A complete solution to the chromatic equivalence class of graph $\overline{\zeta_n^1}$ *

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Abstract

Two graphs are defined to be adjointly equivalent if their complements are chromatically equivalent. By $h(G,x)$ and $P(G,\lambda)$ we denote the adjoint polynomial and the chromatic polynomial of graph G, respectively. A new invariant of graph G, which is the fifth character $R_5(G)$, is given in this paper. Using this invariant and the properties of the adjoint polynomials, we firstly and completely determine the adjoint equivalence class of the graph ζ_n^1. According to the relations between $h(G,x)$ and $P(G,\lambda)$, we also simultaneously determine the chromatic equivalence class of $\overline{\zeta_n^1}$.

Keywords: chromatic equivalence class, adjoint polynomial, the smallest real root, the fifth character.
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1 Introduction

The graphs considered in this paper are finite undirected and simple graphs. We follow the notation of Bondy and Murty[1], unless otherwise stated. For a graph G, let $V(G)$, $E(G)$, $p(G)$, $q(G)$ and $\overline{G}$, respectively, be the set of vertices, the set of edges, the order, the size and the complement of G.

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For a graph G, we denote by $P(G, \lambda)$ the chromatic polynomial of G. A partition $\{A_1, A_2, \cdots, A_r\}$ of $V(G)$, where r is a positive integer, is called an *r-independent partition* of graph G if every A_i is nonempty independent set of G. We denote by $\alpha(G, r)$ the number of r-independent partitions of G. Thus the chromatic polynomial G is $P(G, \lambda) = \sum_{r \geq 1} \alpha(G, r)(\lambda)_r$, where $(\lambda)_r = \lambda(\lambda - 1) \cdots (\lambda - r + 1)$ for all $r \geq 1$. The readers can turn to [19] for details on chromatic polynomials.

Two graphs G and H are said to be *chromatically equivalent*, denoted by $G \sim H$, if $P(G, \lambda) = P(H, \lambda)$. By $[G]$ we denote the equivalence class determined by G under "$\sim$". It is obvious that "$\sim$" is an equivalence relation on the family of all graphs. A graph G is called chromatically unique (or simply $\chi - unique$) if $H \cong G$ whenever $H \sim G$. See [4, 5] for many results on this field.

Definition 1.1. *[7] Let G be a graph with p vertices, the polynomial*

$$h(G, x) = \sum_{i=1}^{p} \alpha(\overline{G}, i)x^i$$

is called its adjoint polynomial.

Definition 1.2. *[7] Let G be a graph and $h_1(G, x)$ be the polynomial with a nonzero constant term such that $h(G, x) = x^{\rho(G)} h_1(G, x)$. If $h_1(G, x)$ is an irreducible polynomial over the rational number field, then G is called* irreducible graph.

Two graphs G and H are said to be *adjointly equivalent*, denoted by $G \sim^h H$, if $h(G, x) = h(H, x)$. Evidently, "$\sim^h$" is an equivalence relation on the family of all graphs. Let $[G]_h = \{H | H \sim^h G\}$. A graph G is said to be *adjointly unique*(or simply h-unique) if $G \cong H$ whenever $G \sim^h H$.

Theorem 1.1. *[3]* (1) *$G \sim^h H$ if and only if $\overline{G} \sim \overline{H}$.*
(2) $[G]_h = \{H | \overline{H} \in [\overline{G}]\}$.
(3) *G is χ-unique if and only if $\overline{G}$ is h-unique.*

Now we define some classes of graphs with order n, which will be used throughout the paper.

(1) C_n(resp. P_n) denotes the cycle (resp. the path) of order n, and write $\mathcal{C} = \{C_n | n \geq 3\}$,$\mathcal{P} = \{P_n | n \geq 2\}$ and $\mathcal{U} = \{U_{1,1,t,1,1} | t \geq 1\}$.

(2) $D_n (n \geq 4)$ denotes the graph obtained from C_3 and P_{n-2} by identifying a vertex of C_3 with a pendent vertex of P_{n-2}.

(3) T_{l_1,l_2,l_3} is a tree with a vertex v of degree 3 such that $T_{l_1,l_2,l_3} - v = P_{l_1} \cup P_{l_2} \cup P_{l_3}$ and $l_3 \geq l_2 \geq l_1$, write $\mathcal{T}_0 = \{T_{1,1,l_3} | l_3 \geq 1\}$ and $\mathcal{T} = \{T_{l_1,l_2,l_3} | (l_1, l_2, l_3) \neq (1, 1, 1)\}$.

(4) $\vartheta = \{C_n, D_n, K_1, T_{l_1,l_2,l_3} | n \geq 4\}$.

(5) $\xi = \{C_r(P_s), Q(r, s), B_{r,s,t}, F_n, U_{r,s,t,a,b}, K_4^-\}$.

(6) $\psi = \{\psi_n^1, \psi_n^2, \psi_n^3(r,s), \psi_n^4(r,s), \psi_n^5(r,s,t), \psi_5^6\}$.

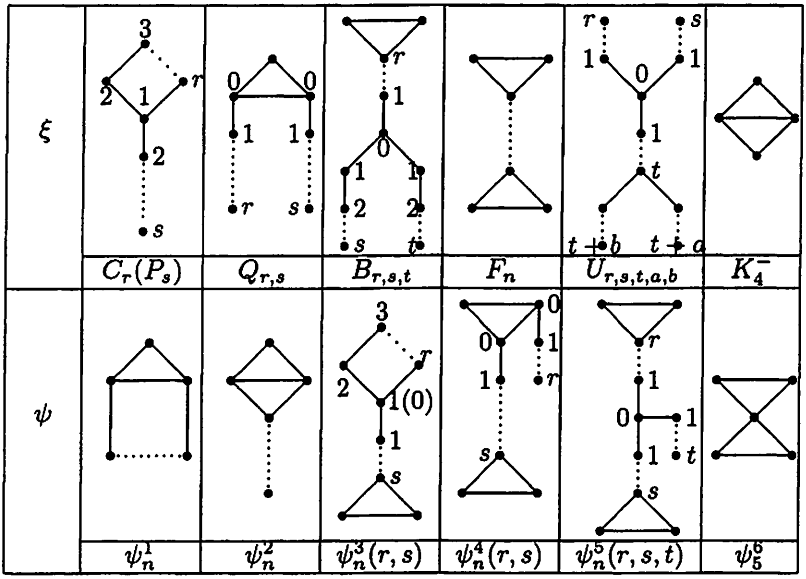


Figure 1 Families of ξ and ψ

(7) $\zeta = \{\zeta_n^1, \zeta_n^2(r,s), \zeta_n^3(r,s,t)\}$.

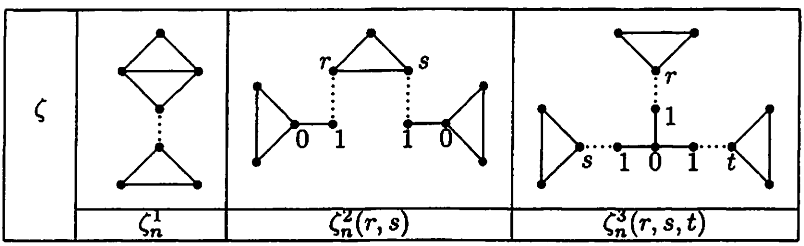


Figure 2 Family of ζ

For convenience, we simply denote $h(G,x)$ by $h(G)$ and $h_1(G,x)$ by $h_1(G)$. By $\beta(G)$ and $\beta_{min}(G)$ we denote the smallest real root and the minimal extremes of the smallest real root of $h(G)$, respectively. Let $d_G(v)$, simply denoted by $d(v)$, be the degree of vertex v. For two graphs G and H, $G \cup H$ denotes the disjoint union of G and H, and mH stands for the disjoint union of m copies. By K_n we denote the complete graph with order n, let $n_G(K_3)$ and $n_G(K_4)$ denote the number of subgraphs isomorphic to K_3 and K_4 in G, respectively. Let $g(x)|f(x)$(resp. $g(x) \nmid f(x)$) denote $g(x)$ divides $f(x)$(resp. $g(x)$ does not divide $f(x)$) and $\partial(f(x))$ denote the degree of $f(x)$. By $(f(x), g(x))$ we denote the largest common factor of $f(x)$ and $g(x)$ on the real field. Let $N_G(v)$ be the neighborhood set of a vertex v.

It is an interesting problem to determine $[G]$ for a given graph G. From Theorem 1.1, it is not difficult to see that the goal of determining $[G]$ can be realized by determining $[\overline{G}]_h$. The determination of $[G]$ for a given graph G has received much attention in [14, 21, 22, 23] recently. In this paper, using the properties of adjoint polynomials, we determine the $[\zeta_n^1]_h$ of graph ζ_n^1, simultaneously, $[\overline{\zeta_n^1}]$ is also determined, where $n \geq 7$.

2 Preliminaries

For a polynomial $f(x) = x^n + b_1x^{n-1} + b_2x^{n-2} + \cdots + b_n$, we define

$$R_1(f(x)) = \begin{cases} -\binom{b_1}{2} + 1, & if\ n=1. \\ b_2 - \binom{b_1-1}{2} + 1, & if\ n \geq 2. \end{cases}$$

For a graph G, we write $R_1(G)$ instead of $R_1(h(G))$.

Definition 2.1. *[2, 7] Let G be a graph with q edges.*

(1) *The first character of a graph G is defined as*

$$R_1(G) = \begin{cases} 0, & if\ q = 0. \\ b_2 - \binom{b_1-1}{2} + 1, & if\ q > 0. \end{cases}$$

(2) *The second character of a graph G is defined as*

$$R_2(G) = b_3(G) - \binom{b_1(G)}{3} - (b_1(G) - 2)\left(b_2(G) - \binom{b_1(G)}{2}\right) - b_1(G),$$

where $b_i(G)(0 \leq i \leq 3)$ is the first four coefficients of $h(G)$.

Lemma 2.1. *[2, 7] Let G be a graph with k components of $G_1, G_2, \cdots, G_k$. Then $h(G) = \prod_{i=1}^k h(G_i)$ and $R_j(G) = \sum_{i=1}^k R_j(G_i)$ for $j = 1, 2$.*

It is obvious that $R_j(G)$ is an invariant of graphs. So, for any two graphs G and H, we have $R_j(G) = R_j(H)$ for $j = 1, 2$ if $h(G) = h(H)$ or $h_1(G) = h_1(H)$.

Lemma 2.2. *[7, 8] Let G be a graph with p vertices and q edges. Denote M the set of the triangles in G and by $M(i)$ the number of triangles which cover the vertex i in G. If the degree sequence of G is $(d_1, d_2, \cdots, d_p)$, then the first four coefficients of $h(G)$ are, respectively,*

(1) $b_0(G) = 1$, $b_1(G) = q$.

(2) $b_2(G) = \binom{q+1}{2} - \frac{1}{2}\sum_{i=1}^{p} d_i^2 + n_G(K_3)$.

(3) $b_3(G) = \frac{q}{6}(q^2+3q+4) - \frac{q+2}{2}\sum_{i=1}^{p} d_i^2 + \frac{1}{3}\sum_{i=1}^{p} d_i^3 - \sum_{ij\in E(G)} d_i d_j - \sum_{i\in M} M(i)d_i + (q+2)n_G(K_3) + n_G(K_4)$, *where* $b_i(G) = \alpha(\overline{G}, p-i)(i = 0, 1, 2, 3)$.

For an edge $e = v_1v_2$ of a graph G, the graph $G * e$ is defined as follow: the vertex set of $G * e$ is $(V(G) - \{v_1, v_2\}) \bigcup v(v \notin G)$, and the edge set of $G * e$ is $\{e' | e' \in E(G), e'$ is not incident with v_1 or $v_2\} \cup \{uv | u \in N_G(v_1) \cap N_G(v_2)\}$, where $N_G(v)$ is the set of vertices of G which are adjacent to v.

Lemma 2.3. *[7] Let G be a graph with $e \in E(G)$. Then*

$$h(G, x) = h(G - e, x) + h(G * e, x),$$

where $G - e$ denotes the graph obtained by deleting the edge e from G.

Lemma 2.4. *[7]* (1) *For $n \geq 2$, $h(P_n) = \sum_{k\leq n} \binom{k}{n-k} x^k$.*

(2) *For $n \geq 4$, $h(D_n) = \sum_{k\leq n}(\frac{n}{k}\binom{k}{n-k} + \binom{k-2}{n-k-3})x^k$.*

(3) *For $n \geq 4$, $m \geq 6$, $h(P_n) = x(h(P_{n-1}) + h(P_{n-2}))$, $h(D_m) = x(h(D_{m-1}) + h(D_{m-2}))$.*

Lemma 2.5. *[20] Let $\{g_i(x)\}$, simply denoted by $\{g_i\}$, be a polynomial sequence with integer coefficients and $g_n(x) = x(g_{n-1}(x) + g_{n-2}(x))$. Then*

(1) *$g_n(x) = h(P_k)g_{n-k}(x) + xh(P_{k-1})g_{n-k-1}(x)$.*

(2) *$h_1(P_n) | g_{k(n+1)+i}(x)$ if and only if $h_1(P_n) | g_i(x)$, where $0 \leq i \leq n$, $n \geq 2$ and $k \geq 1$.*

Lemma 2.6. *[6, 10] Let G be a nontrivial connected graph with n vertices. Then*

(1) *$R_1(G) \leq 1$, and the equality holds if and only if $G \cong P_n(n \geq 2)$ or $G \cong K_3$.*

(2) *$R_1(G) = 0$ if and only if $G \in \vartheta$.*

(3) *$R_1(G) = -1$ if and only if $G \in \xi$, especially, $q(G) = p(G) + 1$ if and only if $G \in \{F_n | n \geq 6\} \cup \{K_4^-\}$.*

(4) *$R_1(G) = -2$ if and only if $G \in \varphi$(see Figure 3) for $q(G) = p(G)$, $G \in \psi$ for $q(G) = p(G) + 1$ and $G \cong K_4$ for $q(G) = p(G) + 2$.*

(5) *$R_1(G) = -3$ if and only if $G \in \phi$ (see Figure 5) for $q(G) = p(G) + 1$ and $G \in \zeta$ for $q(G) = p(G) + 2$.*

(6) *$R_1(G) = -4$ if and only if $G \in \theta$ (see Figure 4) for $q(G) = p(G) + 2$.*

(7) *$R_1(G) = -5$ if and only if $G \in \tau$ for $q(G) = p(G) + 3$.*

Lemma 2.7. *[11] Let G be a connected graph.*

(1) *If $R_1(G) = 0, -1, -2$, then $q(G) - p(G) \leq |R_1(G)|$.*

(2) *If $R_1(G) = -3$, then $q(G) - p(G) \leq |R_1(G) + 1|$.*

(3) *If $R_1(G) \leq -4$, then $q(G) - p(G) < |R_1(G) + 1|$.*

Lemma 2.8. *[20] Let G be a connected graph and H a proper subgraph of G, then*

$$\beta(G) < \beta(H).$$

Lemma 2.9. *[20] Let G be a connected graph. Then*

(1) $\beta(G) = -4$ *if and only if*

$$G \in \{T(1,2,5), T(2,2,2), T(1,3,3), K_{1,4}, C_4(P_2), Q_{1,1}, K_4^-, D_8\} \cup \mathcal{U}.$$

(2) $\beta(G) > -4$ *if and only if*

$$G \in \{K_1, T(1,2,i)(2 \le i \le 4), D_i(4 \le i \le 7)\} \cup \mathcal{P} \cup \mathcal{C} \cup \mathcal{T}_0.$$

Lemma 2.10. *[20] Let G be a connected graph. Then* $-(2+\sqrt{5}) \le \beta(G) < -4$ *if and only if G is one of the following graphs:*

(1) T_{l_1,l_2,l_3} *for* $l_1 = 1, l_2 = 2, l_3 > 5$ *or* $l_1 = 1, l_2 > 2, l_3 > 3$ *or* $l_1 = l_2 = 2, l_3 > 2$ *or* $l_1 = 2, l_2 = l_3 = 3$.

(2) $U_{r,s,t,a,b}$ *for* $r = a = 1$, $(r,s,t) \in \{(1,1,2), (2,4,2), (2,5,3), (3,7,3), (3,8,4)\}$, *or* $r = a = 1$, $s \ge 1$, $t \ge t^*(s,b)$, $b \ge 1$, *where* $(s,b) \ne (1,1)$ *and*

$$t^* = \begin{cases} s+b+2, & if\ s \ge 3. \\ b+3, & if\ s = 2. \\ b, & if\ s = 1. \end{cases}$$

(3) D_n *for* $n \ge 9$.

(4) $C_n(P_2)$ *for* $n \ge 5$.

(5) F_n *for* $n \ge 9$.

(6) $B_{r,s,t}$ *for* $r = 5, s = 1$ *and* $t = 3$, *or* $r \ge 1, s = 1$ *if* $t = 1$, *or* $r \ge 4, s = 1$ *if* $t = 2$, *or* $b \ge c+3, s = 1$ *if* $t \ge 3$.

(7) $G \cong C_4(P_3)$ *or* $Q_{1,2}$.

Corollary 2.1. *[14] If a graph G satisfies* $R_1(G) \le -2$, *then* $\beta(G) < -2-\sqrt{5}$.

Definition 2.2. *[10] Let G be a graph,* $e = v_1v_2 \in E(G)$, *then* $N_G(e)$ *and* $d(e)$ *are defined as follow:*

$$N_G(e) = N_G(v_1) \cup N_G(v_2) - \{v_1, v_2\} \text{ and } d(e) = d_G(e) = |N_G(e)|.$$

Lemma 2.11. *[9] Let* G_1 *be a subgraph of G and* $q(G) \ge q(G_1) \ge 2$, *then* $R_2(G_1) \ge R_2(G)$.

3 The algebraic properties of adjoint polynomials

3.1 The divisibility of adjoint polynomials and the fifth characters of graphs

Lemma 3.1. *[20] For $n, m \geq 2$, $h(P_n) \mid h(P_m)$ if and only if $(n+1)|(m+1)$.*

Theorem 3.1. (1) *For $n \geq 7$, $\rho(\zeta_n^1) = \begin{cases} \frac{n-2}{2}, & if\ n\ is\ even; \\ \frac{n-1}{2}, & otherwise. \end{cases}$*

(2) *For $n \geq 7$, $\partial(\zeta_n^1) = \begin{cases} \frac{n+2}{2}, & if\ n\ is\ even; \\ \frac{n+1}{2}, & otherwise. \end{cases}$*

(3) *For $n \geq 7$, $h(\zeta_n^1) = x(h(\zeta_{n-1}^1) + h(\zeta_{n-2}^1))$.*

Proof. (1) Choosing an edge $e \in E(\zeta_n^1)$ whose deletion brings about a proper subgraph K_4^- of ζ_n^1, and by Lemma 2.3, we have $h(\zeta_n^1) = h(K_4^-)h(D_{n-4}) + xh(K_3)h(D_{n-5})$. Then we obtain, from Lemma 2.4, that

$\rho(K_4^- \cup D_{n-4}) = 2 + \lfloor\frac{n-4}{2}\rfloor$ and $\rho(K_1 \cup K_3 \cup D_{n-5}) = 2 + \lfloor\frac{n-5}{2}\rfloor$.

If n is even, then $\rho(K_4^- \cup D_{n-4}) = \frac{n}{2} > \frac{n-2}{2} = \rho(K_1 \cup K_3 \cup D_{n-5})$,which implies $\rho(\zeta_n^1) = \frac{n-2}{2}$. If n is odd, then we arrive at $\rho(K_4^- \cup D_{n-4}) = \rho(K_1 \cup K_3 \cup D_{n-5})$, which implies $\rho(\zeta_n^1) = \frac{n-1}{2}$. Hence the result holds.

(2) It obviously follows from (1).

(3) Choosing an edge $e \in E(\zeta_n^1)$ whose deletion brings about a proper subgraph K_4^- of ζ_n^1. We have, by Lemma 2.4, that

$$\begin{aligned} h(\zeta_n^1) &= h(K_4^-)h(D_{n-4}) + xh(K_3)h(D_{n-5}) \\ &= h(K_4^-)(xh(D_{n-5}) + xh(D_{n-6})) + xh(K_3)(xh(D_{n-6}) + xh(D_{n-7})) \\ &= x(h(K_4^-)h(D_{n-5}) + xh(K_3)h(D_{n-6})) + x(h(K_4^-)h(D_{n-6}) \\ &\quad +xh(K_3)h(D_{n-7})) \\ &= x(h(\zeta_{n-1}^1) + h(\zeta_{n-2}^1)). \end{aligned}$$

□

Theorem 3.2. *For $n \geq 2$, $m \geq 9$, $h(P_n) \mid h(\zeta_m^1)$ if and only if $n = 2$ and $m = 3k+1$ for $k \geq 2$, or $n = 4$ and $m = 5k+2$ for $k \geq 1$.*

Proof. Let $g_0(x) = -x^4 - 6x^3 - 11x^2 - 8x + \frac{1}{x}$, $g_1(x) = x^4 + 5x^3 + 7x^2 + 5x + 2$ and $g_m(x) = x(g_{m-1}(x) + g_{m-2}(x))$.

Let $q_i(x) = xg_i(x)$ $(0 \leq i \leq 6)$ and $q_i(x) = xh(\zeta_i^1)(i \geq 7)$. Easily to see that $q_m(x) = x(q_{m-1}(x) + q_{m-2}(x))$ and $h_1(P_n)|g_m(x)$ if and only if $h_1(P_n)|q_m(x)$. We can deduce that

$$
\begin{aligned}
q_0(x) &= -x^5 - 6x^4 - 11x^3 - 8x^2 + 1, \\
q_1(x) &= x^5 + 5x^4 + 7x^3 + 5x^2 + 2x, \\
q_2(x) &= -x^5 - 4x^4 - 3x^3 + 2x^2 + x, \\
q_3(x) &= x^5 + 4x^4 + 7x^3 + 3x^2, \\
q_4(x) &= 4x^4 + 5x^3 + x^2, \\
q_5(x) &= x^6 + 8x^5 + 12x^4 + 4x^3, \\
q_6(x) &= x^7 + 8x^6 + 16x^5 + 9x^4 + x^3, \\
q_m(x) &= xh(\zeta_m^1),\ if\ m \geq 7.
\end{aligned} \tag{3.1}
$$

It is obvious that $h_1(P_n)|h(\zeta_m^1)$ if and only if $h_1(P_n)|q_m(x)$. Let $m = (n+1)k + i$, where $0 \leq i \leq n$. From Lemma 2.5, it follows that $h_1(P_n)|q_m(x)$ if and only if $h_1(P_n)|q_i(x)$, where $0 \leq i \leq n$. We distinguish the following two cases:

Case 1 $n \geq 7$.

If $0 \leq i \leq 6$, from (3.1), it is not difficult to verify that $h_1(P_n) \nmid q_i(x)$. If $i \geq 7$, from $i \leq n$, Lemma 2.4 and Theorem 3.1, we have that

$$
\partial(h_1(P_n)) = \lfloor \frac{n}{2} \rfloor\ and\ \partial(h_1(\zeta_i^1)) = \lceil \frac{i+1}{2} \rceil. \tag{3.2}
$$

The following cases are taken into account:

Subcase 1.1 $i = n$.

It follows from (3.2) that $\partial(h_1(\zeta_i^1)) = \partial(h_1(P_n)) + 1$. Assume that $h_1(P_n)| h_1(\zeta_i^1)$, it follows that $h_1(\zeta_i^1) = (x+a)h_1(P_n)$. Note that $R_1(\zeta_i^1) = -3$ and $R_1(P_n) = 1$. So $R_1(x+a) = -4$, which brings about $a = \frac{1 \pm \sqrt{41}}{2}$. This contradicts that a is an integer number. Hence $h_1(P_n) \nmid h_1(\zeta_i^1)$, together with $(h_1(P_n), x^{\alpha(\zeta_i^1)}) = 1$, we have $h_1(P_n) \nmid h(\zeta_i^1)$.

Subcase 1.2 $i = n - 1$.

It follows from (3.2) that $\partial(h_1(\zeta_i^1)) = \partial(h_1(P_n)) = \frac{n}{2}$ if n is even and $\partial(h_1(\zeta_i^1)) = \partial(h_1(P_n)) + 1 = \frac{n+1}{2}$ if n is odd.

Subcase 1.2.1 $\partial(h_1(\zeta_i^1)) = \partial(h_1(P_n))$.

Suppose that $h_1(P_n)|h_1(\zeta_i^1)$, we have $h_1(P_n) = h_1(\zeta_i^1)$, which implies $R_1(P_n) = R_1(\zeta_i^1)$. By Lemma 2.6, we know it is impossible. Hence $h_1(P_n) \nmid h_1(\zeta_i^1)$, together with $(h_1(P_n), x^{\alpha(\zeta_i^1)}) = 1$, we have $h_1(P_n) \nmid h(\zeta_i^1)$.

Subcase 1.2.2 $\partial(h_1(\zeta_i^1)) = \partial(h_1(P_n)) + 1$.

We can turn to Subcase 1.1 for the same contradiction.

Subcase 1.3 $i \leq n - 2$.

It follows by (3.2) that $\partial(h_1(\zeta_i^1)) \leq \partial(h_1(P_n))$. Assume that $h_1(P_n)|h_1(\zeta_i^1)$, we have that $\partial(h_1(\zeta_i^1)) = \partial(h_1(P_n))$ and $h_1(\zeta_i^1) = h_1(P_n)$. So we can turn to Subcase 1.2.1 for the same contradiction.

Case 2 $2 \le n \le 6$.

From (1) of Lemma 2.4 and (3.1), we can verify that $h_1(P_n) = q_i(x)$ if and only if $n = 2$ and $i = 1$, or $n = 4$ and $i = 2$ for $0 \le i \le n \le 7$. From Lemma 2.5, we have that $h_1(P_n)|h(\zeta_m^1)$ if and only if $n = 2$ and $m = 3k+1$, or $n = 4$ and $m = 5k+2$. From $\rho(P_2) = 1$, $\rho(P_4) = 2$ and $\rho(\zeta_m^1) \ge 2$ for $m \ge 7$, we obtain that the result holds. □

Theorem 3.3. *For $m \ge 7$, $h^2(P_2) \nmid h(\zeta_m^1)$, $h^2(P_4) \nmid h(\zeta_m^1)$.*

Proof. Suppose that $h^2(P_2) \mid h(\zeta_m^1)$, from Theorem 3.2, we have that $m = 3k+1$, where $k \ge 2$. Let $g_m(x) = h(\zeta_m^1)$ for $m \ge 7$. By (3) of Theorem 3.1, (1) of Lemma 2.5, it follows that

$$\begin{aligned} g_m(x) &= h(P_2)g_{m-2}(x) + x^2 g_{m-3}(x) \\ &= h^2(P_2)g_{m-4}(x) + 2x^2 h(P_2)g_{m-5}(x) + x^4 g_{m-6}(x) \\ &= h^2(P_2)(g_{m-4}(x) + 2x^2 g_{m-7}(x)) + 3x^4 h(P_2)g_{m-8}(x) + x^6 g_{m-9}(x) \\ &= h^2(P_2)(g_{m-4}(x) + 2x^2 g_{m-7}(x) + 3x^4 g_{m-10}(x)) \\ &\quad +4x^6 h(P_2)g_{m-11}(x) + x^8 g_{m-12}(x) \\ &= \cdots \\ &= h^2(P_2)\sum_{s=1}^{k-2} g_{m-3s-1}(x) + (k-1)x^{2k-4}h(P_2)g_{m+1-3(k-1)}(x) \\ &\quad +x^{2k-2}h(P_2)g_{m-3(k-1)}(x). \end{aligned}$$

According to the assumption and $m = 3k+1$, we arrive at, by (3.1), that

$$h^2(P_2) \mid ((k-1)x^{2k-4}h(P_2)g_5(x) + x^{2k-2}g_4(x))$$

that is

$$h(P_2) \mid ((k-1)x^{2k-4}g_5(x) + x^{2k-2}(4x+1))$$

By direct calculation, we obtain that $k = -2$, which contradicts to $k \ge 2$. Using the similar method, we can also prove $h^2(P_4) \nmid h(\zeta_m^1)$. □

Definition 3.1. *Let G be a graph with q edges. The fifth character of a graph G is defined as follow:*

$$R_5(G) = R_2(G) - R_1(G) + p - q.$$

From Lemmas 2.1 and 2.2, we obtain the following two theorems:

Theorem 3.4. *Let G be a graph with components $G_1, G_2, \cdots, G_k$. Then*

$$R_5(G) = \sum_{i=1}^{k} R_5(G_k).$$

It is obvious that $R_5(G)$ is an invariant of graphs. So, for any two graphs G and H, we have $R_5(G) = R_5(H)$ if $h(G) = h(H)$ or $h_1(G) = h_1(H)$.

Theorem 3.5. (1) $R_5(C_n) = 0$ *for* $n \geq 4$; $R_5(C_3) = -3$; $R_5(K_1) = 1$.
(2) $R_5(B_{r,1,1}) = 4$ *for* $r \geq 1$; $R_5(B_{r,1,t}) = 5$ *for* $r, t > 1$.
(3) $R_5(F_6) = 5$; $R_5(F_n) = 4$ *for* $n \geq 7$; $R_5(K_4^-) = 3$.
(4) $R_5(D_4) = 0$; $R_5(D_n) = 1$ *for* $n \geq 5$; $R_5(T_{1,1,1}) = 0$.
(5) $R_5(T_{1,1,l_3}) = 1$; $R_5(T_{1,l_2,l_3}) = 2$; $R_5(T_{l_1,l_2,l_3}) = 3$ *for* $l_3 \geq l_2 \geq l_1 \geq 2$.
(6) $R_5(C_r(P_2)) = 4$ *for* $r \geq 4$; $R_5(C_4(P_3)) = R_5(Q_{1,2}) = 5$.
(7) $R_5(P_2) = -1$; $R_5(P_n) = -2$ *for* $n \geq 3$.
(8) $R_5(K_4) = 7$; $R_5(\zeta_n^1) = 12$ *for* $n \geq 7$.

Lemma 3.2. *If a graph $G \in \varphi$, then* $9 \leq R_5(G) \leq 14$.

Proof. According to Lemma 2.2, we calculate the fourth coefficients of adjoint polynomials of Family φ. Then

$b_3(\varphi_n^1(r,s)) = b_3(D_{n+1}) - 2(n+1) + t$, where $10 \leq t \leq 11$.
$b_3(\varphi_n^2(r,s,t)) = b_3(D_{n+1}) - 2(n+1) + t$, where $10 \leq t \leq 12$.
$b_3(\varphi_n^3(r,s,t)) = b_3(D_{n+1}) - 2(n+1) + t$, where $10 \leq t \leq 12$.
$b_3(\varphi_n^4(r,s,t,p)) = b_3(D_{n+1}) - 2(n+1) + t$, where $9 \leq t \leq 12$.
$b_3(\varphi_n^5(r,s,t,p,q)) = b_3(D_{n+1}) - 2(n+1) + t$, where $9 \leq t \leq 14$.
$b_3(\varphi_n^6(r,s,t)) = b_3(D_{n+1}) - 2(n+1) + t$, where $9 \leq t \leq 11$.
$b_3(\varphi_n^7(r,s,t,u)) = b_3(D_{n+1}) - 2(n+1) + t$, where $10 \leq t \leq 13$.

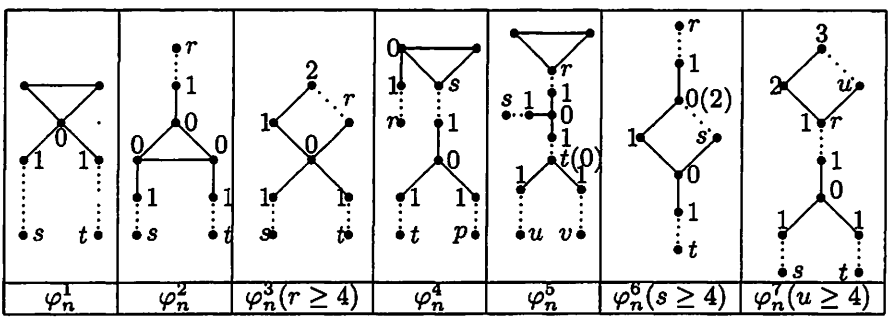


Figure 3 Family of φ

From Definition 2.1, it follows that $7 \leq R_2(G) \leq 12$. Together with Definition 3.1 and Lemma 2.6, we know that the result holds.

□

Lemma 3.3. *[12] Let graph* $G \in \xi \backslash \{F_n, U_{r,s,t,a,b}, K_4^-\}$. *Then*

(1) $R_5(G) = 4$ *if and only if* $G \in \{C_{n-1}(P_2)|n \geq 5\} \cup \{Q_{1,1}\} \cup \{B_{n-5,1,1}|n \geq 7\}$.

(2) $R_5(G) = 5$ *if and only if* $G \in \{C_r(P_s)|r \geq 4, s \geq 3\} \cup \{Q_{1,n-4}|n \geq 6\} \cup \{B_{r,1,t}, B_{1,1,1}|r, t \geq 2\}$.

(3) $R_5(G) = 6$ *if and only if* $G \in \{Q_{r,s}|r, s \geq 2\} \cup \{B_{1,1,t}, B_{r,s,t}|r, s, t \geq 2\}$.

(4) $R_5(G) = 7$ *if and only if* $G \in \{B_{1,s,t}|s, t \geq 2\}$.

Corollary 3.1. *If a graph* $G \in \xi \backslash \{F_n, U_{r,s,t,a,b}, K_4^-\}$, *then* $R_5(G) \geq 4$.

Lemma 3.4. *[12] If a graph* $G \in \psi$, *then*

(1) $R_5(G) = 8$ *if and only if* $G \in \{\psi_n^1\} \cup \{\psi_5^2\} \cup \{\psi_n^3(r, s)|r \geq 4, s \geq 2\} \cup \{\psi_n^4(n-6, 1)|n \geq 8\} \cup \{\psi_n^5(1, s, t)|s, t \geq 2\}$.

(2) $R_5(G) = 9$ *if and only if* $G \in \{\psi_n^2\} \cup \{\psi_n^3(n-3, 1)|n \geq 6\} \cup \{\psi_n^4(r, s)|r, s \geq 2\} \cup \{\psi_7^4(1, 1)\} \cup \{\psi_n^5(1, 1, t), \psi_n^5(r, s, t)|r, s, t \geq 2\} \cup \{\psi_5^6\}$.

(3) $R_5(G) = 10$ *if and only if* $G \in \{\psi_n^4(1, n-6)|n \geq 8\} \cup \{\psi_n^5(r, 1, t)|r, t \geq 2\} \cup \{\psi_8^5(1, 1, 1)\}$.

(4) $R_5(G) = 11$ *if and only if* $G \in \{\psi_n^5(n-7, 1, 1)|n \geq 9\}$.

Corollary 3.2. *If a graph* $G \in \psi$, *then* $R_5(G) \geq 8$.

Lemma 3.5. *[12] Let graph* $G \in \zeta$, *then*

(1) $R_5(G) = 12$ *if and only if* $G \in \{\zeta_n^1|n \geq 8\} \cup \{\zeta_n^2(r, s)|r, s \geq 2\} \cup \{\zeta_n^3(r, s, t)|r, s, t \geq 2\}$.

(2) $R_5(G) = 13$ *if and only if* $G \in \{\zeta_7^1\} \cup \{\zeta_n^2(1, n-8)|n \geq 10\} \cup \{\zeta_n^3(1, s, t)|s, t \geq 2\}$.

(3) $R_5(G) = 14$ *if and only if* $G \in \{\zeta_9^2(1, 1)\} \cup \{\zeta_n^3(1, 1, n-9)|n \geq 11\}$.

(4) $R_5(G) = 15$ *if and only if* $G \in \{\zeta_n^3(1, 1, 1)|n \geq 9\}$.

Corollary 3.3. *If a graph* $G \in \zeta$, *then* $R_5(G) \geq 12$.

Lemma 3.6. *[13] If a graph* $G \in \theta$, *then* $16 \leq R_5(G) \leq 22$.

Lemma 3.7. *[12] If a graph* $G \in \phi$, *then* $12 \leq R_5(G) \leq 17$.

Lemma 3.8. *If a graph* $G \in \tau$, *then* $R_5(G) \geq 17$.

Proof. As a matter of fact, $\pi(G)$ in [10] is actually equal to $R_1(G)$. Moreover, Du [10] gave a recursive method to construct the family π_i consisting of graphs with $R_1(G) = -i$, which is stated as follows:

Suppose that $\pi_{-1}, \pi_0, \pi_1, \cdots, \pi_{i-1}$ have been determined. For each graph $G \in \pi_t$ $(-1 \leq t \leq i-1)$, together with Definition 2.2, we find all the edges e

satisfying $e \notin E(G)$ and $d_{G+e}(e) = i + 1 - t$ to construct the new graph $G + e$ (add vertices where necessary). Such graphs are collected in π_i'. Then we proceed to add all possible edges e with $d(e) = 1$ to each graph in π_i'. In this way, we obtain the graphs in π_i.

Using the above method, we can construct the graphs in Family τ. If $R_1(G) = -5$ and $q(G) = p(G) + 3$, then $i = 5$, $-1 \leq t \leq 4$ and $d_{G+e}(e) = 6 - t$. By Lemmas 2.6 and 2.7 we can do as follows: add the edges e with $d(e) = 5$ to each graph with $R_1(G) = -1$ and $q(G) = p(G) + 1$, and then add the edges e with $d(e) = 1$. These resulting graphs constitutes the family $\mathscr{H}_1$; add the edges e with $d(e) = 4$ to the graph with $R_1(G) = -2$ and $q(G) = p(G) + 1$, and then add the edges e with $d(e) = 1$. These resulting graphs constitutes the family $\mathscr{H}_2$; add the edges e with $d(e) = 3$ to the graph with $R_1(G) = -3$ and $q(G) = p(G) + 2$, and then add the edges e with $d(e) = 1$. These resulting graphs constitutes the family $\mathscr{H}_3$; add the edges e with $d(e) = 2$ to the graph with $R_1(G) = -4$ and $q(G) = p(G) + 2$, and then add the edges e with $d(e) = 1$. These resulting graphs constitutes the family $\mathscr{H}_4$. Clearly, $\tau = \cup_{i=1}^{4} \mathscr{H}_i$ and $min\{R_5(G)|G \in \tau\} = min\{R_5(H)|H \in \mathscr{H}_4\}$. Let $G \in \theta$ and $H \in \mathscr{H}_4 \subset \tau$. From Lemma 2.11, it follows that $R_2(G) \leq R_2(H)$. By Definition 2.1 and 3.1, we know that $R_5(G) + 1 \leq R_5(H)$, which implies $17 = min\{R_5(G)|G \in \theta\} + 1 \leq min\{R_5(H)|H \in \mathscr{H}_4\} = min\{R_5(H)|H \in \tau\}$.

This completes the lemma. □

3.2 The smallest real roots of adjoint polynomials of graphs

An internal x_1x_k−path of a graph G is path $x_1x_2x_3 \cdots x_k$(possibly $x_1 = x_k$) of G such that $d(x_1)$ and $d(x_k)$ are at least 3 and $d(x_2) = d(x_3) = \cdots = d(x_{k-1}) = 2$ (unless $k = 2$).

Lemma 3.9. *[20] Let T be a tree. If uv is an internal path of T and $T \not\cong U(1,1,t,1,1)$ for $t \geq 1$, then $\beta(T) < \beta(T_{xy})$, where $\beta(T_{xy})$ is the graph obtained from T by inserting a new vertex on the edge xy of T.*

Lemma 3.10. *[15, 16, 17]* (1) *For $n \geq 4$, $m \geq 6$, $\beta(K_4) < \beta(F_m) < \beta(D_n) < \beta(C_n) < \beta(P_n)$.*

(2) $\beta_{min}(B_{r,s,t}) \leq \beta_{min}(Q(r,s)) \leq \beta_{min}(C_r(P_s)) \leq \beta_{min}(T_n)$ *for* $n \geq 6$.

(3) $\beta_{min}(\psi_n^5(r,s,t)) \leq \beta_{min}(\psi_n^4(r,s)) \leq \beta_{min}(\psi_n^3(r,s)) \leq \beta_{min}(\psi_n^2) \leq \beta_{min}(\psi_n^1)$ *for* $n \geq 8$.

(4) $\beta_{min}(B_{r,s,t}) = \beta(B_{1,1,n-5})$; $\beta_{min}(Q(r,s)) = \beta(Q(1,n-4))$.

(5) $\beta_{min}(\zeta_n^3) \leq \beta_{min}(\zeta_n^2) \leq \beta_{min}(\zeta_n^1)$.

(6) $\beta_{min}(\psi_n^3(r,s)) = \beta(\psi_n^3(n-3,1))$; $\beta_{min}(\psi_n^4(r,s)) = \beta(\psi_n^4(1,n-6))$; $\beta_{min}(\psi_n^5(r,s,t)) = \beta(\psi_n^5(n-7,1,1))$.

(7) $\beta_{min}(\zeta_n^2(r,s)) = \beta(\zeta_n^2(1,n-8))$; $\beta_{min}(\zeta_n^3(r,s,t)) = \beta(\zeta_n^3(1,1,n-9))$.

(8) $\beta_{min}(\psi_n^1) < \beta(\psi_n^5(1,s,t))$.

Lemma 3.11. (1) *For* $n \geq 7$, $\beta(\zeta_n^1) < \beta(\zeta_{n+1}^1)$.

(2) *For* $n \geq 7$, $r \geq 5$, $m \geq 6$, $\beta(\zeta_n^1) < \beta(Q_{1,1})$; $\beta(\zeta_n^1) < \beta(K_4^-)$; $\beta(\zeta_n^1) < \beta(C_r(P_2))$; $\beta(\zeta_n^1) < \beta(B_{m-5,1,1})$; $\beta(\zeta_n^1) < \beta(F_m)$.

(3) *For* $n \geq 7$, $m \geq 6$, $\beta(\zeta_n^1) < \beta(K_4) = \beta(\psi_5^2)$; $\beta(\zeta_n^1) < \beta(B_{1,1,m-5}) < \beta(C_r(P_s))$.

(4) *For* $n \geq 7$, $m \geq 6$, $\beta(\zeta_n^1) < \beta(Q_{1,m-4})$.

Proof. (1) Using Software Mathematica, we have that

For $n_1 \geq 20$, $\beta(\zeta_7^1) = -5 < \beta(\zeta_8^1) = -4.86906 < \beta(\zeta_9^1) = -4.80535 < \beta(\zeta_{10}^1) = -4.77448 < \beta(\zeta_{11}^1) = -4.75999 < \beta(\zeta_{12}^1) = -4.7534 < \beta(\zeta_{13}^1) = -4.75047 < \beta(\zeta_{14}^1) = -4.74981 < \beta(\zeta_{15}^1) = -4.74862 < \beta(\zeta_{16}^1) = -4.74838 < \beta(\zeta_{17}^1) = -4.74828 < \beta(\zeta_{18}^1) = -4.74823 < \beta(\zeta_{19}^1) = -4.74821 < \beta(\zeta_{n_1}^1) < \beta(\zeta_{n_1+1}^1) < -4.7482$.

(2) From Lemmas 2.9, 2.10 and Corollary 2.1, it is easy to see that the result holds.

(3) It is obvious that ψ_5^2 is a subgraph of ζ_n^1. By Lemma 2.8, we obtain that $\beta(\zeta_n^1) < \beta(\psi_5^2) = \beta(K_4)$; From (2) and (4) of Lemma 3.10, we know that $\beta(B_{1,1,m-5}) < \beta(C_r(P_s))$. From this together with Lemma 2.8, we have $\beta(\zeta_n^1) < \beta(\psi_6^2) = -4.65109$. From $n_1 \geq 8$, $m_1 \geq 14$, $\beta(\zeta_7^1) = -5 < \beta(\zeta_{n_1}^1) < \beta(B_{1,1,m_1-5}) < \beta(B_{1,1,15}) = -4.51729 < \beta(B_{1,1,14}) = -4.51728 < \beta(B_{1,1,13}) = -4.51726 < \beta(B_{1,1,12}) = -4.51721 < \beta(B_{1,1,11}) = -4.51713 < \beta(B_{1,1,10}) = -4.51695 < \beta(B_{1,1,9}) = -4.51658 < \beta(B_{1,1,8}) = -4.51584 < \beta(B_{1,1,7}) = -4.51432 < \beta(B_{1,1,6}) = -4.51119 < \beta(B_{1,1,5}) = -4.50469 < \beta(B_{1,1,4}) = -4.49086 < \beta(B_{1,1,3}) = -4.4605 < \beta(B_{1,1,2}) = -4.39026 < \beta(B_{1,1,1}) = -4.21432$.

(4) For $n_1 \geq 8$, $m_1 \geq 16$, $\beta(\zeta_7^1) = -5 < \beta(\zeta_{n_1}^1) < \beta(Q_{1,m_1-4}) < \beta(Q_{1,11}) = -4.38249 < \beta(Q_{1,10}) = -4.38207 < \beta(Q_{1,9}) = -4.38131 < \beta(Q_{1,8}) = -4.37988 < \beta(Q_{1,7}) = -4.3772 < \beta(Q_{1,6}) = -4.37213 < \beta(Q_{1,5}) = -4.36232 < \beta(Q_{1,4}) = -4.334292 < \beta(Q_{1,3}) = -4.30278 < \beta(Q_{1,2}) = -4.21342$. □

Lemma 3.12. (1) *For* $n \geq 7$, $m \geq 5$, $\beta(\zeta_n^1) < \beta(\psi_m^1) < \beta(\psi_n^5(1,s,t))$.

(2) *For* $n \geq 7$, $m \geq 5$, $\beta(\zeta_n^1) < \beta(\psi_m^2)$.

(3) *For* $n \geq 7$, $m \geq 7$, $\beta(\zeta_n^1) = \beta(\psi_n^3(m-3,1))$ *if and only if* $n = 13$ *and* $m = 9$.

(4) *For* $n \geq 10$, $m \geq 10$, $\beta(\psi_n^4(1, m-6)) < \beta(\zeta_n^1)$; $\beta(\zeta_n^1) < \beta(\psi_n^4(m-6,1))$.

(5) *For* $n \geq 7$, $m \geq 8$, $\beta(\psi_n^5(m-7,1,1)) < \beta(\zeta_n^1)$.

(6) *For* $n \geq 7$, $\beta(\zeta_n^1) < \beta(\psi_5^6)$.

Proof. (1) For $n_1 \geq 8$, $m_1 \geq 6$, $\beta(\zeta_7^1) = -5 < \beta(\zeta_{n_1}^1) < \beta(\psi_{m_1}^1) < \beta(\psi_{18}^1) = -4.61347 < \beta(\psi_{17}^1) = -4.61346 < \beta(\psi_{16}^1) = -4.61345 < \beta(\psi_{15}^1) = -4.61342 < \beta(\psi_{14}^1) = -4.61337 < \beta(\psi_{13}^1) = -4.61325 < \beta(\psi_{12}^1) = -4.613 < \beta(\psi_{11}^1) = -4.61246 < \beta(\psi_{10}^1) = -4.61128 < \beta(\psi_9^1) = -4.60873 < \beta(\psi_8^1) = -4.60212 <$

$\beta(\psi_7^1) = -4.59056 < \beta(\psi_6^1) = -4.56155 < \beta(\psi_5^1) = -4.49086$. From (8) of Lemma 3.8, the result holds.

(2) For $n_1 \geq 8$, $m_1 \geq 18$, $\beta(\zeta_7^1) = -5 < \beta(\zeta_{n_1}^1) < \beta(\psi_{m_1}^2) < \beta(\psi_{17}^2) = -4.74819 < \beta(\psi_{16}^2) = -4.74818 < \beta(\psi_{15}^2) = -4.74815 < \beta(\psi_{14}^2) = -4.7481 < \beta(\psi_{13}^2) = -4.74796 < \beta(\psi_{12}^2) = -4.74766 < \beta(\psi_{11}^2) = -4.74694 < \beta(\psi_{10}^2) = -4.74528 < \beta(\psi_9^2) = -4.74137 < \beta(\psi_8^2) < \beta(\psi_7^2) = -4.70928 < \beta(\psi_6^2) = -4.65109 < \beta(\psi_5^2) = -4.49086$.

(3) For $n_1 \geq 14$, $m_1 \geq 17$, combining with (1) of Lemma 3.11, it follows that $\beta(\zeta_7^1) = -5 < \beta(\zeta_8^1) < \beta(\zeta_9^1) < \beta(\zeta_{10}^1) < \beta(\psi_{m_1}^3(m_1 - 3, 1)) < \beta(\psi_{17}^3(14, 1)) = -4.76349 < \beta(\zeta_{11}^1) < \beta(\psi_{16}^3(13, 1)) = -4.76347 < \beta(\psi_{15}^3(12, 1)) = -4.76343 < \beta(\psi_{14}^3(11, 1)) = -4.76332 < \beta(\psi_{13}^3(10, 1)) = -4.76308 < \beta(\psi_{12}^3(9, 1)) = -4.76251 < \beta(\psi_{11}^3(8, 1)) = -4.76118 < \beta(\psi_{10}^3(7, 1)) = -4.75802 < \beta(\zeta_{12}^1) < \beta(\zeta_{13}^1) = \beta(\psi_9^3(6, 1)) = -4.75047 < \beta(\zeta_{n_1}^1) < \beta(\psi_8^3(5, 1)) = -4.73205 < \beta(\psi_7^3(4, 1)) = -4.68554$.

(4) For $n_1 \geq 10, m_1 \geq 17, m_2 \geq 12, \beta(\zeta_7^1) = -5 < \beta(\zeta_8^1) < \beta(\psi_{m_1}^4(1, m_1 - 6)) < \beta(\psi_{16}^4(1, 10)) = -4.85505 < \beta(\psi_{15}^4(1, 9)) = -4.85498 < \beta(\psi_{14}^4(1, 8)) = -4.85482 < \beta(\psi_{13}^4(1, 7)) = -4.85443 < \beta(\psi_{12}^4(1, 6)) = -4.85347 < \beta(\psi_{11}^4(1, 5)) = -4.85109 < \beta(\psi_{10}^4(1, 4)) = -4.84517 < \beta(\psi_9^4(1, 3)) = -4.83021 < \beta(\zeta_9^1) < \beta(\psi_8^4(1, 2)) = -4.79129 < \beta(\zeta_{n_1}^1) < \beta(\psi_7^3(4, 1)) = \beta(\psi_7^4(1, 1)) = -4.68554; \beta(\zeta_7^1) = -5 < \beta(\zeta_{n_1}^1)) < \beta(\psi_8^4(2, 1)) = -4.56155 < \beta(\psi_9^4(3, 1)) = -4.49086 < \beta(\psi_{10}^4(4, 1)) = -4.4887 < \beta(\psi_{11}^4(5, 1)) = -4.4217 < \beta(\psi_{m_2}^4(m_2 - 6, 1))$

(5) For $n_1 \geq 8$, $m_1 \geq 18$, $\beta(\psi_{m_1}^5(m_1 - 7, 1, 1)) < \beta(\psi_{17}^5(10, 1, 1)) = -5.00991 < \beta(\psi_{16}^5(9, 1, 1)) = -5.00986 < \beta(\psi_{15}^5(8, 1, 1)) = -5.00973 < \beta(\psi_{14}^5(7, 1, 1)) = -5.0094 < \beta(\psi_{13}^5(6, 1, 1)) = -5.00852 < \beta(\psi_{12}^5(5, 1, 1)) = -5.0062 < \beta(\psi_{11}^5(4, 1, 1)) = -5 < \beta(\psi_{10}^5(3, 1, 1)) = -4.98311 < \beta(\psi_9^5(2, 1, 1)) = -4.93543 < \beta(\psi_8^5(1, 1, 1)) = -4.79129 < \beta(\zeta_7^1) = -5 < \beta(\zeta_{n_1}^1) < \beta(\psi_8^5) = -4.79129$.

(6) For $n_1 \geq 8$, $\beta(\psi_5^6) = -6.17508 < \beta(\zeta_7^1) = -5 < \beta(\zeta_{n_1}^1)$. □

Lemma 3.13. (1) *For* $n \geq 10$, $m \geq 9$, $\beta(\zeta_m^2(1, m - 8)) < \beta(\zeta_n^1)$.

(2) *For* $n \geq 7$, $m \geq 10$, $\beta(\zeta_m^3(1, 1, m - 9)) < \beta(\zeta_n^1)$.

Proof. Using Software Mathematica, we have that

(1) For $n_1 \geq 11$, $m_1 \geq 19$, $\beta(\zeta_9^2(1, 1)) = -5.04892 < \beta(\zeta_7^1) = -5 < \beta(\zeta_{10}^2(1, 2)) = -4.9418 < \beta(\zeta_{11}^2(1, 3)) = -4.89307 < \beta(\zeta_{12}^2(1, 4)) = -4.8713 < \beta(\zeta_8^1) = -4.86906 < \beta(\zeta_{13}^2(1, 5)) = -4.86118 < \beta(\zeta_{14}^2(1, 6)) = -4.8579 < \beta(\zeta_{15}^2(1, 7)) = -4.85625 < \beta(\zeta_{16}^2(1, 8)) = -4.8557 < \beta(\zeta_{17}^2(1, 7)) = -4.85529 < \beta(\zeta_{18}^2(1, 8)) = -4.85517 < \beta(\zeta_{m_1}^2(1, m_1 - 8)) < \beta(\zeta_{10}^1) = -4.80535 < \beta(\zeta_{n_1}^1)$.

(2) For $n_1 \geq 8$, $m_1 \geq 20$, $\beta(\zeta_{10}^3(1, 1, 1)) = -5.23607 < \beta(\zeta_{11}^3(1, 1, 2)) = -5.10552 < \beta(\zeta_{12}^3(1, 1, 3)) = -5.04892 < \beta(\zeta_{13}^3(1, 1, 4)) = -5.0254 < \beta(\zeta_{14}^3(1, 1, 5)) = -5.01594 < \beta(\zeta_{15}^3(1, 1, 6)) = -5.01224 < \beta(\zeta_{16}^3(1, 1, 7)) =$

$-5.01082 < \beta(\zeta_{17}^3(1,1,8)) = -5.01027 < \beta(\zeta_{18}^3(1,1,9)) = -5.01006 < \beta(\zeta_{19}^3(1,1,10)) = -5.00998 < \beta(\zeta_{m_1}^3(1,1,m_1-9)) < \beta(\zeta_7^1) = -5 < \beta(\zeta_{n_1}^1)$. □

4 The chromaticity of graph $\overline{\zeta_n^1}$

Lemma 4.1. *[18] For $n \geq 4$, D_n is adjointly unique if and only if $n \neq 4, 8$.*

Lemma 4.2. *Let G be a graph such that $G \sim^h \zeta_n^1$, where $n \geq 7$. Then*
(1) If $n \neq 13$, then G does not contain K_4^- as one of its components.
(2) G does not contain K_4 as one of its components.

Proof. (1) Suppose that $h(K_4^-)|h(\zeta_n^1)$. From Lemma 2.3, we know that $h(\zeta_n^1) = h(K_4^-)h(D_{n-4})+xh(K_3)h(D_{n-5})$. Combining this with $(h(K_4^-), h(K_3)) = 1$, we have that $h(K_4^-)|h(D_{n-5})$, which implies $\beta(D_{n-5}) < \beta(K_4^-)$. If $n \neq 13$, then we have from Lemma 2.9 and 2.10, that $\beta(D_{n-5}) < \beta(K_4^-)$ for $n < 13$; $\beta(K_4^-) < \beta(D_{n-5})$ for $n > 13$. Hence $\beta(D_i) < \beta(K_4^-)$ for $4 \leq i \leq 7$. It contradicts to (2) of Lemma 2.9.

(2) Suppose that $h(K_4)|h(\zeta_n^1)$. From Lemma 2.3, we arrive at $h(\zeta_n^1) = h(\psi_5^2)h(D_{n-5}) + xh(K_4^-)h(D_{n-6}) = h(K_4)h(D_{n-5}) + xh(K_4^-)h(D_{n-6})$. Together with $(h(K_4), h(K_4^-)) = 1$, we have that $h(K_4)|h(D_{n-6})$, which implies $\beta(D_{n-6}) < \beta(K_4)$. By Lemma 2.10 and Corollary 2.1, we obtain that $\beta(K_4) < \beta(D_{n-6})$. This is obviously a contradiction. □

Theorem 4.1. *Let G be a graph such that $G \sim^h \zeta_n^1$, where $n \geq 7$. Then G contains at most two components whose first characters are 1, furthermore, one of both is P_2 and the other is P_4 or one of both is P_2 and the other is C_3.*

Proof. Let G_1 be one of the components of G such that $R_1(G_1) = 1$. From Lemma 2.6, it follows, from Theorem 3.2, that $h(G_1)|h(\zeta_n^1)$ if and only if $G_1 \cong P_2$ and $n = 3k+1$, or $G_1 \cong P_4$ and $n = 5k+2$. According to (1) of Lemma 2.5, we obtain the following equality:

$$h(\zeta_{15k+7}^1) = h(P_{15})h(\zeta_{15(k-1)+7}^1) + xh(P_{14})h(\zeta_{15(k-1)+6}^1) \qquad (4.1)$$

Noting that $\{n|n = 3k+1, k \geq 1\} \cap \{n|n = 5k+2, k \geq 1\} = \{n|n = 15k+7, k \geq 0\}$, we have that

$$h(P_2)h(P_4) \mid h(\zeta_{15(k-1)+7}^1) \qquad (4.2)$$

By Lemma 3.1, we get $h(P_2) \mid h(P_{14})$ and $h(P_4) \mid h(P_{14})$, together with $(h(P_2), h(P_4)) = 1$, which leads to

$$h(P_2)h(P_4) \mid h(P_{14}) \qquad (4.3)$$

From (4.1) to (4.3), we obtain $h(P_2)h(P_4) \mid h(\zeta^1_{15k+7})$. Noting $h(P_4) = h(K_1 \cup C_3)$, we also have $h(P_2)h(C_3) \mid h(\zeta^1_{15k+7})$, together with Theorem 3.3, so the theorem holds. □

Theorem 4.2. *Let G be a graph such that $G \sim^h \zeta^1_n$, where $n \geq 9$.*

(1) If $n = 13$, then $[G]_h = \{\zeta^1_{13}, K_4^- \cup \psi^3_9(6,1)\}$.

(2) If $n \neq 13$, then $[G]_h = \{\zeta^1_n\}$.

Proof. (1) When $n = 13$, let graph G satisfy $h(G) = h(\zeta^1_{13})$. From Lemmas 2.1, 2.2 and 2.6, we obtain that $q(G) - p(G) = 2$ and $R_1(G) = -3$. We distinguish the following cases:

Case 1 G is a connected graph.

By $R_5(G) = R_5(\zeta^1_{13}) = 12$ and (1) of Lemma 3.5, we have that $G \in \mathscr{G} = \{\zeta^1_{13}\} \cup \{\zeta^2_{13}(r,s) | r+s = 6,\ 1 \leq r,s \leq 5\} \cup \{\zeta^3_{13}(r,s,t) | r+s+t = 6,\ 1 \leq r,s,t \leq 4\}$. By calculation, we have that $\zeta^1_{13} \in [G]_h$.

Case 2 G is not a connected graph.

By calculation, we have $h(G) = h(\zeta^1_{13}) = x^6(x+1)(x+4)(x^5+10x^4+33x^3+42x^2+18x+2)$. Let $h(G) = h(\zeta^1_{13}) = x^6 f_1(x) f_2(x) f_3(x)$, where $f_1(x) = x+1$, $f_2(x) = x+4$ and $f_3(x) = x^5+10x^4+33x^3+42x^2+18x+2$. Noting that $R_1(f_1(x)) = 1$ and $b_1(f_1(x)) = 1$, from Lemma 2.6, we obtain that $f_1(x) = h_1(P_2)$ if $f_1(x)$ is a factor of adjoint polynomial of some graph. If P_2 is a component of G, then let $G = P_2 \cup G_1$, we arrive at $h_1(f_2(x)f_3(x)) = x^6+14x^5+73x^4+174x^3+186x^2+74x+8$, which implies $R_1(G_1) = R_1(f_2(x)f_3(x)) = -4$ and $q(G_1) - p(G_1) = 3$. From (3) of Lemma 2.7, we know that it is impossible. Noting that $R_1(f_2(x)) = 1$ and $b_1(f_2(x)) = 4$, from Lemma 2.6, we obtain that $f_2(x) = h_1(T_{1,1,1,1})$ if $f_2(x)$ is a factor of adjoint polynomial of some graph. Let $G = T_{1,1,1,1} \cup G_1$, then we arrive at $h_1(f_1(x)f_3(x)) = x^6+11x^5+43x^4+75x^3+60x^2+20x+2$, which implies $R_1(G_1) = R_1(f_1(x)f_3(x)) = -1$ and $q(G_1) - p(G_1) = 2$. It is impossible by Lemma 2.7. According to $R_1(f_1(x)f_2(x)) = -1$, $b_1(f_2(x)) = 5$ and (3) of Lemma 2.6, we obtain that $f_1(x) = h_1(K_4^-)$ if $f_1(x)$ is a factor of adjoint polynomial of some graph.

Subcase 2.1 K_4^- is not a component of G.

Since G is not connected, then the expression of G is $G = aK_1 \cup G_1$, where $a \geq 1$ and G_1 is connected. It is not difficult to obtain that $q(G_1) - p(G_1) \geq 3$. We conclude, from Lemma 2.7, that $q(G_1) - p(G_1) \leq 2$. Thus this brings about a contradiction.

Subcase 2.2 K_4^- is a component of G.

Let $G = K_4^- \cup G_1$, where $h_1(G_1) = x^5+10x^4+33x^3+42x^2+18x+2$. The following cases are taken into account:

Subcase 2.2.1 G_1 is a connected graph.

Noting that $R_1(G_1) = -2$ and $q(G_1) = p(G_1) + 1 = 10$, we have from Lemma 2.6, that $G_1 \in \psi$. Then we consider that $G_1 \in \{\psi^1_9, \psi^2_9, \psi^3_9(6,1), \psi^3_9(5,2),$

$\psi_9^3(4,3), \psi_9^4(3,1), \psi_9^4(2,2), \psi_9^4(1,3), \psi_9^5(1,1,2), \psi_9^5(2,1,1)\}$. By calculation, $K_4^- \cup \psi_9^3(6,1) \in [G]_h$.

Subcase 2.2.2 G_1 is not a connected graph.

It follows that $G = K_4^- \cup aK_1 \cup G_1$, where $a \geq 1$ and $h_1(G_1) = x^5+10x^4+33x^3+42x^2+18x+2$. It is not difficult to get that $q(G_1)-p(G_1) \geq 2$. Remarking that $R_1(G_1) = -2$, we obtain from Lemma 2.7 that $q(G_1) - p(G_1) \leq 2$, which results in $q(G_1)-p(G_1) = 2$. Thus we conclude, from Lemma 2.6, that $G_1 \cong K_4$, a=1. By calculation, $G = K_4^- \cup K_1 \cup K_4 \notin [G]_h$.

(2) When $n \geq 7, n \neq 13$, let $G = \bigcup_{i=1}^t G_i$. From Lemma 2.1, we have that

$$h(G) = \prod_{i=1}^{t} h(G_i) = h(\zeta_n^1), \tag{4.4}$$

which results in $\beta(G) = \beta(\zeta_n^1) \in (-\infty, -2-\sqrt{5})$ by Corollary 2.1. Let s_i denote the number of components G_i such that $R(G_i) = -i$, where $i \geq -1$. From Theorem 4.2, Lemmas 2.1 and 2.2, it follows that $0 \leq s_{-1} \leq 2$ and

$$R_1(G) = \sum_{i=1}^{t} R_1(G_i) = -3 \; and \; q(G) = p(G)+2, \tag{4.5}$$

which implies

$$\begin{aligned} &-5 \leq R_1(G_i) \leq 1, \\ &s_{-1} = s_1 + 2s_2 + 3s_3 + 4s_4 + 5s_5 - 3, \\ &\sum_{-5 \leq R_1(G) \leq 0} (q(G_i) - p(G_i)) = s_{-1}. \end{aligned} \tag{4.6}$$

Let $\cup_{T\in\mathcal{T}_0} T_{l_1,l_2,l_3} = (\cup_{T\in\mathcal{T}_1} T_{1,1,l_3}) \cup (\cup_{T\in\mathcal{T}_2} T_{1,l_2,l_3}) \cup (\cup_{T\in\mathcal{T}_3} T_{l_1,l_2,l_3})$, $\mathcal{T}_1 = \{T_{1,1,l_3} | l_3 \geq 2\}$, $\mathcal{T}_2 = \{T_{1,l_2,l_3} | l_3 \geq l_2 \geq 2\}$, $\mathcal{T}_3 = \{T_{l_1,l_2,l_3} | l_3 \geq l_2 \geq l_1 \geq 2\}$, $\mathcal{T}_0 = \mathcal{T}_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$, the tree T_{l_1,l_2,l_3} is denoted by T for short, $A = \{i | i \geq 4\}$ and $B = \{j | j \geq 5\}$.

We distinguish the following cases by $0 \leq s_{-1} \leq 2$:

Case 1 $s_{-1} = 0$.

It follows, from (4.6), that

$$s_5 = s_4 = 0 \; and \; s_1 + 2s_2 + 3s_3 = 3. \tag{4.7}$$

We distinguish the following cases by (4.7):

Subase 1.1 $s_3 = 1$ and $s_2 = s_1 = 0$.

From Lemmas 2.1 and 2.6, we set

$$G = G_1 \cup (\cup_{i\in A} C_i) \cup (\cup_{j\in B} D_j) \cup fD_4 \cup aK_1 \cup bT_{1,1,1} \cup (\cup_{T\in\mathcal{T}_0} T_{l_1,l_2,l_3}), \tag{4.8}$$

where $R_1(G_1) = -3$.

Recalling that $q(G) = p(G) + 2$, we obtain that $q(G_1) - p(G_1) \geq 2$. By (2) of Lemma 2.7, it follows that $q(G_1) - p(G_1) \leq 2$. Then $q(G_1) - p(G_1) = 2$, which implies $a = b = |\mathcal{T}_1| = |\mathcal{T}_2| = |\mathcal{T}_3| = 0$ and $G_1 \in \zeta$. Hence $G = G_1 \cup (\cup_{i\in A} C_i) \cup (\cup_{j\in B} D_j) \cup fD_4$. From Theorems 3.4 and 3.5, we arrive at $R_5(G) = R_5(\zeta_n^1) = 12 = R_5(G_1) + |B|$, which leads to $|B| = 0$ and $R_5(G_1) = 12$ by Corollary 3.3. Then $G = G_1 \cup (\cup_{i\in A} C_i) \cup fD_4$. In terms of (1) of Lemma 3.10 and (3) of Lemma 3.11, we obtain that $\beta(G) = \beta(G_1) = \beta(\zeta_n^1)$, which results in $G_1 \cong \zeta_m^1$ by (5) of Lemma 3.10. By (1) of Lemma 3.11, we know that $m = n$ and $|A| = f = 0$. So $G \cong \zeta_n^1$.

Subase 1.2 $s_3 = 0$ and $s_2 = s_1 = 1$.

From Lemma 2.6 and (4.5), let

$$G = G_1 \cup G_2 \cup (\cup_{i\in A} C_i) \cup (\cup_{j\in B} D_j) \cup fD_4 \cup aK_1 \cup bT_{1,1,1} \cup (\cup_{T\in \mathcal{T}_0} T_{l_1,l_2,l_3}). \tag{4.9}$$

where $R_1(G_1) = -1$, $R_1(G_2) = -2$.

By Theorems 3.4 and 3.5, we arrive at

$$R_5(G) = R_5(\zeta_n^1) = 12 = R_5(G_1) + R_5(G_2) + |B| + a + |\mathcal{T}_1| + 2|\mathcal{T}_2| + 3|\mathcal{T}_3|. \tag{4.10}$$

From (2) of Lemma 4.2, we obtain that $G_2 \ncong K_4$. Together with Lemma 2.6 and (4.5), we have that $q(G_1) - p(G_1) = 1$, $q(G_2) - p(G_2) = 1$ and $a = b = |\mathcal{T}_1| = |\mathcal{T}_2| = |\mathcal{T}_3| = 0$, which implies $G_1 \in \{F_m, K_4^-\}$, $G_2 \in \psi$. By (1) of Lemma 4.2 and (4.10), $G_1 \cong F_m$ and $R_5(G_2) = 8 - |B|$, which leads to $|B| = 0$ and $R_5(G_2) = 8$ by Corollary 3.2. Together with (4.9) and Lemma 3.4, we obtain that $G = F_m \cup G_2 \cup (\cup_{i\in A} C_i) \cup fD_4$, where $G_2 \in \{\psi_n^1\} \cup \{\psi_5^2\} \cup \{\psi_n^3(r,s)\} \cup \{\psi_n^4(n-6,1)\} \cup \{\psi_n^5(1,s,t)\}$. According to (1) of Lemma 3.10 and (2) of Lemma 3.11, we know that $\beta(G) = \beta(G_2)$. By Lemma 3.12, $\beta(G) = \beta(\zeta_n^1) = \beta(G_2)$ if and only if $n = 13$ and $m = 9$. Then $G = F_m \cup \psi_9^3(6,1) \cup (\cup_{i\in A} C_i) \cup fD_4$, which contradicts to $p(G) = 13$.

Subcase 1.3 $s_3 = s_2 = 0$ and $s_1 = 3$.

Without loss of generality, let

$$G = (\bigcup_{i=1}^{3} G_i) \cup (\cup_{i\in A} C_i) \cup (\cup_{j\in B} D_j) \cup fD_4 \cup aK_1 \cup bT_{1,1,1} \cup (\cup_{T\in \mathcal{T}_0} T_{l_1,l_2,l_3}), \tag{4.11}$$

where $R_1(G_1) = R_1(G_2) = R_1(G_3) = -1$.

From Theorems 3.4 and 3.5, we have that

$$R_5(G) = R_5(\zeta_n^1) = 12 = \sum_{i=1}^{3} R_5(G_i) + |B| + a + |\mathcal{T}_1| + 2|\mathcal{T}_2| + 3|\mathcal{T}_3|. \tag{4.12}$$

Recalling that $q(G) = p(G) + 2$, we distinguish the following subcases:

Subcase 1.3.1 $q(G_i) - p(G_i) = 1 (i = 1, 2, 3)$.

From Lemma 2.6, (4.5), (4.11) and (1) of Lemma 4.2, we arrive at $G_i \cong F_m (i = 2, 3, 4)$ and $a + b + |\mathcal{T}_1| + 2|\mathcal{T}_2| + 3|\mathcal{T}_3| = 1$. If $b = 0$, then we have, from (4.12), that $12 = 3R_5(F_m) + |B| + 1$, which contradicts to $R_5(F_m) = 4$. If $b = 1$, then we obtain, from (4.12), that $12 = 3R_5(F_m) + |B|$, which leads to $G = F_m \cup F_m \cup F_m \cup (\cup_{i \in A} C_i) \cup fD_4 \cup T_{1,1,1}$. From (1) of Lemma 3.10, Lemma 2.9 and Corollary 2.1, $\beta(\zeta_n^1) = \beta(G) = \beta(F_m)$, which contradict to $\beta(G) = \beta(\zeta_n^1) < \beta(F_m)$ by (2) of Lemma 3.11.

Subcase 1.3.2 $q(G_i) - p(G_i) = 1 (i = 1, 2)$, $q(G_3) = p(G_3)$.

Using Lemma 2.6, (4.5), (4.11) and (1) of Lemma 4.2, we obtain that $G_i \cong F_m (i = 1, 2)$, $G_3 \in \xi$ and $a = b = |\mathcal{T}_1| = |\mathcal{T}_2| = |\mathcal{T}_3| = 0$. By (4.12), we have $R_5(G_3) = 12 - 2R_5(F_m) - |B| = 4 - |B|$, which implies that $G = F_m \cup F_m \cup G_3 \cup (\cup_{i \in A} C_i) \cup fD_4$, $R_5(G_3) = 4$. From (1) of Lemma 3.3, it follows that $G_3 \in \{C_{n-1}(P_2)\} \cup \{Q_{1,1}\} \cup \{B_{n-5,1,1}\}$.

As stated above, we have, from Lemma 2.10 and Corollary 2.1, that $\beta(G) = \beta(G_3)$. From (2) of Lemma 3.11, we arrive at $\beta(\zeta_n^1) = \beta(G) < \beta(G_3)$. This is also a contradiction.

Case 2 $s_{-1} = 1$.

It follows, from (4.6), that $s_5 = 0$ and $s_1 + 2s_2 + 3s_3 + 4s_4 = 4$, which brings about the following subcases:

Subcase 2.1 $s_4 = 1, s_3 = s_2 = s_1 = 0$.

Without loss of generality, let

$$G = G_1 \cup G_2 \cup (\cup_{i \in A} C_i) \cup (\cup_{j \in B} D_j) \cup fD_4 \cup aK_1 \cup bT_{1,1,1} \cup (\cup_{T \in \mathcal{T}_0} T_{l_1,l_2,l_3}), \tag{4.13}$$

where $G_1 \in \{P_2, P_4, C_3\}$, $R_1(G_2) = -4$.

From Theorems 3.4 and 3.5, we obtain that

$$R_5(G) = R_5(\zeta_n^1) = 12 = \sum_{i=1}^{2} R_5(G_i) + |B| + a + |\mathcal{T}_1| + 2|\mathcal{T}_2| + 3|\mathcal{T}_3|. \tag{4.14}$$

We distinguish the following subcases:

Subcase 2.1.1 $G_1 \cong P_2$ or $G_1 \cong P_4$.

Recalling that $q(G) = p(G) + 2$, we obtain that $q(G_2) - p(G_2) \geq 3$. From Lemma 2.7 and (4.13), it follows that $q(G_2) - p(G_2) < 3$. It is a contradiction.

Subcase 2.1.2 $G_1 \cong C_3$.

It is obvious that $q(G_2) - p(G_2) \geq 2$ by (4.5) and (4.13). By Lemma 2.7, it follows that $q(G_2) - p(G_2) < 3$. Then $G_2 \in \theta$ and $a = b = |\mathcal{T}_1| = |\mathcal{T}_2| = |\mathcal{T}_3| = 0$, which implies that $R_5(G_2) = 15 - |B| \leq 15$. It contradicts to $G_2 \in \theta$ by Lemma 3.6.

Subcase 2.2 $s_4 = s_2 = 0, s_3 = s_1 = 1$.

Without loss of generality, we set

$$G = \left(\bigcup_{i=1}^{3} G_i\right) \cup (\cup_{i\in A} C_i) \cup (\cup_{j\in B} D_j) \cup f D_4 \cup a K_1 \cup b T_{1,1,1} \cup (\cup_{T\in \mathcal{T}_0} T_{l_1,l_2,l_3}), \tag{4.15}$$

where $G_1 \in \{P_2, P_4, C_3\}$, $R_1(G_1) = -1$, $R_1(G_2) = -3$.

Using Theorems 3.4 and 3.5, it follows that

$$R_5(G) = R_5(\zeta_n^1) = 12 = \sum_{i=1}^{3} R_5(G_i) + |B| + a + |\mathcal{T}_1| + 2|\mathcal{T}_2| + 3|\mathcal{T}_3| \tag{4.16}$$

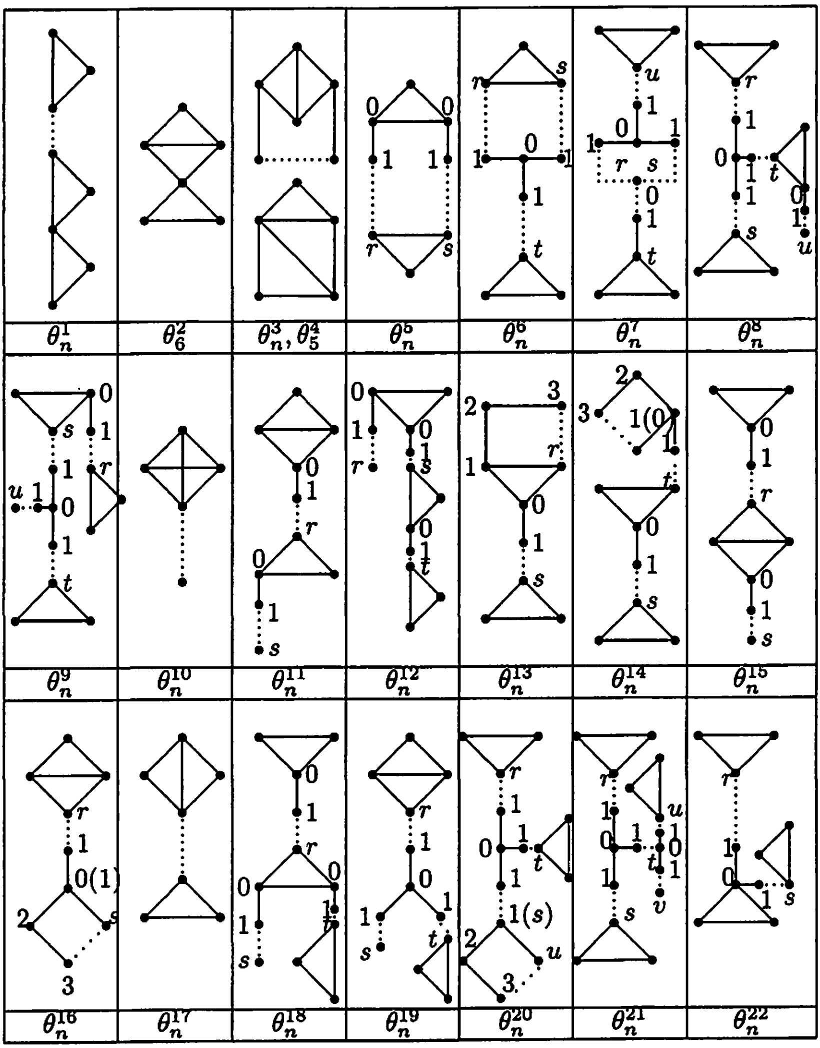


Figure 4 Family of θ

Subcase 2.2.1 $G_1 \cong P_2$ or $G_1 \cong P_4$.

Recalling that $q(G) = p(G) + 2$, we arrive at $G_2 \cong F_m$, $G_3 \in \zeta$ and $a = b = |\mathcal{T}_1| = |\mathcal{T}_2| = |\mathcal{T}_3| = 0$ by Lemmas 2.6, 4.2 and (4.15). Combining these with (4.16), we have if $G_1 \cong P_2$, then $R_5(G_3) = 9 - |B| \leq 9$, which contradicts to $G_3 \in \zeta$ by Corollary 3.3. If $G_1 \cong P_4$, then $R_5(G_3) = 10 - |B| \leq 10$, which also contradicts to $G_3 \in \zeta$ by Corollary 3.3.

Subcase 2.2.2 $G_1 \cong C_3$.

In terms of (4.5), we have the following three subcases to consider:

Subcase 2.2.2.1 $q(G_2) - p(G_2) = 1, q(G_3) - p(G_3) = 2$.

From Lemmas 2.6, 4.2, (4.5) and (4.15), $a + b + |\mathcal{T}_1| + 2|\mathcal{T}_2| + 3|\mathcal{T}_3| = 1$, $G_2 \cong F_m$ and $G_3 \in \zeta$. By (4.16), we have

if $b = 0$, then $R_5(G_3) = 14 - R_5(F_m) - |B| \leq 10 - |B| \leq 10$, which contradict to $G_3 \in \zeta$ by Corollary 3.3.

if $b = 1$, then $R_5(G_3) = 15 - R_5(F_m) - |B| \leq 11 - |B| \leq 11$, which contradicts to $G_3 \in \zeta$ by Corollary 3.3.

Subcase 2.2.2.2 $q(G_2) = p(G_2), q(G_3) - p(G_3) = 2$.

It is a obvious that $G_2 \in \xi$, $G_3 \in \zeta$ and $a = b = |\mathcal{T}_1| = |\mathcal{T}_2| = |\mathcal{T}_3| = 0$ by Lemmas 2.6 and (4.15). Using Corollary 3.1 and (4.16), we have $R_5(G_3) = 15 - R_5(G_2) - |B| \leq 11 - |B| \leq 11$, which contradicts to $G_3 \in \zeta$.

Subcase 2.2.2.3 $q(G_2) - p(G_2) = 1, q(G_3) - p(G_3) = 1$.

Applying Lemmas 2.6, 4.2 and (4.15), we have that $G_2 \cong F_m$, $G_3 \in \phi$ and $a = b = |\mathcal{T}_1| = |\mathcal{T}_2| = |\mathcal{T}_3| = 0$. Then $12 = -3 + R_5(F_m) + R_5(G_3) + |B|$, that is $R_5(G_3) = 11 - |B| \leq 11$, which contradicts to $G_3 \in \phi$ by Lemma 3.7.

Subcase 2.3 $s_4 = s_3 = s_1 = 0, s_2 = 2$.

Without loss of generality, let

$$G = (\bigcup_{i=1}^{3} G_i) \cup (\cup_{i\in A} C_i) \cup (\cup_{j\in B} D_j) \cup f D_4 \cup a K_1 \cup b T_{1,1,1} \cup (\cup_{T\in\mathcal{T}_0} T_{l_1,l_2,l_3}), \tag{4.17}$$

where $G_1 \in \{P_2, P_4, C_3\}$, $R_1(G_2) = R_1(G_3) = -2$.

In terms of Theorems 3.4 and 3.5, we arrive at

$$R_5(G) = R_5(\zeta_n^1) = 12 = \sum_{i=1}^{3} R_5(G_i) + |B| + a + |\mathcal{T}_1| + 2|\mathcal{T}_2| + 3|\mathcal{T}_3| \tag{4.18}$$

Subcase 2.3.1 $G_1 \cong P_2$ or $G_1 \cong P_4$.

It is obvious that $\sum_{i=2}^{3}(q(G_i) - p(G_i)) \geq 3$ by (4.5) and (4.17). From Lemmas 2.6 and 4.2, it follows that $\sum_{i=2}^{3}(q(G_i) - p(G_i)) \leq 2$. This is obviously a contradiction.

Subcase 2.3.2 $G_1 \cong C_3$.

Using Lemmas 2.1, 2.6, 4.2 and (4.17), it is not difficult to see that $q(G_i) - p(G_i) = 1 (i = 2, 3)$, which implies $G_2, G_3 \in \psi$ and $a = b = |\mathcal{T}_1| = |\mathcal{T}_2| = |\mathcal{T}_3| = 0$. Then $12 = -3 + R_5(G_2) + R_5(G_3) + |B|$. Hence $R_5(G_3) = 15 - R_5(G_2) - |B| \leq 7 - |B| \leq 7$, which contradicts to $G_3 \in \psi$ by Corollary 3.2.

Subcase 2.4 $s_4 = s_3 = 0$, $s_2 = 1$, $s_1 = 2$.

Without loss of generality, we set

$$G = (\bigcup_{i=1}^{4} G_i) \cup (\cup_{i \in A} C_i) \cup (\cup_{j \in B} D_j) \cup f D_4 \cup a K_1 \cup b T_{1,1,1} \cup (\cup_{T \in \mathcal{T}_0} T_{l_1, l_2, l_3}), \tag{4.19}$$

where $G_1 \in \{P_2, P_4, C_3\}$, $R_1(G_2) = R_1(G_3) = -1$, $R_1(G_4) = -2$.

From Theorems 3.4 and 3.5, we obtain the following equality:

$$R_5(G) = R_5(\zeta_n^1) = 12 = \sum_{i=1}^{4} R_5(G_i) + |B| + a + |\mathcal{T}_1| + 2|\mathcal{T}_2| + 3|\mathcal{T}_3| \tag{4.20}$$

Subcase 2.4.1 $G_1 \cong P_2$ or $G_1 \cong P_4$.

From Lemmas 2.6, 4.2 and (4.5), we obtain that $q(G_i) - p(G_i) = 1 (i = 2, 3, 4)$, which implies $G_i \cong F_m (i = 2, 3)$, $G_4 \in \psi$ and $a = b = |\mathcal{T}_1| = |\mathcal{T}_2| = |\mathcal{T}_3| = 0$. If $G_1 \cong P_2$, then $R_5(G_4) = 13 - 2R_5(F_m) - |B| \leq 5 - |B| \leq 5$, which contradicts to $G_4 \in \psi$ by Corollary 3.2. If $G_1 \cong P_4$, then $R_5(G_4) = 14 - 2R_5(F_m) - |B| \leq 6 - |B| \leq 6$, which also contradicts to $G_4 \in \psi$ by Corollary 3.2.

Subcase 2.4.2 $G_1 \cong C_3$.

By (4.5), the following three subcases will be discussed:

Subcase 2.4.2.1 $q(G_i) - p(G_i) = 1 (i = 2, 3, 4)$.

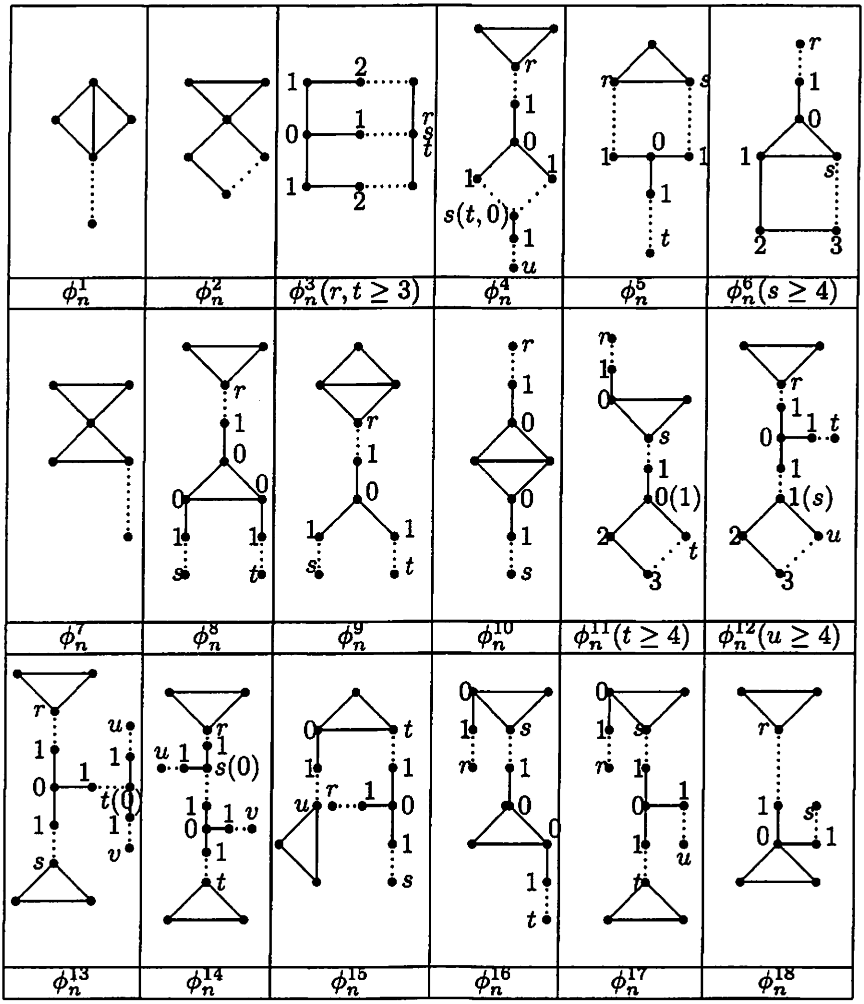
It is easy to see that $G_i \cong F_m (i = 2, 3)$, $G_4 \in \psi$ and $a + b + |\mathcal{T}_1| + 2|\mathcal{T}_2| + 3|\mathcal{T}_3| = 1$ by Lemmas 2.6, 4.2 and (4.19), which implies $|\mathcal{T}_3| = |\mathcal{T}_2| = 0$ and $0 \leq b \leq 1$. If $b = 0$, then we obtain, from (4.20), that $12 = -3 + 2R_5(F_m) + R_5(G_4) + |B| + 1$. Therefore $R_5(G_4) = 6 - |B| \leq 6$, which contradicts to $G_4 \in \psi$ by Corollary 3.2. We can get the same contradiction for the case of $b = 1$.

Subcase 2.4.2.2 $q(G_2) = p(G_2)$, $q(G_i) - p(G_i) = 1 (i = 3, 4)$.

It is obvious that $G_2 \in \xi$, $G_3 \cong F_m$, $G_4 \in \psi$ and $a = b = |\mathcal{T}_1| = |\mathcal{T}_2| = |\mathcal{T}_3| = 0$ by Lemmas 2.6, 4.2 and (4.19). From these together with (4.20), we have $12 = -3 + R_5(G_2) + R_5(F_m) + R_5(G_4) + |B|$, that is $R_5(G_4) = 11 - R_5(G_2) - |B| \leq 7 - |B| \leq 7$, which contradicts to $G_4 \in \psi$ by Corollary 3.2.

Subcase 2.4.2.3 $q(G_i) - p(G_i) = 1 (i = 2, 3)$, $q(G_4) = p(G_4)$.

By Lemmas 2.6, 4.2 and (4.19), we have $G_i \cong F_m (i = 2, 3)$, $G_4 \in \varphi$ and $a = b = |\mathcal{T}_1| = |\mathcal{T}_2| = |\mathcal{T}_3| = 0$. Combining these with (4.20), we have $12 = -3 + 2R_5(F_m) + R_5(G_4) + |B|$, that is $R_5(G_4) = 7 - |B| \leq 7$, which contradicts to $G_4 \in \varphi$ by Lemma 3.2.



Figue 5 Family of ϕ

Subcase 2.5 $s_4 = s_3 = s_2 = 0$, $s_1 = 4$.
Without loss of generality, let

$$G = (\bigcup_{i=1}^{5} G_i) \cup (\cup_{i\in A} C_i) \cup (\cup_{j\in B} D_j) \cup fD_4 \cup aK_1 \cup bT_{1,1,1} \cup (\cup_{T\in \mathcal{T}_0} T_{l_1,l_2,l_3}), \tag{4.21}$$

where $G_1 \in \{P_2, P_4, C_3\}$, $R_1(G_i) = -1 (i = 2, 3, 4, 5)$.

From Theorems 3.4 and 3.5, we arrive at

$$R_5(G) = 12 = \sum_{i=1}^{5} R_5(G_i) + |B| + a + |\mathcal{T}_1| + 2|\mathcal{T}_2| + 3|\mathcal{T}_3|. \tag{4.22}$$

Subcase 2.5.1 $G_1 \cong P_2$ or $G_1 \cong P_4$.

Recalling that $q(G) = p(G) + 2$, we have the following two cases to be considered:

Subcase 2.5.1.1 $q(G_i) - p(G_i) = 1 (i = 2, 3, 4, 5)$.

From Lemmas 2.6, 4.2 and (4.21), we get that $G_i \cong F_m (i = 2, \cdots, 5)$ and $a + b + |\mathcal{T}_1| + 2|\mathcal{T}_2| + 3|\mathcal{T}_3| = 1$. From these together with (4.22), if $b = 0$, then $12 = R_5(G_1) + 4R_5(F_m) + |B| + 1$. Hence $R_5(G_1) = -5 - |B| \leq -5$, which contradicts to $R_5(P_2) = -1$ and $R_5(P_4) = -2$. We can get the same contradition by the same reason for the case of $b = 1$.

Subcase 2.5.1.2 $q(G_i) - p(G_i) = 1 (i = 2, 3, 4)$, $q(G_5) = p(G_5)$.

It is easy to see that $G_i \cong F_m (i = 2, 3, 4)$, $G_5 \in \xi$ and $a = b = |\mathcal{T}_1| = |\mathcal{T}_2| = |\mathcal{T}_3| = 0$ by Lemmas 2.6, 4.2 and (4.21). From (4.22), it follows that $12 = R_5(G_1) + 3R_5(F_m) + R_5(G_5) + |B|$, which implies $R_5(G_1) = -R_5(G_5) - |B| \leq -8$ by Corollary 3.2. It contradicts to $R_5(P_2) = -1$ and $R_5(P_4) = -2$.

Subcase 2.5.2 $G_1 \cong C_3$.

We distinguish the following three cases by (4.5).

Subcase 2.5.2.1 $q(G_i) - p(G_i) = 1 (i = 2, 3, 4, 5)$.

From Lemmas 2.6, 4.2 and (4.21), we obtain that $G_i \cong F_m (i = 2, \cdots, 5)$ and $a + b + |\mathcal{T}_1| + 2|\mathcal{T}_2| + 3|\mathcal{T}_3| = 2$, which implies that $|\mathcal{T}_3| = 0$ and $0 \leq b \leq 2$. We only consider the case of $b = 2$, other cases can be similarly discussed. If $b = 2$, then $12 = -3 + 4R_5(F_m) + |B|$, which contradicts to $R_5(F_m) = 4$.

Subcase 2.5.2.2 $q(G_i) - p(G_i) = 1 (i = 2, 3, 4)$, $q(G_5) = p(G_5)$.

It is obvious that $G_i \cong F_m (i = 2, 3, 4)$, $G_5 \in \xi$ and $a + b + |\mathcal{T}_1| + 2|\mathcal{T}_2| + 3|\mathcal{T}_3| = 1$ by Lemmas 2.6, 4.2 and (4.21). If $b = 1$, then we obtain, from (4.22), that $12 = -3 + 3R_5(F_m) + R_5(G_5) + |B|$, which implies $R_5(G_5) \leq 3 - |B| \leq 3$. It contradicts to $G_5 \in \xi$. We can get the same contradition for the case of $b = 0$.

Subcase 2.5.2.3 $q(G_i) - p(G_i) = 1 (i = 2, 3)$, $q(G_i) = p(G_i) (i = 4, 5)$.

From Lemmas 2.6, 4.2 and (4.21), we obtain that $G_i \cong F_m (i = 2, 3)$, $G_4, G_5 \in \xi$ and $a = b = |\mathcal{T}_1| = |\mathcal{T}_2| = |\mathcal{T}_3| = 0$. In the light of Corollary 3.1, $R_5(G_4) \geq 4$. From these together with (4.22), $R_5(G_5) = 15 - 2R_5(F_m) - R_5(G_4) - |B| \leq 3 - |B|$, which contradicts to $G_5 \in \xi$ by Corollary 3.1.

Case 3 $s_{-1} = 2$.

It follows, from (4.6), that $s_1 + 2s_2 + 3s_3 + 4s_4 + 5s_5 = 5$, which brings about the following cases:

Subcase 3.1 $s_5 = 1$, $s_4 = s_3 = s_2 = s_1 = 0$.

Without loss of generality, we set

$$G = P_2 \cup G_1 \cup G_2 \cup (\cup_{i \in A} C_i) \cup (\cup_{j \in B} D_j) \cup f D_4 \cup a K_1 \cup b T_{1,1,1} \cup (\cup_{T \in \mathcal{T}_0} T_{l_1, l_2, l_3}), \tag{4.23}$$

where $G_1 \in \{P_4, C_3\}$, $R_1(G_2) = -5$.

Applying Theorems 3.4 and 3.5, it follows that

$$R_5(G) = R_5(\zeta_n^1) = 12 = -1 + \sum_{i=1}^{2} R_5(G_i) + |B| + a + |\mathcal{T}_1| + 2|\mathcal{T}_2| + 3|\mathcal{T}_3| \tag{4.24}$$

Subcase 3.1.1 $G_1 \cong P_4$.

Recalling that $q(G) = p(G) + 2$, we have that $q(G_2) - p(G_2) \geq 4$. By (3) of Lemma 2.7, we arrive at $q(G_2) - p(G_2) < 4$. Thus this products a contradiction.

Subcase 3.1.2 $G_1 \cong C_3$.

It is obvious that $q(G_2) - p(G_2) \geq 3$ by (4.5) and (4.23). By (3) of Lemma 2.7, we arrive at $q(G_2) - p(G_2) < 4$. Then $q(G_2) - p(G_2) = 3$, which implies $G_2 \in \tau$ by Lemma 2.6. From (4.24), it follows that $R_5(G_2) = 16 - |B| \leq 16$, which contradicts to Lemma 3.8.

Subcase 3.2 $s_5 = s_3 = s_2 = 0$, $s_4 = s_1 = 1$.

Without loss of generality, let

$$G = P_2 \cup (\bigcup_{i=1}^{3} G_i) \cup (\cup_{i \in A} C_i) \cup (\cup_{j \in B} D_j) \cup f D_4 \cup a K_1 \cup b T_{1,1,1} \cup (\cup_{T \in \mathcal{T}_0} T_{l_1,l_2,l_3}), \tag{4.25}$$

where $G_1 \in \{P_4, C_3\}$, $R_1(G_2) = -1$, $R_1(G_3) = -4$.

From Theorems 3.4 and 3.5, we arrive at

$$R_5(G) = R_5(\zeta_n^1) = 12 = -1 + \sum_{i=1}^{3} R_5(G_i) + |B| + a + |\mathcal{T}_1| + 2|\mathcal{T}_2| + 3|\mathcal{T}_3| \tag{4.26}$$

Subcase 3.2.1 $G_1 \cong P_4$.

By (4.5) and (4.25), we have that $\sum_{i=2}^{3}(q(G_i) - p(G_i)) \geq 4$. From Lemmas 2.6 and 2.7, it follows that $\sum_{i=2}^{3}(q(G_i) - p(G_i)) \leq 3$. Thus this brings about a contradiction.

Subcase 3.2.2 $G_1 \cong C_3$.

From Lemmas 2.6, 4.2, (4.5) and (4.25), we can obtain that

$$G = P_2 \cup C_3 \cup G_2 \cup G_3 \cup (\cup_{i \in A} C_i) \cup (\cup_{j \in B} D_j) \cup f D_4,$$

where $G_3 \in \theta$, $G_2 \cong F_m$. From (4.26), we arrive at $12 = -1 - 3 + R_5(F_m) + R_5(G_3) + |B|$, which leads to $R_5(G_3) = 12 - |B| \leq 12$, which contradicts to $G_3 \in \theta$ by Lemma 3.6.

Subcase 3.3 $s_5 = s_4 = s_1 = 0$, $s_3 = s_2 = 1$.

Without loss of generality, we set

$$G = P_2 \cup (\bigcup_{i=1}^{3} G_i) \cup (\cup_{i \in A} C_i) \cup (\cup_{j \in B} D_j) \cup f D_4 \cup a K_1 \cup b T_{1,1,1} \cup (\cup_{T \in \mathcal{T}_0} T_{l_1,l_2,l_3}), \tag{4.27}$$

where $G_1 \in \{P_4, C_3\}$, $R_1(G_2) = -2$, $R_1(G_3) = -3$.

Using Theorems 3.4 and 3.5, we obtain the following equality:

$$R_5(G) = R_5(\zeta_n^1) = 12 = -1 + \sum_{i=1}^{3} R_5(G_i) + |B| + a + |\mathcal{T}_1| + 2|\mathcal{T}_2| + 3|\mathcal{T}_3|. \tag{4.28}$$

Subcase 3.3.1 $G_1 \cong P_4$.

Recalling that $q(G) = p(G) + 2$, we have that $\sum_{i=2}^{3}(q(G_i) - p(G_i)) \geq 4$. From Lemmas 2.6, 2.7 and 4.2, $\sum_{i=2}^{3}(q(G_i) - p(G_i)) \leq 3$. Thus this brings about a contradiction.

Subcase 3.3.2 $G_1 \cong C_3$.

From (4.5) and (4.27), we obtain that $\sum_{i=2}^{3}(q(G_i)-p(G_i)) \geq 3$. Combining with Lemma 2.7, we have that $\sum_{i=2}^{3}(q(G_i)-p(G_i)) = 3$, which implies $G_2 \in \psi$, $G_3 \in \zeta$ and $a = b = |\mathcal{T}_1| = |\mathcal{T}_2| = |\mathcal{T}_3| = 0$. In terms of (4.28), we have that $12 = -1 - 3 + R_5(G_2) + R_5(G_3) + |B|$. By Corollary 3.2, $R_5(G_2) \geq 8$. Hence $R_5(G_3) \leq 8 - |B| \leq 8$, which contradicts to $G_3 \in \zeta$ by Corollary 3.3.

Subcase 3.4 $s_5 = s_4 = s_2 = 0$, $s_1 = 2$, $s_3 = 1$.

Without loss of generality, let

$$G = P_2 \cup (\bigcup_{i=1}^{4} G_i) \cup (\cup_{i\in A} C_i) \cup (\cup_{j\in B} D_j) \cup f D_4 \cup a K_1 \cup b T_{1,1,1} \cup (\cup_{T\in\mathcal{T}_0} T_{l_1,l_2,l_3}), \tag{4.29}$$

where $G_1 \in \{P_4, C_3\}$, $R_1(G_2) = R_1(G_3) = -1$, $R_1(G_4) = -3$.

From Theorems 3.4 and 3.5, we arrive at

$$R_5(G) = R_5(\zeta_n^1) = 12 = -1 + \sum_{i=1}^{4} R_5(G_i) + |B| + a + |\mathcal{T}_1| + 2|\mathcal{T}_2| + 3|\mathcal{T}_3|. \tag{4.30}$$

Subcase 3.4.1 $G_1 \cong P_4$.

Using Lemma 2.6, 4.2, (4.5) and (4.29), we have that $G_i \cong F_m (i = 2, 3)$, $G_4 \in \zeta$ and $a = b = |\mathcal{T}_1| = |\mathcal{T}_2| = |\mathcal{T}_3| = 0$. From these together with (4.30), we obtain that $12 = -1-2+|B|+2R_5(F_m)+R_5(G_4)$. Then $R_5(G_4) = 7-|B| \leq 7$, which contradicts to $G_4 \in \zeta$ by Corollary 3.3.

Subcase 3.4.2 $G_1 \cong C_3$.

Recalling that $q(G) = p(G) + 2$, we have the following three subcases to consider:

Subcase 3.4.2.1 $q(G_i) - p(G_i) = 1 (i = 2, 3)$ and $q(G_4) - p(G_4) = 2$.

It is easy to see that $G_2, G_3 \cong F_m$, $G_4 \in \zeta$ and $a+b+|\mathcal{T}_1|+2|\mathcal{T}_2|+3|\mathcal{T}_3| = 1$. If $b = 0$, then we obtain, from (4.30), that $12 = -1 - 3 + 2R_5(F_m) + R_5(G_4) + |B| + 1$. Thus $R_5(G_4) \leq 7 - |B| \leq 7$, which contradicts to $G_4 \in \zeta$. If $b = 1$, then we have, from (4.30), that $12 = -1-3+2R_5(F_m)+R_5(G_4)+|B|$. Hence $R_5(G_4) \leq 8 - |B| \leq 8$, which also contradicts to $G_4 \in \zeta$.

Subcase 3.4.2.2 $q(G_i) - p(G_i) = 1 (i = 2, 3, 4)$.

It is obvious that $G_i \cong F_m (i = 2, 3)$, $G_4 \in \phi$ and $a = b = |\mathcal{T}_1| = |\mathcal{T}_2| = |\mathcal{T}_3| = 0$ by (4.5), (4.29), Lemmas 2.6 and 4.2. Combining with (4.30), we have $12 = -1 - 3 + 2R_5(F_m) + R_5(G_4) + |B|$. Then $R_5(G_4) = 8 - |B| \leq 8$, which contradicts to $G_4 \in \phi$ by Lemma 3.7.

Subcase 3.4.2.3 $q(G_2) - p(G_2) = 1, q(G_3) = p(G_3)$ and $q(G_4) - p(G_4) = 2$.

Applying Lemmas 2.6, 4.2 and (4.29), we have $G_2 \cong F_m$, $G_3 \in \xi$, $G_4 \in \zeta$ and $a = b = |\mathcal{T}_1| = |\mathcal{T}_2| = |\mathcal{T}_3| = 0$. From these together with (4.30), $12 = -1 - 3 + |B| + R_5(F_m) + R_5(G_3) + R_5(G_4)$. By Corollary 3.1, $R_5(G_3) \geq 4$. Hence $R_5(G_4) = 12 - R_5(G_3) - |B| \leq 8 - |B| \leq 8$, which contradicts to $G_4 \in \zeta$ by Corollary 3.3.

Subcase 3.5 $s_5 = s_4 = s_3 = 0$, $s_2 = 2$, $s_1 = 1$.

Without loss of generality, we set

$$G = P_2 \cup (\bigcup_{i=1}^{4} G_i) \cup (\cup_{i \in A} C_i) \cup (\cup_{j \in B} D_j) \cup f D_4 \cup a K_1 \cup b T_{1,1,1} \cup (\cup_{T \in \mathcal{T}_0} T_{l_1, l_2, l_3}), \tag{4.31}$$

where $G_1 \in \{P_4, C_3\}$, $R_1(G_2) = -1$, $R_1(G_3) = R_1(G_4) = -2$.

Applying Theorems 3.4 and 3.5, it follows that

$$R_5(G) = R_5(\zeta_n^1) = 12 = -1 + \sum_{i=1}^{4} R_5(G_i) + |B| + a + |\mathcal{T}_1| + 2|\mathcal{T}_2| + 3|\mathcal{T}_3|. \tag{4.32}$$

Subcase 3.5.1 $G_1 \cong P_4$.

Recalling that $q(G) = p(G) + 2$, we know that $\sum_{i=2}^{4}(q(G_i) - p(G_i))) \geq 4$. From Lemmas 2.6 and 4.2, it follows that $\sum_{i=2}^{4}(q(G_i) - p(G_i)) \leq 3$. It is a contradiction.

Subcase 3.5.2 $G_1 \cong C_3$.

Applying (4.5), we have that $\sum_{i=2}^{4}(q(G_i) - p(G_i)) \geq 3$. From Lemmas 2.6 and 4.2, it follows that $\sum_{i=2}^{4}(q(G_i) - p(G_i)) \leq 3$. Then $\sum_{i=2}^{4}(q(G_i) - p(G_i)) = 3$, which leads to $G_2 \cong F_m$, $G_i \in \psi (i = 3, 4)$ and $a = b = |\mathcal{T}_1| = |\mathcal{T}_2| = |\mathcal{T}_3| = 0$. Combining these with (4.32), we arrive at $12 = -1 - 3 + R_5(F_m) + R_5(G_3) + R_5(G_4) + |B|$. By Corollary 3.2, $R_5(G_3) \geq 8$. Then $R_5(G_4) = 12 - R_5(G_3) - |B| \leq 4 - |B| \leq 4$, which contradicts to $G_4 \in \psi$.

Subcase 3.6 $s_5 = s_4 = s_3 = 0$, $s_2 = 1$, $s_1 = 3$.

Without loss of generality, let

$$G = P_2 \cup (\bigcup_{i=1}^{5} G_i) \cup (\cup_{i \in A} C_i) \cup (\cup_{j \in B} D_j) \cup f D_4 \cup a K_1 \cup b T_{1,1,1} \cup (\cup_{T \in \mathcal{T}_0} T_{l_1, l_2, l_3}), \tag{4.33}$$

where $G_1 \in \{P_4, C_3\}$, $R_1(G_2) = R_1(G_3) = R_1(G_4) = -1$, $R_1(G_5) = -2$.

From Theorems 3.2 and 3.3, we arrive at

$$R_5(G) = R_5(\zeta_n^1) = 12 = -1 + \sum_{i=1}^{5} R_5(G_i) + |B| + a + |\mathcal{T}_1| + 2|\mathcal{T}_2| + 3|\mathcal{T}_3|. \tag{4.34}$$

Subcase 3.6.1 $G_1 \cong P_4$.

From (4.5), we know that $\sum_{i=2}^{4}(q(G_i)-p(G_i)) \geq 4$. From Lemmas 2.6 and 4.2, it follows that $\sum_{i=2}^{4}(q(G_i) - p(G_i) \leq 4$. Then $\sum_{i=2}^{4}(q(G_i) - p(G_i)) = 4$, which implies $G_i \cong F_m(i = 2,3,4)$, $G_5 \in \psi$ and $a = b = |\mathcal{T}_1| = |\mathcal{T}_2| = |\mathcal{T}_3| = 0$. In the light of (4.34), we obtain that $R_5(G_5) = 15 - 3R_5(F_m) - |B| \leq 3 - |B| \leq 3$, which contradicts to $G_5 \in \psi$.

Subcase 3.6.2 $G_1 \cong C_3$.

Recalling that $q(G) = p(G) + 2$, the following three subcases will be discussed:

Subcase 3.6.2.1 $q(G_i) - p(G_i) = 1(i = 2,3,4,5)$.

From Lemmas 2.6, 4.2, (4.5) and (4.33), it follows that $G_i \cong F_m(i = 2,3,4)$, $G_5 \in \psi$ and $a + b + |\mathcal{T}_1| + 2|\mathcal{T}_2| + 3|\mathcal{T}_3| = 1$. If $b = 0$, then we obtain, from (4.34), that $12 = -1 - 3 + 3R_5(F_m) + R_5(G_5) + |B| + 1$. Hence $R_5(G_4) = 3 - |B| \leq 3$, which contradicts to $G_5 \in \psi$. If $b = 1$, then we have, from (4.34), that $12 = -1 - 3 + 3R_5(F_m) + R_5(G_5) + |B|$. Then $R_5(G_4) = 4 - |B| \leq 4$, which also contradicts to $G_5 \in \psi$.

Subcase 3.6.2.2 $q(G_i) - p(G_i) = 1(i = 2,3,5)$, $q(G_4) = p(G_4)$.

It is easy to see that $G_i \cong F_m(i = 2,3)$, $G_4 \in \xi$, $G_5 \in \psi$ and $a = b = |\mathcal{T}_1| = |\mathcal{T}_2| = |\mathcal{T}_3| = 0$ by Lemmas 2.6, 4.2, (4.5) and (4.33). From (4.34), we obtain that $12 = -1 - 3 + 2R_5(F_m) + R_5(G_4) + R_5(G_5) + |B|$, which results in $R_5(G_5) = 8 - R_5(G_4) - |B| \leq 4 - |B| \leq 4$ by Corollary 3.1. It contradicts to $G_5 \in \psi$ by Corollary 3.2.

Subcase 3.6.2.3 $q(G_i) - p(G_i) = 1(i = 2,3,4)$ and $q(G_5) = p(G_5)$.

From Lemmas 2.6, 4.2, (4.5) and (4.33), it follows that $G_i \cong F_m(i = 2,3,4)$, $G_5 \in \varphi$ and $a = b = |\mathcal{T}_1| = |\mathcal{T}_2| = |\mathcal{T}_3| = 0$. Combining these with (4.34), we arrive at $12 = -1 - 3 + 3R_5(F_m) + R_5(G_5) + |B|$. Then $R_5(G_5) = 4 - |B| \leq 4$, which contradicts to $G_5 \in \varphi$ by Lemma 3.2.

Subcase 3.7 $s_5 = s_4 = s_3 = s_2 = 0$, $s_1 = 5$.

Without loss of generality, let

$$G = P_2 \cup (\bigcup_{i=1}^{6} G_i) \cup (\cup_{i \in A} C_i) \cup (\cup_{j \in B} D_j) \cup f D_4 \cup a K_1 \cup b T_{1,1,1} \cup (\cup_{T \in \mathcal{T}_0} T_{l_1,l_2,l_3}), \tag{4.35}$$

where $G_1 \in \{P_4, C_3\}$, $R_1(G_i) = -1(i = 2, \cdots, 6)$.

From Theorems 3.4 and 3.5, we arrive at

$$R_5(G) = 12 = -1 + \sum_{i=1}^{6} R_5(G_i) + |B| + a + |\mathcal{T}_1| + 2|\mathcal{T}_2| + 3|\mathcal{T}_3|. \tag{4.36}$$

We only consider the case of $G_1 \cong P_4$, the case of $G_1 \cong C_3$ can be similarly discussed.

Subcase 3.7.1 $q(G_i) - p(G_i) = 1 (i = 2, 3, \cdots, 6)$

It is obvious that $G_i \cong F_m (i = 2, 3, \cdots, 6)$ and $a+b+|\mathcal{T}_1|+2|\mathcal{T}_2|+3|\mathcal{T}_3| = 1$ by Lemmas 2.6, 4.2, (4.5) and (4.35). If $b = 0$, then we obtain, from (4.36), that $12 = -1 - 2 + 5R_5(F_m) + |B| + 1$, which contradicts to $R_5(F_m) = 4$. We can get the same contradiction for the case of $b = 1$.

Subcase 3.7.2 $q(G_i) - p(G_i) = 1 (i = 2, 3, 4, 5)$, $q(G_6) = p(G_6)$.

From Lemmas 2.6, 4.2 and (4.35), we arrive at $G_i \cong F_m (i = 2, 3, 4, 5)$, $G_6 \in \xi$ and $a = b = |\mathcal{T}_1| = |\mathcal{T}_2| = |\mathcal{T}_3| = 0$. From these together with (4.36), $12 = -1 - 2 + 4R_5(F_m) + R_5(G_6) + |B|$, which leads to $R_5(G_6) = -1 - |B| \leq -1$. It contradicts to $G_6 \in \xi$ by Corollary 3.1.

This completes the proof of the theorem. □

Corollary 4.1. *If $n \geq 7$, graph ζ_n^1 is adjoint uniqueness if and only if $n \neq 13$.*

Corollary 4.2. *If $n \geq 7$, the chromatic equivalence class of $\overline{\zeta_n^1}$ only contains the complements of graphs that are in Theorem 4.2.*

Corollary 4.3. *If $n \geq 7$, graph $\overline{\zeta_n^1}$ is chromatic uniqueness if and only if $n \neq 13$.*

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