Planar graphs without 5- and 8-cycles and adjacent triangles are 3-colorable

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Abstract. In this paper we prove that every planar graph without 5- and 8-cycles and without adjacent triangles is 3-colorable.

1 Introduction

In 1959 Grötsch [9] proved that every planar graph without 3-cycles is 3-colorable. Steinberg [13] conjectured that every planar graph without 4- and 5-cycles is 3-colorable. In fact, there exist 4-critical planar graphs which have only 4-cycles but no 5-cycles or only 5-cycles but no 4-cycles [1]. In 1990, Erdős proposed the following relaxed conjecture: every planar graph without cycles of size $\{4, 5, ..k\}, k \geq 5$, is 3-colorable. Abbott and Zhou [1] proved that the above conjecture holds for $k = 11$. Borodin [3] improved the result by showing that the result holds for $k = 10$. Borodin [2] and Sanders and Zhao [12] further improved the result showing that $k = 9$. To date, the best known result is by Borodin et al [4], where it is shown that any planar graph without cycles of length in $\{4, 5, 6, 7\}$ is 3-colorable. Xiaofang, Chen and Wang [15] showed that a planar graph without cycles of length $4, 6, 7$ and 8 is 3-colorable. Chen, Raspaud and Wang [8] showed that a planar graph without cycles of length $4, 6, 7$ and 9 is 3-colorable. Zhang and Wu [17] showed that every planar graph without $4, 5, 6$ and 9-cycles is 3-choosable. Wang and Chen [14] proved that every planar graph without 4, 6, and 8-cycles and without adjacent triangles is 3-colorable. We showed recently [11] that planar graphs without 4, 5, and 8-cycles are 3-colorable. In this article, we strengthen the result by showing that planar graphs without 5- and 8-cycles and without adjacent triangles are 3-colorable.

Another problem somewhat related to Steinberg's conjecture is the Havel's conjecture [10]. In 1969, Havel [10] posed the following problem:

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Does there exist a constant d such that every planar graph with the minimum distance between triangles at least d is 3-colorable? Some of the recent results on Havel's problems are that every planar graph without 3-cycles at distance less than d and without 5-cycles is 3-colorable ($d = 4$ [6] and $d = 3$ [5], [16]). Borodin et al [7] proved that a planar graph without adjacent triangles and without 5- and 7-cycles is 3-colorable. In this paper, we intend to prove the following result:

Theorem 1. *Every planar graph without* 5- *and* 8-*cycles and without adjacent triangles is* 3-*colorable.*

We use $\mathcal{G}$ to denote the class of planar graphs without 5- and 8-cycles and without adjacent triangles. Let C_i denote an i-cycle. A 6-cycle is bad if the interior of the cycle has a partition into 4-cycles. A 9-cycle is bad if the interior of the cycle has a partition into 6- and 3-cycles or a partition into 4- and 7-cycles. We call a cycle of length $\{3, 4, 6, 7, 9, 10\}$ that is not bad as good cycle. We would prove a stronger version of Theorem 1 as given below:

Theorem 2. *Let G be a planar embedding of a graph in $\mathcal{G}$. Let D be an arbitrary good cycle of G. Then every proper* 3-*coloring of D can be extended to a* 3-*coloring of the whole graph G.*

Assuming that Theorem 2 holds, we can easily establish Theorem 1. Suppose $G \in \mathcal{G}$, namely, G contains no 5- and 8-cycles and adjacent triangles. If G does not contain any 3-cycle, then G is 3-colorable by Grötsch theorem [9]. Hence, there is a 3-cycle (C_3). There cannot be any internal chord of C_3 and it has a proper 3-coloring (ϕ). By Theorem 2, ϕ can be extended to both inside and outside of C_3 to make a proper 3-coloring of G.

Only simple graphs are considered in this paper. A plane graph is a particular drawing of a planar graph in the Euclidean plane. For a plane graph G, we denote its vertices, edges, faces and maximum degree by $V(G)$, $E(G)$, $F(G)$, and $\Delta(G)$ respectively. We use k-vertex, k^+-vertex, k^--vertex, $>k$-vertex, $<k$-vertex to denote a vertex of degree k, at least k, or at most k, greater than k, less than k respectively. Similarly, we can define k-face, k^+-face, k^--face, $>k$-face, $<k$-face. We say that two cycles or faces are adjacent if they share at least one common edge. For $f \in F(G)$, we use $b(f)$ to denote the boundary walk of f. If $u_1, u_2, \ldots, u_n$ are the boundary vertices of f in the clockwise order, we write $f = [u_1 u_2 \ldots u_n]$. Given two vertices u and v in a cycle C, let $C[u, v]$ denote the path of C in the clockwise order from u to v (including u and v), and let $C(u, v) = C[u, v] \setminus \{u, v\}$. A cycle

C in a plane graph G is called separating if $int(C) \neq \emptyset$ and $ext(C) \neq \emptyset$, where $int(C)$ and $ext(C)$ represent the sets of vertices located inside and outside C, respectively.

2 Proof of Theorem 2

Assume that G is a minimal (least number of vertices and edges) counterexample to Theorem 2. Without loss of generality, assume that the outside face f_0 is of degree $3, 4, 6, 7, 9$ or 10 such that a proper 3-coloring ϕ of the boundary vertices of f_0 cannot be extended to the whole graph G. This implies that there exists at least one vertex in the interior of $b(f_0)$.

We write C as the boundary walk of f_0, i.e., $C = b(f_0)$. Other faces in G different from f_0 are called the internal faces. The vertices in C are called the outer vertices and other vertices the internal vertices. An internal 3-vertex incident to a 3-face is called bad. It is easy to note that if $v \in int(C)$ and C is good, then C cannot become bad in $G - v$.

Claim 1. *G does not contain a separating good cycle.*

Proof. Suppose that G has such a separating cycle C_i. Then we can extend ϕ to $G - int(C_i)$ by the minimality of G. Subsequently, we delete the (possible) chords from C_i and extend the 3-coloring of C_i induced by ϕ to $G - ext(C_i)$ (this is possible due to the minimality of G). □

Claim 2. *G is 2-connected.*

Proof. Assume that C contains a cut vertex u. Assume that B is an end block with a cut vertex $u \in V(G) \setminus V(C)$. Due to minimality of G, we can extend ϕ to $G - (B - u)$, then 3-color B, and thus obtain an extension of ϕ to G. □

Claim 3. *Each 2-vertex in G belongs to C; no 2-vertex in C is incident to a 3-face.*

Proof. Let G contains a 2-vertex $v \in V(G) \setminus V(C)$. Let e be one of the edges incident on v. Then we have a 3-coloring f of $G - e$. If the end-vertices of e receive different colors, adding e does not create any violation. On the other hand, if the end-vertices of e have the same color, we can give v the third color (i.e., different from the colors of its neighbors) and add the edge e. thus we have a valid 3-coloring of G. So G cannot be a counter example.

If a 2-vertex v in C is incident to a 3-face, we can extend ϕ to $G-e$ (due to minimality of G) and then recolor v with a color different from the colors of its neighbors in G. □

Claim 4. *No good cycle of length* $\{4, 6, 7\}$ *in* G *has a non-triangular chord. In particular, if* C *is a good cycle and the boundary of the external face, it has no chord at all.*

Proof. If G contains a cycle of length 4 to 7 with a non-triangular chord, then it is easy to show that G must contain two adjacent triangles or a 5-cycle or C is bad, contradicting the assumption. Suppose that C has a chord e. If e cuts a 3-cycle C_3 from C, then C_3 forms a 3-face by Claim 1, which contradicts Claim 3. Otherwise, it follows that $|C| = 9$, or 10 by the previous argument.

Assume that $|C| = 9$. Since G contains no 5, 8-cycles, e cuts C into two cycles $C^1 = C_4$ and $C^2 = C_7$. In this case, C is a bad cycle, contradicting the assumption.

Assume that $|C| = 10$. Since G contains no 5, 8-cycles, e cuts C into two cycles $C^1 = C_6$ and $C^2 = C_6$. If both $int(C^1)$ and $int(C^2)$ are empty, then it is straightforward to derive that G is 3-colorable. Otherwise, at least one of C^1 and C^2 is a separating cycle, which contradicts Claim 1. □

Claim 5. *No* 3*-face is adjacent to a* k*-face for* $k = 3, 4, 7$.

Proof. Suppose that G contains a 3-face f adjacent to a k-face $f' = [v_1 v_2 \cdots v_k]$ for some $k \in \{3, 4, 7\}$. Note that $k \neq 3$ by assumption. When $k = 4$, then either there is a 5-cycle or there are two adjacent triangles.

Assume that $k = 7$. If f' and f have two common boundary edges, then G has an internal 2-vertex (contradicting Claim 3) or there is a 5-cycle. So we may suppose that $f = [v_1 u v_2]$. If u does not belong to $b(f')$, then a 8-cycle $u v_2 \cdots v_8 v_1 u$ is present in G, which is impossible. So, $u \in b(f')$. If $u = v_3$ then G has an internal 2-vertex (contradicting Claim 3) or there is a 5-cycle. If $u = v_4$, a 5-cycle $v_1 v_4 v_5 v_6 v_7 v_1$ is established. If $u = v_5$, a 5-cycle $v_1 v_2 v_3 v_4 v_5 v_1$ is established. We always get a contradiction. We can give a similar proof for $u = v_6$ or $u = v_7$. This proves Claim 5. □

Claim 6. *No* 4*-face can be adjacent to a* 6*-face.*

Proof. If two such faces have two or three common edges, it is easy to show that there is a 5-cycle or an internal 2-vertex. So, suppose that there are two such adjacent faces $f = [v_1 v_2 v_3 v_4]$ and $f' = [v_1 v_2 u_1 u_2 u_3 u_4]$ with $v_1 v_2$ as a common edge. Then there is a 8-cycle if u_1, u_2, u_3 and u_4 do not coincide with other vertices of f. If any of these vertices coincides with other vertices of f, then either there is a contradiction to Claim 1, or there is a 5-cycle or there is an internal 2-vertex. □

Claim 7. *Consider a 3-vertex v. In this case, v cannot be incident with three 4-faces, provided there is a vertex that does not belong to these faces.*

Proof. In these cases, it is easy to establish contradiction to Claim 1. □

Claim 8. *There cannot be three consecutive 4-faces adjacent to a face.*

Proof. In this case, there is a cycle of length 8, contradicting assumption. □

Claim 9. *Let C be a good cycle. For $v_1, v_2 \in C$ and $x \notin C$, if $xv_1, xv_2 \in E(G)$, then $v_1v_2 \in E(C)$.*

Proof. Assume on the contrary that v_1v_2 does not belong to $E(C)$. Let l denote the number of edges in sector $C[v_1, v_2]$ i.e., $|C[v_1, v_2]| = l \leq |C[v_2, v_1]|$. Then $2 \leq l \leq 5$, by $|C| \leq 10$. Let $C^1 = C[v_1, v_2] \cup v_2xv_1$ and $C^2 = C[v_2, v_1] \cup v_1xv_2$. Then C^1 is an $(l+2)$-cycle and C^2 is a $(|C|-l+2)$-cycle. Since G contains no 5, and 8-cycles, $l \neq 3$.

Assume that $l = 2$. Then C^1 is a 4-cycle and C^2 is a $|C|$-cycle. Thus, $|C| \neq 5, 8$. By Claim 1, neither C^1 nor C^2 is separating, if it is good. As $d(x) \geq 3$ and there are no two adjacent triangles, C^2 must have non-triangular chords. In that case, by Claim 4, C^2 is a 9-cycle. If C^2 is good, the internal non-triangular chord divides C^2 into two cycles of length 4 and 7, and hence C^2 is bad. This is a contradiction. If C^2 is bad (cycle partitioned into 6 and 3-cycles) then as $d(x) \geq 3$, C^1 along with an adjacent 6-cycle (inside C^2) will create an 8-cycle in G. If C^2 is bad and partitioned into 4 and 7-cycles, C^1 along with an adjacent 6-cycle (inside C^2) will make C a bad cycle. This is a contradiction to the assumption.

Assume that $l = 4$. Then C^1 is a 6-cycle and C^2 is a $(|C| - 2)$-cycle. Thus, $|C| \neq 7, 10$. By Claim 1, neither C^1 nor C^2 is separating. When $|C| = 8$, $|C^2| = 6$. It is easy to see that C^2 cannot have a chord in this case without violating assumption that there is no cycle in $\{5, 8\}$ and Claim 6. The same holds true for C^1. When $|C| = 9$, $|C^2| = 7$. It is easy to see that C^2 cannot have a chord in this case without violating assumption that there is no cycle in $\{5, 8\}$. C^2 cannot have a non-triangular chord due to Claim 4. If C^2 has a triangular chord then there is an 8-cycle, contradicting assumption.

Assume that $l = 5$. C^1 is a 7-cycle and C^2 is a $(|C| - 3)$-cycle. Thus, $|C| \neq 8$. By Claim 1, neither C^1 nor C^2 is separating if it is good. When $|C| = 7$, $|C^2| = 4$. We have already seen in earlier paragraphs that there is a contradiction when $|C^1| = 4$, $|C^2| = 7$. By symmetry, there is a contradiction in the current case too. When $|C| = 9$, $|C^2| = 6$. We have already seen in earlier paragraphs that there is a contradiction when $|C^1| = 6$, $|C^2| = 7$. Again, by symmetry, there is a contradiction in the current case too. □

Now, we shall make G into smaller graphs by identifying vertices. In doing so, we should be sure that we do not

i. identify two vertices of C (because then C is not a cycle anymore),
ii. create an edge between two vertices of C colored the same (for otherwise our precoloring ϕ of C would be destroyed),
iii. create loops,
iv. create multiple edges,
v. create adjacent triangles,
vi. create cycles of length 5 or 8, and
vii. make C a bad cycle.

Claim 10. *G has no 4-cycle other than C.*

Proof. By Claim 1, G has no separating 4-cycle. Of course, G has no 4-cycle with just one edge inside. So suppose $f = wxyz$ is a face inside C. Identifying x with z within f cannot violate (i). Suppose $x, z \in C$. Clearly, x and z are not consecutive along C as otherwise, it violates assumption of no two adjacent triangles. This implies by Claim 9 that none of w and y can be internal. By Claim 4, no edge of f can be a chord of C. It follows from Claim 5 that the only obstacle for (i) is the trivial case of $G = C = wxyz$.

Next suppose (ii) is an obstacle for identifying x with z. Without loss of generality, $x \in C$, $z \notin C$, and there is an edge zv_i such that $v_i \in C$, where v_i is not adjacent to x along C. If y is on C, by Claim 9, it must be adjacent to v_i. Similarly, if w is on C, by Claim 9, it must be adjacent to v_i. This implies adjacent triangles. Hence, both w and y must be internal. Now $int(C) \setminus \{w, y, z\}$ is partitioned into the closed interiors of three cycles: $C^1 = v_i zyxP^1$, $C^2 = v_i zwxP^2$, and $F = wxyz$, where $C = P^1 \cup P^2$ and $6 \leq |C^1| \leq |C^2|$. If $|C^1| = 6$, then $|C^2| = |C|$. We have a 6-cycle (C^1) adjacent to a 4-cycle (F), implying a 8-cycle, a contradiction to the assumption. If $|C^1| = 9$, and hence $|C^2| = 6$, there is a contradiction by symmetry to the earlier case. If $|C^1| = 7$, then we have a separating 7-cycle $C^1 \cup F \setminus \{y\}$, contradicting Claim 1.

The property (iii) follows from the absence of adjacent 3-cycles in G; (iv) from Claim 5; and (v) due to the absence of 3-cycles adjacent to 4-cycles in G.

Suppose we have created a 5- or 8-cycle $C' = xv_1 \cdots v_k$, where y is non-strictly inside C'. Hence, $k \in \{4, 7\}$. If $k = 4$, then y cannot actually coincide with any one of v_i's because then there is a 5-cycle in G. But then there is a separating 7-cycle contradicting Claim 1.

If $k = 7$, y must coincide with one of the v_i's, else there is a separating cycle $(zwxv_1 \cdots v_k)$ of length 10. If y coincides with one of the v_i's, then there is 4-face adjacent to a 6-face contradicting Claim 6.

Finally, suppose collapsing the 4-face f by identifying x with z makes C bad. This means C is a 6- or 9-cycle. Let S' be a bad partition of G'. If $x * z$ is a vertex inside a cell of S', then S' is also a bad partition of C in G, implying C as bad, a contradiction. So suppose $x * z$ is a vertex of S' in G'. There can be at most two such cells created, say C_y and C_w denoting a cell of S' that contains y and w respectively nonstrictly inside. The cell C_y cannot be 3-cell, as then there is a 5-cycle in G. If y is a vertex of C_y and the cell is a 6-cell, there is a 8-cycle in G.

Similarly, cell C_w cannot be a 3- or 6-cell. Hence, the identification of vertices cannot make C a bad 9-cycle partitioned into 6- and 3-cycles. So let us assume that C is a bad 9-cycle (partitioned into 4- and 7-cycles) or a bad 6-cycle (i.e., partitioned into 4-cycles). In both the cases, it is easy to see that either there is a bad 6-cycle (three 4-cycles mutually adjacent) in G before the identification or there is a 8-cycle (three 4-cycles pairwise adjacent but not mutually adjacent to the other two) in G, contradicting assumption. Hence Claim 10 is proved. □

Claim 11. *Let C be a good cycle. For $v_1, v_2 \in C$, if $v_1x, xy, yv_2 \in E(G)$ and $x, y \in int(C)$, then $v_1v_2 \in E(C)$.*

Proof. Assume on the contrary that v_1v_2 does not belong to $E(C)$. Let l denote the number of edges in sector $C[v_1, v_2]$ i.e., $|C[v_1, v_2]| = l \leq |C[v_2, v_1]|$. Then $2 \leq l \leq 5$, by $|C| \leq 10$. Let $C^1 = C[v_1, v_2] \cup v_2xyv_1$ and $C^2 = C[v_2, v_1] \cup v_2yxv_1$. Then C^1 is an $(l+3)$-cycle and C^2 is a $(|C| - l + 3)$-cycle.

Assume that $l = 2$. Then C^1 is a 5-cycle, contradicting assumption.

Assume that $l = 3$. Then C^1 is a 6-cycle and C^2 is a $|C|$-cycle. Let us assume that C^1 is good. So it cannot be separating. First assume that C^1 does not have any chord. If C^2 is good, it cannot be separating by Claim 1. Hence, as $d(x), d(y) \geq 3$, there must be at least two chords of C^2. If

$|C^2| = 6$, there is a 5- or 8-cycle (contradicting the assumption) or there is a 6-cycle adjacent to a 4-cycle, contradicting Claim 6. If $|C^2| = 7$, either there is a 5-cycle or there is a 6-cycle adjacent to a 4-cycle, contradicting Claim 6. When $|C^2| = 9$, either there is a 5- or 8-cycle (contradiction to the assumption) or there is a 6-cycle adjacent to a 4-cycle (contradicting Claim 6), or C is a bad 9-cycle (contradiction to the assumption of the Claim) or there is a separating good cycle of length 9. Finally if $|C^2| = 10$, again either there is a 5- or 8-cycle or there is a 6-cycle adjacent to a 4-cycle. If C^2 is bad, then we cannot have $d(x) \geq 3$ and $d(y) \geq 3$, contradicting Claim 3. So we assume that C^1 has an internal chord. The only possible chord divides it into two 4-faces. Let us assume that C^2 is good. Hence, as $d(x), d(y) \geq 3$, there must be at least one chord of C^2 with one end at x or y. If $|C^2| = 6$, there is a 5- or 8-cycle (contradicting the assumption) or C is partitioned into 4-cycles, making C bad. If $|C^2| = 7$, there is a 5-cycle contradicting the assumption. When $|C^2| = 9$, either there is a 5- or 8-cycle or there is a separating good cycle of length 9 or C is a bad 9-cycle (contradiction to the assumption of the Claim). Finally if $|C^2| = 10$, again either there is a 5- or 8-cycle or there is a 6-cycle adjacent to a 4-cycle, contradicting Claim 6. If C^2 is bad, then we have 6-cycle adjacent to a 4-cycle, contradicting Claim 6.

Let us assume that C^1 is bad. If $|C^2| = 6$, there is a 5- or 8-cycle (contradicting the assumption) or there is a 6-cycle adjacent to a 4-cycle, contradicting Claim 6 or C is partitioned into 4-cycles, making C bad. If $|C^2| = 7$, either there is a 5-cycle or there is a 6-cycle adjacent to a 4-cycle, contradicting Claim 6. When $|C^2| = 9$, either there is a 5- or 8-cycle or there is a separating good cycle of length 9 or C is a bad 9-cycle (contradiction to the assumption of the Claim). Finally if $|C^2| = 10$, again either there is a 5- or 8-cycle or there is a 6-cycle adjacent to a 4-cycle. If C^2 is bad, then we cannot have $d(x) \geq 3$ and $d(y) \geq 3$, contradicting Claim 3.

When $l = 4$, C^1 is a 7-cycle and C^2 is a $|C| - 1$ cycle. By Claim 6, neither C^1 nor C^2 is separating(unless bad). First assume that C^1 does not have any chord. If C^2 is good, it cannot be separating by Claim 1. Hence, as $d(x), d(y) \geq 3$, there must be at least two chords of C^2. There are two possibilities of good C^2: a 6-cycle or a 9-cycle. In both the cases, we can establish that there is a cycle of length in $\{5, 8\}$. When C^2 is bad, there is a contradiction as either $d(x) = 2$ or $d(y) = 2$ or there is 8-cycle C is bad. Next we assume that C^1 has an internal chord. The only possible chord divides it into 6- and 3-faces. By Claim 1, C^2 (when good) cannot be separating. Let us assume that C^2 is good. Hence, as $d(x), d(y) \geq 3$, there must be at least one chord of C^2 with one end at x or y. If $|C^2| = 6$ and is good, there is a 5- or 8-cycle (contradicting the assumption). If C^2 is bad, then we have 6-cycle adjacent to a 4-cycles, hence a 8-cycle contradicting assumption. When $|C^2| = 9$ and C^2 is good, either there is a 5- or 8-cycle. If C^2 is bad, then either we have a 6-cycle adjacent to a 4-cycles, hence a 8-cycle (contradicting assumption).

When $l = 5$, then C^1 is a 8-cycle, a contradiction. □

Claim 12. *G has no 6-faces other than C.*

Proof. Suppose $f = wxyzpq$ is a face inside C. By Claim 4, f has at least one internal vertex. Let y be an internal vertex. Identifying x with z within f cannot violate (i). Suppose $x, z \in C$. Clearly, x and z are not consecutive along C as otherwise, it violates assumption of no 5-cycle. This implies by Claim 9 that y cannot be internal, a contradiction.

Next suppose (ii) is an obstacle for identifying x with z. W.l.o.g., $x \in C, z \notin C$, and there is an edge zv_i such that $v_i \in C$, where v_i is not adjacent to x along C. If w is on C, by Claim 11, it must be adjacent to v_i. This creates a 5-cycle ($zpqwv_i$), contradicting assumption. Similarly p cannot be on C. If q is on C, by Claim 11, q is adjacent to v_i. In this case, f along with the 4-cycle $zpqv_i$ creates a 8-cycle, contradicting assumption. Hence, all of y, p, q and w must be internal. Now $int(C)$ can be partitioned into the closed interiors of three cycles: $C^1 = v_i zyxP^1$, $C^2 = v_i zpqwxP^2$, and $F = wxyzpq$, where $C = P^1 \cup P^2$. Due to Claim 11, either $|C^1| = 4$ or $|C^2| = 6$. For the former case, there is a 6-face adjacent to a 4-face, contradicting Claim 6. For the latter case, $C^2 \cup F \setminus \{w, p, q\}$ is a separating 4-cycle, contradicting Claim 1. The property (iii) follows from the absence of adjacent 5-cycles in G. The property (iv) is true else there is a separating good cycle of length at most 6 contradicting Claim 1. The property (v) is true due to the absence of 8-cycles in G. Suppose we have created a 5- or 8-cycle $C' = xv_1 \cdots v_k$, where $y \in int(C')$ and $k \in \{4, 7\}$. If $k = 4$, then there is a separating 7-cycle if y does not belong $b(C')$. However, y cannot actually coincide with one of v_i's as then there is a 5-cycle in G or f is adjacent to a 4-cycle contradicting Claim 6. If $k = 7$ and y does not coincide with on the of v_i's, there is a separating cycle ($zwxv_1 \cdots v_k$) of length 10, contradicting Claim 1. If y coincides with one of v_i's, then the only possible case without creating a 5- or 8-cycle is when y coincides with v_2 or v_6. In both the cases there is a 3-cycle incident at y. Also, due to Claim 11, x and v_i are adjacent. Hence there is a 4-cycle which is adjacent to the 3-cycle incident at y. This contradicts Claim 5. Finally, suppose collapsing the 6-face f by identifying x with z makes C bad. This means C is a 6 or 9-cycle. Let S' be a bad partition of G'. If $x * z$ is a vertex inside a cell of S', then S' is also a bad partition of C in G, implying C as bad, a contradiction. So suppose $x * z$ is a vertex of S' in G'. C_y (C_w) denotes a cell of S' that contains $y(w)$ nonstrictly inside. There can be at most two such cells created, C_y and C_w

as described above. The cell C_y cannot be 3-cell, as then there is a 5-cycle in G. If the cell is a 6-cell, then there is a 8-cycle in G. The cell C_w cannot be 3-cell, as then there is a 5-cycle in G. If the cell is a 6-cell, then there is a 8-cycle in G. Hence, the identification of vertices cannot make C a bad 9-cycle partitioned into 6- and 3-cycles. So let us assume that C is a bad 9-cycle (partitioned into 4- and 7-cycles) or a bad 6-cycle (i.e., partitioned into 4-cycles). In both the cases, it is easy to see that there is a 8-cycle (a 4-cycle adjacent to a 6-cycle) in G, contradicting the assumption. Hence Claim 12 is proved. □

We use the definition of good path as in [14]. A path $P = v_1v_2v_3v_4$ in the interior of C is called good if the following properties hold:

a. $d(v_i) = 3\ \forall i = 1, 2, 3, 4$;
b. $\ldots xPx' \ldots$ is on the boundary of a face;
c. there is a triangle $[uv_1v_2]$ with $u \neq x$;
d. $tv_3, t'v_4 \in E(G)$, where $t \neq x'$ and $t' \neq x'$.

Obviously, when $t = t'$, a good path is just a tetrad as defined in [4].

Claim 13. *G does not contain a tetrad P.*

Proof. Suppose on the contrary that such a tetrad P exists in G. Let G' denote the graph obtained from G by deleting vertices v_1, v_2, v_3 and v_4 and identifying x and t. It is easy to see that G' contains no 5 and 8-faces. In order to show that $G' \in \mathcal{G}$, we have the following argument. We first notice that G' has neither loops nor multiple edges. Indeed, if G' has a loop, then x is adjacent to t in G which leads to a 5-cycle $xtv_3v_2v_1x$. If G' has multiple edges, then both x and t are adjacent to a common vertex y so that a separating good 6-cycle $xytv_3v_2v_1x$ is established or there are two adjacent triangles in G.

Next, we claim that G' does not contain a separating cycle of length 5 or 8. In fact, if $C^* = xy_1y_2\ldots y_kt$ is a separating cycle in G', where $k \in \{4, 7\}$, then $C' = xy_1y_2\ldots y_ktv_3v_2v_1x$ is a cycle of length 9 or 12 in G. When $|C'| = 9$, u does not belong C' as there will be either a pair of adjacent cycles or a 5-cycle in G in this case. Thus, C' separates v_4 from u in G, which contradicts Claim 1 unless C' is bad. If C' is bad, there is a 6-cycle adjacent to two 3-cycles. This implies presence of 8-cycle, contradicting assumption. If C' is a 12-cycle, and u belongs to C', then there is either a 5-cycle, or a 8-cycle or two adjacent 3-faces. Hence, u does not belong to C'. Again, it is easy to see by enumeration that there is either a 5-cycle, or a 8-cycle or two adjacent 3-faces (establishing contradiction) or a separating 9-cycle. By using logic as before, we can establish a contradiction.

We need to prove that identifying x and t cannot damage the coloring of C. If this is not true, then we either identify two vertices of C colored differently, or insert an edge between two vertices of C colored by the same color. This means that the total distance from x and t to C is at most 1, that is, at least one of x and t lies on C. Without loss of generality, assume that $t \in C$ and let $C = u_1u_2 \ldots u_{|C|}u_1$, where the subscripts increase in the clockwise order. Suppose that $u_{|C|}$ is a vertex of C nearest to x. Since $|C| \in \{4, 6, 7, 9, 10\}$, C is split by $u_{|C|}$ and t into two paths, P_1 and P_2, one of which, say $P_1 = u_{|C|}u_1 \ldots u_jt$, consists of at most five edges. Thus, P_1 and the path $tv_3v_2v_1xu_{|C|}$ yield a cycle of length at most 10. Since $xv_1v_2v_3v_4x'$ is on the boundary of a face, $C' = u_{|C|}u_1u_2 \ldots u_jtv_3v_2v_1xu_{|C|}$ separates u from v_4, contradicting Claim 1 if the cycle C' is good. If C' is bad, there is a 6-cycle with two adjacent triangles, implying a 8-cycle, or a 4-cycle adjacent to a triangle, implying a 5-cycle, a contradiction.

Suppose the modification makes C bad. Let S' be a bad partition of G'. If $x * t$ is a vertex inside a cell of S', then S' is also a bad partition of C in G (insertion of vertices and edges into a face preserves the bad structure), implying C as bad, a contradiction. So suppose $x * t$ is a vertex of S' in G'. There can be at most two such cells created, say C_u and C_{v_4} denoting a cell of S' that contains u and v_4 respectively nonstrictly inside. The cell C_u cannot be a 3-cell or a 4-cell, as then there is a 5 or 8-cycle in G. If u is a vertex of C_u and the cell is a 4 or 6-cell, then either there are adjacent 3-cycles, or there is a 5-cycle in G. Similarly, if u is a vertex of C_u and the cell is a 7-cell, then either there are adjacent 3-cycles, or there is a 5 or 8-cycle in G. Let us assume that $C_u = xz_1 \cdots z_k$, and u is in $int(C_u)$ and C_{v_4} does not exist. In this case, there is a neighbor r (different from v_1 and v_2) of u which is nonstrictly inside the cycle $C'_u = xv_1v_2v_3tz_1 \cdots z_k$. If $|C_u|$ is 6 or 7, either there are adjacent 3-cycles, or there is a 5-cycle in G or C'_u is bad. The bad cells of C'_u along with triangles uv_1v_2 and tv_3v_4 and all the bad cells of S' except C_u will imply that C is bad. Similarly we reach a contradiction if we assume that v_4 is in $int(C_{v_4})$ and C_u does not exist. If u is a vertex of $int(C_u)$ and v_4 is a vertex of $int(C_{v_4})$, we can find a bad partition of C by using bad cells of C'_u and C'_{v_4}, triangles uv_1v_2, tv_3v_4 and the bad cells of S' except those in C_{v_4} and C_u.

Finally, we prove that any 3-coloring ϕ of G' can be extended to a 3-coloring of G in the following way. We first color v_4 and v_3 in succession, and then properly color v_1 and v_2. Since x and t have the same color, x and v_3 must have different colors, therefore the required coloring exists. □

3 Discharging

We use discharging to prove that there is no G satisfying the properties of the minimal counterexample as established in the previous section. Since by Euler's formula $|V(G)| - |E(G)| + |F(G)| = 2$ and $\sum_{v\in V(G)} d(v) = \sum_{f\in F(G)} d(f) = 2|E(G)|$,

$$\sum_{v\in V(G)} (d(v) - 6) + \sum_{f\in F(G)} (2d(f) - 6) = -12. \qquad (1)$$

We define a charge function w by $w(v) = d(v) - 6$ for each vertex $v \in V(G)$, $w(f) = 2d(f) - 6$ for each internal face $f \in \{F(G) \setminus f_0\}$, and $w(f_0) = d(f_0) + \frac{9}{2}$. It follows from identity (1) that the total sum of charge is equal to -7.5. We intend to design appropriate discharging rules and redistribute charges so that once the discharging is finished, a new charge function w' is produced. The discharging rules maintain that the total charge is kept fixed in the discharging process. Nevertheless, after the discharging is complete, the new charge function $w'(x)$ satisfies the following properties:

1. $w'(x) \geq 0 \ \forall x \in V(G) \cup F(G)$;
2. there exists some $x^* \in V(G) \cup F(G)$ such that $w'(x^*) > 0$.

This leads to the following obvious contradiction,

$$0 < \sum_{x\in V(G)\cup F(G)} w'(x) = \sum_{x\in V(G)\cup F(G)} w(x) = -7.5. \qquad (2)$$

Our discharging rules are as follows:

R0. Each 3-face $f = xyz$ receives $\frac{1}{3}$ from each adjacent face.

R1. Every 3-vertex $v \notin C$ receives $\frac{1}{3}$ from each incident face, unless v is incident with one 3-face, in which case v receives $\frac{1}{2}$ from each of the two $>$ 3-faces.

R2. Every 2-vertex receives $\frac{5}{3}$ from the external face, and $\frac{1}{3}$ from the other adjacent (i.e. internal) face.

R3. The external face f_0 gives 1 to each incident vertex of degree at least 3.

R4. Let v_1, v_2, v_3 be consecutive vertices of external face f_0 with $d(v_2) \geq 4$. Then v_2 gives 1 to each incident face not incident with edges v_1v_2 and v_2v_3. Furthermore, if the internal face receiving 1 is a 3-face (v_2xy) where x and y do not belong to f_0, then it passes the 1 to the neighboring internal face.

R5. Each 9^+-face $f \neq f_0$ gives $\frac{d(f)-8}{2}$ to f_0.

Claim 14. *For all $v \in V(G)$, $w'(v) \geq 0$.*

Proof. Let us assume that v does not belong to C. If $d(v) = 3$ and v is not incident with a 3-face, $w'(v) = 3-4+3\times\frac{1}{3} = 0$. If $d(v) = 3$ and v is incident with a 3-face, $w'(v) = 3-4+2\times\frac{1}{2} = 0$. If $d(v) \geq 4$, $w'(v) = w(v) \geq 0$.

Now suppose $v \in C$. If $d(v) = 2$ then by (R2), $w'(v) = 2-4+\frac{5}{3}+\frac{1}{3} = 0$. If $d(v) = 3$, by (R3), $w'(v) = 3-4+1 = 0$. If $d(v) \geq 4$, $w'(v) = d(v)-4-1\times(d(v)-3) = 0$, by (R3) and (R4). □

Claim 15. *For all $f \in F(G)\setminus C$, $w'(f) \geq 0$.*

Proof. If $d(f) = 3$, then $w'(f) \geq 3-4+3\times\frac{1}{3} = 0$ or $3-4+\frac{2}{3}+2\times\frac{1}{6} = 0$, by (R0).

If f appears in (R4), then it may have additional charge, hence $w'(f) \geq 0$.

We have already shown that $d(f) \neq 4$ and $d(f) \neq 6$.

Let us consider the case of $d(f) = 7$. Let $f = v_1v_2v_3v_4v_5v_6v_7$. Assume f does not have any vertex common with C. Due to absence of 8-cycle, f cannot be adjacent to a 3-face. So, $w'(f) \geq 7-4-7\times\frac{1}{3} > 0$. If f has any vertex or edge common with C, then it is easy to show that $w'(f) \geq 0$.

Let us consider the case of $d(f) = 9$. Due to Claim 10, there cannot exist any 4-face adjacent to f except for the case of external 4-face. Also no 7-face can be adjacent to a 3-face. Since there is no tetrad, $w'(f) = 9-4-6\times\frac{1}{3}-6\times\frac{1}{2} \geq 0$ or $w'(f) = 9-4-3\times\frac{1}{3}-6\times\frac{1}{2} \geq 0$.

Let us consider the case of $d(f) \geq 10$. Due to Claim 10, there cannot exist any 4-face adjacent to f except for the case of external 4-face. we can partition the donation of f to the vertices by (R1), (R2) and to the edges by (R0) into $d(f)$ groups so that the total donation per group is at most $\frac{3}{5}$. This is easy to see. As there is no tetrad, in the worst case we have a set of five consecutive vertices receiving charges of 0, $\frac{1}{2}$, $\frac{1}{2}$, $\frac{1}{2}$, $\frac{1}{2}$ and the consecutive edges receiving $\frac{1}{3}$, 0, $\frac{1}{3}$, 0 and $\frac{1}{3}$. Hence, total average discharge per group is $\frac{3\times\frac{1}{3}+4\times\frac{1}{2}}{5} = \frac{3}{5}$. Hence, $w'(f) = d(f)-4-d(f)\times\frac{3}{5} \geq 0$. □

Claim 16. $w'(f_0) > 0$.

Proof. If f_0 is the outer face of G, then $d(f_0) \in \{3,4,6,7,9,10,\cdots\}$. Since G is different from C, and G is 2-connected, it follows that C has at least two ≥ 3-vertices.

Hence total discharge from f_0 is $d(f_0)+4-\frac{5}{3}\times(d(f_0)-2)-2\times1-1\times1/3 = \frac{2}{3}\times(7.5-d(f_0))$. So $w'(f_0)\geq 0$ for $f_0\leq 7$. Since there are no adjacent triangles, and no internal 4-cycle, no 5- and 8-cycle, there is an internal non-triangular face with at least 4 internal vertices. This implies an internal face of length at least $d(f_0)-2+6$, i.e., $d(f_0)+4$. This face gives at least $(d(f_0)+4-8)\times\frac{1}{2}$, i.e., $(d(f0)-4)\times\frac{1}{2}$ to f_0. Hence, $w'(f_0) = d(f_0)+4-\frac{5}{3}\times(d(f_0)-2)-2\times1-1\times\frac{1}{3}+(d(f_0)-4)\times\frac{1}{2} = \frac{1}{6}\times(18-d(f_0))$.

For the case, there is no 2-vertex, $w'(f_0)\geq d(f_0)+4-1\times d(f_0)-\frac{1}{3}\times\frac{1}{2}\times d(f_0)) = \frac{1}{6}\times(24-d(f_0))$. This implies $w'(f_0)>0$. □

4 Conclusion

To date, the best known result towards Steinberg's conjecture is by [4] that states that any planar graph without cycles of length in $\{4,5,6,7\}$ is 3-colorable. In this article, we show that the 3 colorability holds true for any planar graph without cycles of length in $\{5,8\}$ and without adjacent triangles.

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