1 \leq j \leq m \text{ and } i = 3 \end{cases}$$

$$f(v_4^{(j)})+f(v_1^{(j)}) = \begin{cases} 3m+2j-1 & \begin{cases} if\ 1 \le j \le \frac{m-1}{2} & m \ge 3 \text{ and } m \text{ is odd.} \\ if\ 1 \le j \le \frac{m}{2} & m \ge 2 \text{ and } m \text{ is even.} \end{cases} \\ -m+2j-1 & \begin{cases} if\ m = 1 & j = 1 \\ if\ \frac{m-1}{2} < j \le m & m \ge 3 \text{ and } m \text{ is odd} \\ if\ \frac{m}{2} < j \le m & m \ge 2 \text{ and } m \text{ is even} \end{cases} \end{cases}$$

It is easy to verify that the edge labels are distinct and $f(x) + f(y) \in \{0, 1, 2, 3, \ldots, (4m-1)\}$ for every edge $xy \in E(G)$, Hence f is a felicitous labeling. □

Remark 2.6. *Result 2.5 proves a particular case of the conjecture given in [4] by R. Figueroa-Centeno et al., for any value of m with $n = 4$.*

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