Some identities involving the generalized harmonic numbers

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Abstract In this paper, we discuss the properties of a class of generalized harmonic numbers $H_{n,r}$. Using Riordan arrays and generating functions, we establish some identities involving $H_{n,r}$. Furthermore, we investigate certain sums related to harmonic polynomials $H_n(z)$. In particular, using the Riordan array method, we explore interesting relationships between these polynomials, the generating Stirling polynomials, the Bernoulli polynomials and the Cauchy polynomials. Finally, we obtain the asymptotic expansion of certain sums involving $H_{n,r}$.

Keywords :
Harmonic numbers; Riordan arrays; Harmonic polynomials; Generalized Stirling polynomials; Cauchy polynomials; Bernoulli polynomials

1. Introduction

The harmonic numbers are defined by

$$H_0 = 0, H_n = \sum_{k=1}^{n} \frac{1}{k}, \quad for \quad n = 1, 2, \cdots,$$

and the generating funtion of H_n is

$$\sum_{n=0}^{\infty} H_n t^n = \frac{-\ln(1-t)}{1-t}$$

The harmonic numbers H_n play an important role in number theory and has been generalized by several authors (see for example [1,2,3,4]). In

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this paper, we consider a class of generalized harmonic numbers $H_{n,r}$. The definition of $H_{n,r}$ ([3]) is

$H_{n,0} = 1$ and $H_{n,r} = \sum_{1\leq n_1<\cdots<n_r\leq n} \frac{1}{n_1 n_2 \cdots n_r}$ *for* $n, r \geq 1$.

The generating funtion of $H_{n,r}$ is

$$\sum_{n=0}^{\infty} \frac{(r+1)!}{n+1} H_{n,r} t^n = \frac{(-\ln(1-t))^{r+1}}{t}. \tag{1.1}$$

From (1.1) we have

$$[t^n] \frac{(-\ln(1-t)^r}{1-t} = r! H_{n,r}. \tag{1.2}$$

The concept of Riordan arrays has been introduced in [5]. A Riordan array is an infinite, lower triangular array $D = (d_{n,k})_{n,k\in N}$ defined by a couple of formal power series: $D = \Re(d_{n,k}) = (g(t), f(t))$, such that

$$d_{n,k} = [t^n] g(t)(tf(t))^k, \quad \forall n \in \mathbb{N}. \tag{1.3}$$

Basically, the concept of a Riordan array is used in a constructive way to find the generating funtion of many combinatorial sums. For any sequence h_k having $h(x)$ as its generating funtion, we have

$$\sum_{k=0}^{n} d_{n,k} h_k = [t^n] g(t) h(tf(t)). \tag{1.4}$$

The paper is organized as follows. In Section 2, we investigate the properties of $H_{n,r}$. In Section 3, we obtain some identities for harmonic polynomials $H_n(z)$ and several polynomials by means of Riordan array. In Section 4, we give the asymptotic expansion of certain sums related to $H_{n,r}$ and α-Cauchy numbers (generalized Lah numbers and binomials coefficients), where r is fixed.

For convenience, we recall some definitions involved in the paper. The generalized Stirling numbers of the second kind $S(n, k; r)$ have the following exponential generating function:

$$\sum_{n\geq k} S(n, k; r) \frac{t^n}{n!} = e^{rt} \frac{(e^t - 1)^k}{k!}. \tag{1.5}$$

The generalized harmonic numbers $H_{n,r}$ may be related to the Bernoulli polynomials $B_n(z)$ and generalized Bernoulli polynomials $B_n^{(\alpha)}(z)$, which are defined by:

$$\sum_{n=0}^{\infty} B_n(z) \frac{t^n}{n!} = \frac{te^{zt}}{e^t - 1}, \tag{1.6}$$

$$\sum_{n=0}^{\infty} B_n^{(\alpha)}(z)\frac{t^n}{n!} = (\frac{t}{e^t-1})^{\alpha} e^{zt}. \tag{1.7}$$

The generalized Lah numbers $L(n,k;r)$ are given by:

$$\sum_{n\geq k} L(n,k;r)\frac{t^n}{n!} = (1+t)^r \frac{(\frac{-t}{1+t})^k}{k!}. \tag{1.8}$$

2. Some identities involving $H_{n,r}$

Let us define $\mathcal{H} = [H_{n,r}]_{n\geq r\geq 0}$ to be an infinite lower triangular array. It is easy to show that $\mathcal{H}$ does not constitute a Riordan array but

$$\bar{\mathcal{H}} = \Re(r!H_{n,r}) = (\frac{1}{1-t}, \frac{-\ln(1-t)}{t}) \tag{2.1}$$

is a Riordan array. Thus we obtain

$$H_{n,r} = \frac{1}{r!}[t^n]\frac{(-\ln(1-t))^r}{1-t}.$$

From the generating function of $H_{n,r}$ and generalized Stirling numbers of the second kind $S(n,k;r)$, we have

Theorem 2.1 Let $r \geq 0$, $n \geq 0$ and $k \geq 0$, then

$$\sum_{h=0}^{n} n!\binom{n}{k}^{-1} H_{n,h}S(h,k;r) = (n+r)_{n-k},$$

where $(n)_k = n(n-1)\cdots(n-k+1)$ is the falling factorial.

Proof. From (1.3), (1.5) we have

$$\begin{aligned}
&\sum_{h=0}^{n} h!H_{n,h}\frac{k!}{h!}S(h,k;r) \\
&= [t^n]\frac{1}{1-t}[e^{ry}(e^y-1)^k|y=-\ln(1-t)] \\
&= [t^n]\frac{1}{1-t}\frac{1}{(1-t)^r}(\frac{t}{1-t})^k \\
&= [t^{n-k}]\frac{1}{(1-t)^{r+k+1}} \\
&= \binom{n+r}{n-k}.
\end{aligned}$$

Which completes the proof. □

Theorem 2.2 Let $B_n(z)$ be Bernoulli polynomials given by (1.6), then

$$\sum_{k=0}^{n}\sum_{r=0}^{k} H_{k,r}B_r(z)\frac{(-1)^{n-k}c_{n-k}^{z}}{(n-k)!} = \delta_{n,0}.$$

Proof. From (1.3), (1.6) we have

$$\sum_{n\geq 0}\sum_{r=0}^{n} H_{n,r}B_r(z)t^n = \frac{-\ln(1-t)}{t(1-t)^z}. \tag{2.2}$$

For the α-Cauchy numbers of the first kind c_n^α have the following exponential generating function:

$$\sum_{n\geq 0}\frac{c_n^\alpha}{n!}t^n = \frac{t(1+t)^\alpha}{\ln(1+t)}.$$

Furthermore we can get

$$\sum_{n\geq 0}\frac{(-1)^n c_n^\alpha}{n!}t^n = \frac{t(1-t)^\alpha}{-\ln(1-t)}. \tag{2.3}$$

Then (2.3), (2.4) we have

$$\begin{aligned}
&\sum_{k=0}^{n}\sum_{r=0}^{k} H_{k,r}B_r(z)\frac{(-1)^{n-k}c_{n-k}^{z}}{(n-k)!}\\
&= \sum_{k=0}^{n}[t^k]\frac{-\ln(1-t)}{t(1-t)^z}[t^{n-k}]\frac{-t(1-t)^z}{-\ln(1-t)}\\
&= [t^n]\frac{-\ln(1-t)}{t(1-t)^z}\frac{-t(1-t)^z}{-\ln(1-t)} = \delta_{n,0}.
\end{aligned}$$

Which completes the proof. □

For generalized Bernoulli polynomials $B_n^{(\alpha)}(z)$ have the following identity.

Theorem 2.3 Let $B_n^{(\alpha)}(z)$ be generalized Bernoulli polynomials given by (1.7), then

$$\sum_{r=0}^{n} H_{n,r}B_r^{(z)}(z) = H_{n+z,z}z!,$$

$$\sum_{r=0}^{n} r!H_{n,r}S_r(z) = H_{n+z,z}(z-1)!,$$

where $S_n(z)$ has been defined by the generating function[1]

$$z\sum_{n=0}^{\infty} S_n(z)t^n = (\frac{te^t}{e^t-1})^z. \tag{2.4}$$

Proof. Firstly, from (1.3), (1.7) we have

$$\begin{aligned}
&\sum_{r=0}^{n} H_{n,r}B_r^{(z)}(z) = \sum_{r=0}^{n} r!H_{n,r}\frac{B_r^{(z)}(z)}{r!} \\
&= [t^n]\frac{1}{1-t}[(\frac{y}{e^y-1})^z \cdot e^{zy}|y=-\ln(1-t)] \\
&= [t^n]\frac{1}{1-t}\frac{(-\ln(1-t))^z(1-t)^z}{t^z}\frac{1}{(1-t)^z} \\
&= [t^{n+z}]\frac{(-\ln(1-t))^z}{1-t} = H_{n+z,z}z!.
\end{aligned}$$

Secondly, from (1.3) and (2.5) we have

$$\begin{aligned}
&z\sum_{r=0}^{n} r!H_{n,r}S_r(z) \\
&= [t^n]\frac{1}{1-t}[(\frac{ye^y}{e^y-1})^z|y=-\ln(1-t)] \\
&= [t^n]\frac{1}{1-t}\frac{(-\ln(1-t))^z}{(1-t)^z}\frac{(1-t)^z}{t^z} \\
&= [t^{n+z}]\frac{(-\ln(1-t))^z}{1-t} = H_{n+z,z}z!.
\end{aligned}$$

Which completes the proof. □

Theorem 2.4 The generalized harmonic numbers $H_{n,r}$ may be expressed by means of the harmonic numbers H_n:

$$rH_{n,r} = \sum_{k=1}^{n} H_k \cdot \frac{B_{n-k,r-1}(1,1,2!,3!,\cdots)}{(n-k)!},$$

where the exponential Bell polynomials $B_{n,k}$ are defined by:

$$\sum_{n\geq k} B_{n,k}\frac{t^n}{n!} = \frac{1}{k!}(\sum_{m\geq 1} x_m\frac{t^m}{m!})^k, \quad k=0,1,2,\cdots.$$

Proof. From the generating function of (1.1) we have

$$\begin{aligned}
\sum_{n=0}^{\infty} r!H_{n,r}t^n &= \frac{(-\ln(1-t))^r}{1-t} \\
&= \frac{-\ln(1-t)}{1-t}(-\ln(1-t))^{r-1} \\
&= \sum_{n=1}^{\infty} H_n t^n \cdot (\sum_{n=1}^{\infty} \frac{t^n}{n})^{r-1} \\
&= \sum_{n=1}^{\infty} H_n t^n \cdot (\sum_{n=1}^{\infty} (n-1)!\frac{t^n}{n!})^{r-1} \\
&= \sum_{n=1}^{\infty} H_n t^n \cdot (\sum_{n=1}^{\infty} (r-1)!B_{n,r-1}(1,1,2!,3!,\cdots)\frac{t^n}{n!}) \\
&= \sum_{n\geq 1}(\sum_{k=1}^{n} H_k \cdot \frac{(r-1)!B_{n-k,r-1}(1,1,2!,3!,\cdots)}{(n-k)!})t^n .
\end{aligned}$$

Comparing the coefficients of t^n on both sides of, we obtain

$$rH_{n,r} = \sum_{k=1}^{n} H_k \cdot \frac{B_{n-k,r-1}(1,1,2!,3!,\cdots)}{(n-k)!}.$$

□

3. Harmonic polynomials of degree n

In this section, we develop the polynomials generating the harmonic numbers H_n with the generating function $\frac{-\ln(1-t)}{1-t}$.

For $n \in N_0$, let us define the polynomials $H_n(z)$ in $z \in \mathbb{C}$ of degree n by

$$\sum_{n=0}^{\infty} H_n(z)t^n = \frac{-\ln(1-t)}{t(1-t)^{1-z}}, \tag{3.1}$$

with the alternative representation

$$H_n(z+1) = \sum_{r=0}^{n} \frac{r+1}{n+1}(-1)^r H_{n,r} z^r . \tag{3.2}$$

Theorem 3.1 The harmonic polynomials $H_n(z)$ may be expressed by means of the associated Stirling cycle numbers:

$$H_n(z) = \sum_{k=0}^{n} \frac{k+1}{(n+1)!} \left[\begin{matrix} n+1 \\ k+1 \end{matrix} \right] (1-z)^k,$$

where the associated Stirling cycle number array

$$\Re\left(\frac{(k+1)!}{(n+1)!} \left[\begin{matrix} n+1 \\ k+1 \end{matrix} \right]\right) = \left(\frac{-\ln(1-t)}{t}, \frac{-\ln(1-t)}{t}\right). \tag{3.3}$$

Proof. Let $h(t) = e^{(1-z)t} = \sum_{n\geq 0} \frac{(1-z)^n t^n}{n!}$, from (3.3), (1.1), (1.3) we have

$$\begin{aligned}
&\sum_{k=0}^{n} \frac{k+1}{(n+1)!} \left[\begin{matrix} n+1 \\ k+1 \end{matrix} \right] (1-z)^k \\
&= \sum_{k=0}^{n} \frac{(k+1)!}{(n+1)!} \left[\begin{matrix} n+1 \\ k+1 \end{matrix} \right] \frac{(1-z)^k}{k!} \\
&= [t^n] \frac{-\ln(1-t)}{t} [e^{(1-z)y} | y = -\ln(1-t)] \\
&= [t^n] \frac{-\ln(1-t)}{t} \cdot \frac{1}{(1-t)^{1-z}} = H_n(z) .
\end{aligned}$$

Which completes the proof. □

Let $z = 0$ in the Theorem 3.1, we have the following corollary.

Corollary 3.1 For the harmonic numbers H_n, we have

$$H_{n+1} = \sum_{k=0}^{n} \frac{k+1}{(n+1)!} \left[\begin{matrix} n+1 \\ k+1 \end{matrix} \right].$$

Let $P_{n,k}(z)$ are the generalized Stirling polynomials of the first kind defined by:

$$P_{n,k}(z) = \sum_{j=k+1}^{n} (-z)^{j-k-1} \binom{j-1}{k} \left[\begin{matrix} n \\ j \end{matrix} \right], \tag{3.4}$$

with the alternative representation:

$$P_{n,k}(z) = \frac{(-1)^k}{k!} \left(\frac{\partial^{n-1}}{\partial t^{n-1}} \frac{[\ln(1-t)]^k}{(1-t)^{1-z}} |_{t=0}\right).$$

Now, let us define the infinite lower triangular matrice $P(z)$ by:

$$P(z) = [\frac{k!}{n!} P_{n+1,k}(z)]_{n,k\in N_0}.$$

It is easy to show that $P(z)$ may be expressed by the Riordan array:

$$P(z) = (\frac{1}{(1-t)^{1-z}}, \frac{-\ln(1-t)}{t}). \tag{3.5}$$

Theorem 3.2 Let $H_n(z)$ be harmonic polynomials given by (3.1).Then

$$H_n(z+1) = \sum_{k=0}^{n} \frac{P_{n+1,k}(z)B_k}{n!},$$

where B_n is a Bernoulli number, $P_{n,k}(z)$ are the generalized stirling polynomials of the first kind.

Proof. From (1.3) and (3.5) we have

$$\begin{aligned}
&\sum_{k=0}^{n} \frac{P_{n+1,k}(z)B_k}{n!} \\
&= \sum_{k=0}^{n} \frac{k!P_{n+1,k}(z)B_k}{n!k!} \\
&= [t^n]\frac{1}{(1-t)^{1-z}}[\frac{y}{e^y-1}|y = -\ln(1-t)] \\
&= [t^n]\frac{1}{(1-t)^{1-z}} \cdot \frac{-\ln(1-t)}{\frac{1}{1-t}-1} \\
&= [t^n]\frac{-\ln(1-t)}{t(1-t)^{-z}} = H_n(z+1).
\end{aligned}$$

From (3.4) we can obtain

$$H_n(z+1) = \sum_{k=0}^{n}\sum_{j=k+1}^{n+1} (-z)^{j-k-1}\binom{j-1}{k}\left[{n \atop j} \right]\frac{P_{n+1,k}(z)B_k}{n!}.$$

□

We list more new identities for the harmonic polynomials $H_n(z)$ with related to some other polynomials.

Theorem 3.3 The harmonic polynomials $H_n(z)$ may be expressed by means of the Bernoulli polynomials

$$H_n(1-z) = \sum_{r=0}^{n} H_{n,r}B_r(z),$$

where $H_{n,r}$ be generalized harmonic numbers given by (1.2).

Proof. From (1.2), (1.3) and (1.6) we have

$$\begin{aligned}\sum_{r=0}^{n} H_{n,r}B_r(z) &= \sum_{r=0}^{n} r!H_{n,r}\frac{B_r(z)}{r!} \\ &= [t^n]\frac{1}{1-t}[\frac{ye^{yz}}{e^y-1}|y=-\ln(1-t)] \\ &= [t^n]\frac{1}{1-t}\frac{-\ln(1-t)}{t}\frac{1-t}{(1-t)^z} \\ &= [t^n]\frac{-\ln(1-t)^z}{t(1-t} = H_n(1-z).\end{aligned}$$

Which completes the proof.

□

Theorem 3.4 The harmonic polynomials $H_n(z)$ may be expressed by means of the generalized Stirling polynomials of the first kind

$$H_{n-1}(z+1) = \sum_{k=1}^{n} \frac{kP_{n+1,k}(z)}{n!}.$$

Proof. Let $h(t) = te^t = \sum_{n\geq 1} \frac{t^n}{(n-1)!}$, and from (1.3), (3.5) we have

$$\begin{aligned}\sum_{k=1}^{n} \frac{k!P_{n+1,k}(z)}{n!}\frac{1}{(k-1)!} &= [t^n]\frac{1}{(1-t)^{1-z}}[ye^y|y=-\ln(1-t)] \\ &= [t^n]\frac{1}{(1-t)^{1-z}}\frac{-\ln(1-t)}{1-t} \\ &= [t^{n-1}]\frac{-\ln(1-t)}{t(1-t)^{-z}} = H_{n-1}(z+1).\end{aligned}$$

Which completes the proof. □

Let us consider the Cauchy polynomials of degree n of the second type $\phi_n^{(2)}(z)$ defined by $\phi_n^{(2)}(z) = \int_0^1 [x-z]_n dx$. and the exponential generating function of $\phi_n^{(2)}(z)$ is

$$\sum_{n=0}^{\infty} \phi_n^{(2)}(z)\frac{t^n}{n!} = \frac{-t}{(1-t)^{1-z}\ln(1-t)}. \tag{3.6}$$

we may define the Cauchy polynomials of degree of n of the first type $\phi_n^{(1)}(z)$ by $\phi_n^{(1)}(z) = \int_0^1 (x-z)_n dx$. and the exponential generating function of $\phi_n^{(1)}(z)$ is

$$\sum_{n=0}^{\infty} \phi_n^{(1)}(z) \frac{t^n}{n!} = \frac{t}{(1+t)^z \ln(1+t)}. \qquad (3.7)$$

In this section, we show that the harmonic polynomials are closely related to the Cauchy polynomials.

Theorem 3.5 The Cauchy polynomials of both types, $\phi_n^{(1)}(z)$ and $\phi_n^{(2)}(z)$, are connected in the following way:

$$\sum_{k=0}^{n} (-1)^n L(n,k;r) \phi_k^{(1)}(z) = \phi_n^{(2)}(r+z). \qquad (3.8)$$

Proof. From (1.9), we note that the generalized Lah numbers array $(L(n,k;r))_{n,k\in N}$ may be expressed by:

$$\Re(L(n,k;r)) = ((1+t)^r, \frac{-1}{1+t})$$

and from (1.3), (3.6), (3.7) we have

$$\begin{aligned}
&\sum_{k=0}^{n} L(n,k;r) \phi_k^{(1)}(z) \\
&= n! \sum_{k=0}^{n} \frac{k!}{n!} L(n,k;r) \frac{\phi_k^{(1)}(z)}{k!} \\
&= n!\,[t^n]\,(1+t)^r [\frac{y}{(1+y)^z \ln(1+y)} | y = \frac{-t}{1+t}] \\
&= n![t^n] \frac{t}{\ln(1+t)(1+t)^{1-r-z}} \\
&= (-1)^n n! [t^n] \frac{-t}{\ln(1-t)(1-t)^{1-r-z}} = (-1)^n \phi_n^{(2)}(r+z).
\end{aligned}$$

Which completes the proof.

□

Let $z = 0$ in the Theorem 3.5, we have the following corollary.

Corollary 3.2

$$\sum_{k=0}^{n} (-1)^n L(n,k;r) c_k = \phi_n^{(2)}(r),$$

where c_n is the Cauchy numbers of the first kind.

By means of the method of coefficients we have the following identities:

Theorem 3.6 Let $H_n(z)$ be the harmonic polynomials, then

$$\sum_{k=0}^{n}\binom{n}{k}k!H_k(z)\phi_{n-k}^{(2)}(z)=(n-2z+1)_n,$$

$$\sum_{k=0}^{n}\frac{(-1)^{n-k}H_k(z)\phi_{n-k}^{(1)}(z)}{(n-k)!}=1.$$

Proof. From (1.1), (3.6) and (3.7) we have

$$\begin{aligned}
&\sum_{k=0}^{n}H_k(z)\frac{\phi_{n-k}^{(2)}(z)}{(n-k)!}\\
&=\sum_{k=0}^{n}([t^k]\frac{-\ln(1-t)}{t(1-t)^{1-z}})[t^{n-k}]\frac{-t}{(1-t)^{1-z}\ln(1-t)}\\
&=[t^n]\frac{-\ln(1-t)}{t(1-t)^{1-z}}\frac{-t}{(1-t)^{1-z}\ln(1-t)}\\
&=[t^n]\frac{1}{(1-t)^{2-2z}}=\binom{n-2z+1}{n}.
\end{aligned}$$

which completes the proof.By the same method we can prove(*ii*). □

Let $z=0$ in the Theorem 3.6, we obtain the following corollary.

Corollary 3.3

$$\sum_{k=0}^{n}\binom{n}{k}k!H_{k+1}\hat{c}_{n-k}=(n+1)!,$$

$$\sum_{k=0}^{n}\frac{(-1)^{n-k}H_{k+1}c_{n-k}}{(n-k)!}=1,$$

where $c_n, \hat{c}_n$ are the Cauchy numbers of both kinds.

4. Asymptotic value involving $H_{n,r}$

In this section, we give asymptotic expansions of certain sums involving $H_{n,r}$. α-Cauchy numbers of the first kind are defined by:

$$\sum_{n=0}^{\infty}c_n^{\alpha}t^n=\frac{t(1+t)^{\alpha}}{\ln(1+t)}.$$

In the following, we give the asymptotic value of certain sums related to $H_{n,r}$ and α-Cauchy numbers of the first kind.At first, we recall a lemma.

Lemma([7]) Let α be a real number and

$$L(z) = \ln\frac{1}{1-z}.$$

When $n \to \infty$,
$[z^n](1-z)^\alpha L^k(z) \sim \frac{1}{\Gamma(-\alpha)} n^{-\alpha-1}\ln^k n, \quad (\alpha \not\subseteq \{0,1,2,\cdots\})$
$[z^n](1-z)^m L^k \sim (-1)^m km! n^{-m-1}\ln^{k-1} n, \quad (m \in \mathbb{Z}_{\geq 0}, k \in \mathbb{Z}_{\geq 1})$.

Now we give the asymtotic expansions of certain sums involving $H_{n,r}$ using above lemma.

Theorem 4.1 Let α be a real number and $\alpha \not\subseteq \{0,1,2\cdots\}$, we have

$$\sum_{k=0}^{n} r! H_{k,r} \frac{(-1)^{n-k} c_{n-k}^\alpha}{(n-k)!} \sim \frac{1}{\Gamma(-\alpha)}(n-1)^{-\alpha}\ln^{r-1}(n-1).$$

Proof. From (1.2), (2.3) we have

$$\begin{aligned}
&\sum_{k=0}^{n} r! H_{n,r} \frac{(-1)^{n-k} c_{n-k}^\alpha}{(n-k)!} \\
&= \sum_{k=0}^{n} ([t^k]\frac{(-\ln(1-t))^r}{1-t})[t^{n-k}]\frac{-t(1-t)^\alpha}{\ln(1-t)} \\
&= [t^n]\frac{(-\ln(1-t))^r}{1-t}\frac{-t(1-t)^\alpha}{\ln(1-t)} \\
&= [t^{n-1}](-\ln(1-t))^{r-1}(1-t)^{\alpha-1}.
\end{aligned}$$

Application of the above Lemma([7]) ,we have

$$\sum_{k=0}^{n} r! H_{k,r} \frac{(-1)^{n-k} c_{n-k}^\alpha}{(n-k)!} \sim \frac{1}{\Gamma(-\alpha)}(n-1)^{-\alpha}\ln^{r-1}(n-1).$$

□

Let $\alpha = 0$ in the Theorem 4.1, we have the following corollary.

Corollary 4.1 $\sum_{k=0}^{n} r! H_{k,r} \frac{(-1)^{n-k} c_{n-k}}{(n-k)!} \sim \ln^{r-1}(n-1)$.

Theorem 4.2 Let $r \in Z, h \geq 1$ and $h \in Z$, then

$$\sum_{k=1}^{n} \frac{k! L(n,k;r)}{n!}(-1)^{n-k} h! H_{k,h} \sim (-1)^{r+h+1} h(r+1)! n^{-r}\ln^{h-1} n.$$

Proof. It follows from (1.2), (1.3), (1.8) and Lemma, we have

$$\sum_{k=1}^{n} \frac{k!L(n,k;r)}{n!}(-1)^{n-k}h!H_{k,h}$$
$$= [t^n](1-t)^{r+1}(\ln\frac{1}{1-t})^h(-1)^h \sim (-1)^{r+h+1}h(r+1)!n^r \ln^{h-1} n.$$

□

Let $r = 0$ in the Theorem 4.2, we have the following corollary.

Corollary 4.2 $\sum_{k=1}^{n} \frac{k!L_{n,k}}{n!}(-1)^{n-k}h!H_{k,h} \sim (-1)^{h+1}h\ln^{h-1} n.$

For the generalized combinatorial coefficients $\binom{n+ak}{m+bk}$, we have the following Riordan array :$\Re(\binom{n+ak}{m+bk}) = (\frac{t^m}{(1-t)^{m+1}}, \frac{t^{b-a}}{t(1-t)^b})$

Theorem 4.3 For combinatorial coefficients we have:

$$\sum_{j=0}^{n}\binom{j+ak}{m+bk}r!H_{n-j,r} \sim \frac{\ln^r(n-m-k(b-a))}{\Gamma(m+kb+2)}(n-m-k(b-a))^{m+bk+1}.$$

Proof.

$$\sum_{j=0}^{n}\binom{j+ak}{m+bk}r!H_{n-j,r} = \sum_{j=0}^{n}([t^j]\frac{t^m}{(1-t)^{m+1}}(\frac{t^{b-a}}{(1-t)^b})^k)[t^{n-j}]\frac{(-\ln(1-t))^r}{1-t}$$
$$= [t^n]\frac{t^m}{(1-t)^{m+1}}(\frac{t^{b-a}}{(1-t)^b})^k\frac{(-\ln(1-t))^r}{1-t}$$
$$= [t^{n-m-k(b-a)}]\frac{(-\ln(1-t))^r}{(1-t)^{m+kb+2}}$$
$$\sim \frac{\ln^r(n-m-k(b-a))}{\Gamma(m+kb+2)}(n-m-k(b-a))^{m+bk+1}.$$

□

Let $a = b = 1, m = 0$ in the Theorem 4.3, we have the following corollary.

Corollary 4.3 When $a = b = 1, m = 0$ we have

$$\sum_{j=0}^{n}\binom{j+k}{k}r!H_{n-j,r} \sim \frac{n^{k+1}\ln^r n}{(k+1)!}.$$

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