The Total Detection Numbers of Graphs

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Abstract

Let G be a connected graph of size at least 2 and $c : E(G) \to \{0, 1, \ldots, k-1\}$ an edge coloring (or labeling) of G using k colors (where adjacent edges may be assigned the same color). For each vertex v of G, the color code of v with respect to c is the k-tuple $\text{code}(v) = (a_0, a_1, \cdots, a_{k-1})$, where a_i is the number of edges incident with v that are labeled i $(0 \leq i \leq k-1)$. The labeling c is called a detectable labeling if distinct vertices in G have distinct color codes. The value $\text{val}(c)$ of a detectable labeling c of a graph G is the sum of the colors assigned to the edges in G. The total detection number $\text{td}(G)$ of G is defined by $\text{td}(G) = \min\{\text{val}(c)\}$, where the minimum is taken over all detectable labelings c of G. Thus if G is a connected graph of size $m \geq 2$, then $1 \leq \text{td}(G) \leq \binom{m}{2}$. We present characterizations of all connected graphs G of size $m \geq 2$ for which $\text{td}(G) \in \left\{1, \binom{m}{2}\right\}$. The total detection numbers of complete graphs and cycles are also investigated.

Keywords: vertex-distinguishing coloring, detectable labeling, detection number, total detection number.

AMS subject classification: 05C15, 05C78.

1 Introduction

We refer to the book [6] for graph-theoretical notation and terminology not described in this paper.

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Let G be a connected graph of size at least 2 and $c : E(G) \to \{0, 1, \ldots, k-1\}$ an edge coloring (or labeling) of G for some integer $k \geq 2$ (where adjacent edges may be assigned the same color). If c uses k colors, then c is a k-labeling. The *color code* of a vertex v of G (with respect to c) is the ordered k-tuple

$$\text{code}_c(v) = (a_0, a_1, \cdots, a_{k-1}) \quad (\text{or simply } \text{code}_c(v) = a_0 a_1 \cdots a_{k-1}),$$

where a_i is the number of edges incident with v that are labeled i for $0 \leq i \leq k-1$. Thus,

$$\sum_{i=0}^{k-1} a_i = \deg_G v. \tag{1}$$

If the labeling c is clear, we use code(v) to denote the color code of a vertex v. The labeling c is called a *detectable labeling* of G if distinct vertices of G have distinct color codes; that is, for every two vertices of G, there exists a color such that the numbers of incident edges assigned that color are different for these two vertices. Thus, a detectable labeling is a vertex-distinguishing edge labeling. The *detection number* $\det(G)$ of G is the minimum positive integer k for which G has a detectable k-labeling. A detectable labeling of a graph G using $\det(G)$ colors is called a *minimum detectable labeling* of G. Since there is no nontrivial irregular graph (a graph in which no two distinct vertices have the same degree), every detectable labeling of a graph must use at least two colors by (1). Thus, $\det(G) \geq 2$ for every connected graph G of size at least 2. Detectable labelings have been studied in [1, 2, 3, 4, 5], sometimes with different terminology and notation.

For a detectable labeling $c : E(G) \to \{0, 1, \ldots, k-1\}$ of a graph G, define the *value* val(c) of c by

$$\text{val}(c) = \sum_{e \in E(G)} c(e).$$

The *total detection number* td(G) of G is then defined by

$$\text{td}(G) = \min\{\text{val}(c)\},$$

where the minimum is taken over all detectable labelings c of G. Thus in the case of the detection number $\det(G)$ of G, we minimize the number of colors used in a detectable labeling of G; while in the case of the total detection number td(G) of G, we minimize the sum of colors of the edges of G used in a detectable labeling of G (which may or may not be a minimum detectable labeling).

2 Examples and some observations

To illustrate the concept described in Section 1, let us first determine the total detection number of a 5-cycle C_5. Figure 1(a) shows a 3-edge labeling

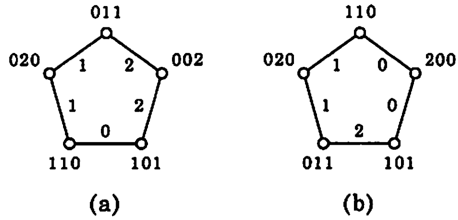

Figure 1: Detectable labelings of C_5

of C_5 using the three colors 0, 1, and 2, where each vertex is labeled by its color code. Since no two vertices have the same code, this is an example of a detectable labeling of C_5. In fact, this is a minimum detectable labeling of C_5 since an arbitrary 2-edge labeling of C_5 induces at most three distinct codes, namely 20, 11, and 02, and so it cannot be a detectable labeling of C_5. That is, $\det(C_5) \geq 3$ and the labeling in Figure 1(a) confirms that $\det(C_5) = 3$. The detection numbers of cycles in general are presented in [5].

Theorem 2.1 [5] *Let $n \geq 3$ be an integer and $p = \lceil \sqrt{n/2} \rceil$. Then*

$$\det(C_n) = \begin{cases} 2p-1 & \text{if } 2(p-1)^2 < n \leq p(2p-1) \\ 2p & \text{if } p(2p-1) < n \leq 2p^2. \end{cases}$$

The labeling in Figure 1(b) is another minimum detectable labeling of C_5, which is obtained from the labeling in Figure 1(a) by interchanging the colors 0 and 2. The values of these labelings in Figure 1(a) and (b) are 6 and 4, respectively. Thus, $\text{td}(C_5) \leq 4$. In order to verify that $\text{td}(C_5) = 4$, assume, to the contrary, that there is a detectable labeling c' whose value is less than 4. Since c' must use at least three colors by Theorem 2.1, assume that $c' : E(C_5) \to \{0,1,2\}$ and there are three edges that are assigned the color 0 by c'. Then we may further assume that $C_5 = (v_1, v_2, \ldots, v_5, v_1)$ and $c'(v_1v_2) = c'(v_3v_4) = 0$. However then, $\text{code}_{c'}(v_2) = \text{code}_{c'}(v_3)$ regardless of the color of v_2v_3, which contradicts the assumption that c' is a detectable labeling of C_5. Therefore, there is no such c' and we conclude that $\text{td}(C_5) = 4$.

The above discussion gives us a useful observation on detectable labelings of cycles and paths.

Observation 2.2 *Let G be a cycle or a path. If e_1, e_2, e_3 are three consecutive edges in G, then $c(e_1) \neq c(e_3)$ for every detectable labeling c of G.*

As another example, let us also look at some detectable labelings of C_8. By Theorem 2.1, every detectable labeling of C_8 uses at least four colors.

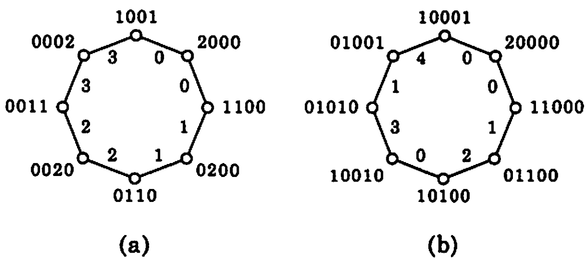

Figure 2: Detectable labelings of C_8

Thus, the labeling in Figure 2(a) is a minimum detectable labeling of C_8 while the one in Figure 2(b) is not. On the other hand, the values of these labelings are 12 and 11, respectively. In fact, $\text{td}(C_8) = 11$ and there is *no* detectable 4-labeling of C_8 whose value equals 11. In order to see this, consider an arbitrary detectable labeling $c : E(C_8) \to \{0, 1, \ldots, k-1\}$, where then $k \geq 4$. By Observation 2.2, one can verify that at most three edges in C_8 can be assigned the same color by c. Therefore, if $k \geq 5$, then $\text{val}(c) \geq 0+0+0+1+1+2+3+4 = 11$. When $k = 4$, that is, when c is a minimum detectable labeling of C_8, we further show that no three edges can be assigned the same color. If there are three edges that are labeled the same, then let $C_8 = (v_1, v_2, \ldots, v_8, v_1)$ and we may assume that $c(v_1v_2) = c(v_2v_3) = c(v_5v_6) = A \in \{0, 1, 2, 3\}$ by Observation 2.2. Then each of the remaining five edges must be assigned one of the colors in $\{0, 1, 2, 3\} - \{A\}$. However, this implies that at least two of the four vertices v_1, v_3, v_5, and v_6 have the same code, which is impossible. Therefore, $\text{val}(c) \geq 0+0+1+1+2+2+3+3 = 12$ when c uses only four colors, as claimed.

The following are simple yet important observations on detectable labelings of graphs.

Observation 2.3 *If c is a detectable k-labeling of a graph G, then a k-labeling obtained from c by permuting some of the k colors is also a detectable labeling of G.*

Observation 2.4 *If a graph G contains ℓ end-vertices, then every detectable labeling of G uses at least ℓ colors.*

Let G be a connected graph of order $n \geq 3$ and size m. A labeling assigning 0 to every edge of G is not a detectable labeling of G and so $\text{td}(G) \geq 1$. If $n = 3$, then $G \in \{C_3, P_3\}$ and it is straightforward to verify that $\det(C_3) = \text{td}(C_3) = 3$ while $\det(P_3) = \text{td}(P_3) + 1 = 2$. For $n \geq 4$, let T be a spanning tree of G with $E(T) = \{e_0, e_1, e_2, \ldots, e_{n-2}\}$, where say e_0 is a pendant edge. Then a labeling $c : E(G) \to \{0, 1, \ldots, n-2\}$ defined by $c(e_i) = i$ for $0 \leq i \leq n-2$ and $c(e) = 0$ for each $e \in E(G) - E(T)$ is a detectable labeling of G with $\text{val}(c) = \binom{n-1}{2}$. Therefore,

$$2 \leq \det(G) \leq n - 1 \leq m \tag{2}$$

and

$$1 \leq \text{td}(G) \leq \binom{n-1}{2} \leq \binom{m}{2} \tag{3}$$

if G is of order $n \geq 4$. Thus, $1 \leq \text{td}(G) \leq 3$ if G is of order 3 or 4. Figure 3 shows all connected graphs G of order 3 and 4 along with detectable labelings c such that $\text{val}(c) = \text{td}(G)$. Note that $\text{td}(G) \in \{1, 3\}$ for each G.

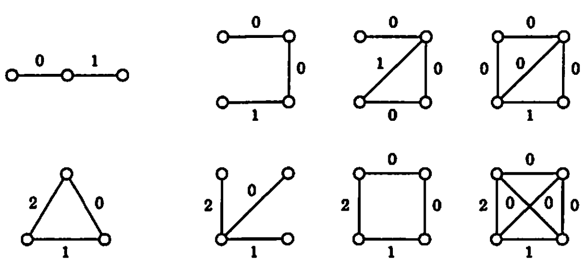

Figure 3: Detectable labelings of connected graphs of order 3 and 4

We make another useful observation.

Observation 2.5 *If G is a connected graph of size $m \geq 2$, then* $\det(G) =$ *m if and only if $G = C_3$ or G is a star.*

Proof. The result certainly holds for the connected graphs of order 3. Thus, let G be a connected graph of order at least 4. If G is a star, then

$\det(G) = m$ by Observation 2.4. For the converse, observe first that if $\det(G) = m$, then G must be a tree by (2). Suppose then that G is a tree that is not a star. We show that $\det(G) < m$. Let $E(G) = \{e_1, e_2, \ldots, e_m\}$. Since $\text{diam}(G) \geq 3$, there are two adjacent edges, say e_{m-1} and e_m, at most one of which is a pendant edge. Then a labeling $c : E(G) \to \{0, 1, \ldots, m-2\}$ defined by $c(e_i) = i$ for $1 \leq i \leq m-2$ and $c(e_{m-1}) = c(e_m) = 0$ is a detectable $(m-1)$-labeling of G and so $\det(G) \leq m-1$. ■

Since $\text{td}(P_3) = 1$ and $\text{td}(C_3) = 3$, it follows that (3) holds for every connected graph G of size $m \geq 2$. In fact, an edge labeling of G that assigns each edge a distinct color is a detectable labeling of G. If c is a detectable labeling of G whose value is less than $\binom{m}{2}$, then some edges in G are assigned the same color by c. Thus, $\text{val}(c) \leq 0 + 0 + 1 + 2 + \cdots + (m-2) = \binom{m-1}{2}$. Furthermore, if c is a detectable labeling of G whose value is less than $\binom{m-1}{2}$, then $\text{val}(c) \leq 0 + 0 + 1 + 1 + 2 + 3 + \cdots + (m-3) = \binom{m-2}{2} + 1$. A *double star* is a tree whose diameter equals 3.

Proposition 2.6 *Let G be a connected graph of size $m \geq 2$. Then* $1 \leq \text{td}(G) \leq \binom{m}{2}$. *Furthermore, each of the following holds.*

(a) $\text{td}(G) = 1$ *if and only if G contains two adjacent vertices x and y such that* (i) $\deg x \neq \deg y$ *and* (ii) *no two vertices in $V(G) - \{x, y\}$ have the same degree.*

(b) $\text{td}(G) = \binom{m}{2}$ *if and only if $G = C_3$ or G is a star.*

(c) $\text{td}(G) = \binom{m-1}{2}$ *if and only if $G = C_4$ or G is a double star.*

(d) $\text{td}(G) = \binom{m-2}{2} + 1$ *if and only if $G \in \{C_5, P_5, P_6\}$.*

Proof. The result certainly holds for $n = 3, 4$ and so suppose that $n \geq 5$. We first verify (a). First suppose that $\text{td}(G) = 1$ and let $c : E(G) \to \{0, 1\}$ be a detectable labeling of G with $\text{val}(c) = 1$. Let $e = xy$ be the unique edge labeled 1. For each vertex v in G, observe that $\text{code}(v) = (\deg v - 1, 1)$ if $v \in \{x, y\}$ and $\text{code}(v) = (\deg v, 0)$ otherwise. Therefore, $\deg x \neq \deg y$ and no two vertices in $V(G) - \{x, y\}$ have the same degree since c is a detectable labeling.

For the converse, suppose that x and y are adjacent vertices having distinct degrees and no two vertices in $V(G) - \{x, y\}$ have the same degree in G. Then the labeling $c : E(G) \to \{0, 1\}$ such that $c(e) = 1$ if and only if $e = xy$ is a detectable labeling of G whose value equals 1. Thus, $\text{td}(G) \leq \text{val}(c) = 1$, which shows that $\text{td}(G) = 1$.

For (b)–(d), it is straightforward to verify that $\text{td}(G) = \binom{m}{2}$ if G is a star, $\text{td}(G) = \binom{m-1}{2}$ if G is a double star, and $\text{td}(G) = \binom{m-2}{2} + 1$ if $G \in \{C_5, P_5, P_6\}$.

For the converse, let G be a connected graph of order $n \geq 5$ and size m that is neither a star nor a double star and $G \notin \{C_5, P_5, P_6\}$. Then one can verify that G contains one of the graphs H shown in Figure 4 as a subgraph. If, for example, G contains P_7 as a subgraph, then observe that

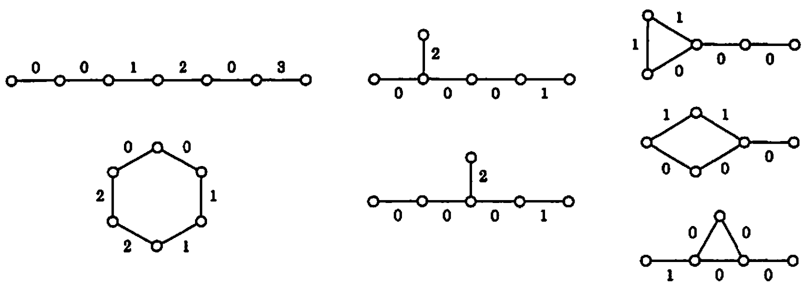

Figure 4: Subgraphs H of G in the proof of Proposition 2.6

$m \geq 6$ and a labeling $c : E(G) \to \{0, 1, \ldots, m-3\}$ of G assigning (i) the colors 0, 1, 2, and 3 to the edges belonging to P_7 as shown in Figure 4 and (ii) the colors $4, 5, \ldots, m-3$ to the remaining $m-6$ edges (if $m \geq 7$) is a detectable labeling of G with $\text{val}(c) = \binom{m-2}{2}$. Thus, $\text{td}(G) \leq \binom{m-2}{2}$. In fact, we can similarly verify that there exists a detectable labeling c of G such that the edges belonging to H are labeled as shown in Figure 4 and each of the remaining edges in $E(G) - E(H)$ is assigned a distinct color with $\text{val}(c) \leq \binom{m-2}{2}$. Hence, $\text{td}(G) \leq \binom{m-2}{2}$ in each case. ■

Note that Proposition 2.6(a) and (b) show the sharpness of the bounds in (3). Also, one can verify that the edge labeling c of H in Figure 4 has the property that $\text{val}(c) = \text{td}(H)$ for each H. In particular, $\text{td}(P_7) = \text{td}(C_6) = 6$.

If $c : E(G) \to \{0, 1, \ldots, k-1\}$ is a detectable k-labeling of a graph G of size m, then for each integer i with $0 \leq i \leq k-1$ there exists an edge e_i such that $c(e_i) = i$. Also, let $\bar{c} : E(G) \to \{0, 1, \ldots, k-1\}$ be another labeling of G such that $c(e) + \bar{c}(e) = k-1$ for each $e \in E(G)$. Then $\bar{c}$ is also a detectable k-labeling of G by Observation 2.3. Furthermore, $\text{val}(c) + \text{val}(\bar{c}) = m(k-1)$ and so

$$\text{td}(G) \leq \min\{\text{val}(c), \text{val}(\bar{c})\} \leq m(k-1)/2.$$

Observation 2.7 *Let G be a connected graph of size $m \geq 2$. If $c : E(G) \to \{0, 1, \ldots, k-1\}$ is a detectable k-labeling of G for which* $\text{val}(c) = \text{td}(G)$, *then*

$$\binom{k}{2} \leq \text{val}(c) = \text{td}(G) \leq m(k-1)/2. \quad (4)$$

In fact,

$$\binom{\det(G)}{2} \leq \text{td}(G) \leq m(\det(G) - 1)/2. \quad (5)$$

The bounds in (5) are sharp. For the lower bound, those graphs G with $\text{td}(G) = 1$ certainly have the desired property. In fact, more can be said; for each integer $d \geq 2$, there is a connected graph whose detection number and total detection number equal d and $\binom{d}{2}$, respectively. In order to see this, let G be a connected graph of order d and consider its *corona*, $\text{cor}(G)$, which is the graph obtained from G by adding a pendant edge at each vertex of G. Then $\det(\text{cor}(G)) \geq d$ by Observation 2.4. Furthermore, a labeling $c : E(G) \to \{0, 1, \ldots, d-1\}$ assigning the colors $0, 1, \ldots, d-1$ to the d pendant edges and the color 0 to the remaining edges belonging to G is a detectable d-labeling of $\text{cor}(G)$ with $\text{val}(c) = \binom{d}{2}$. Hence, $\det(\text{cor}(G)) = d$ and $\text{td}(\text{cor}(G)) = \binom{d}{2}$, as claimed.

Observe also that the upper bound equals the lower bound if $\det(G) = m$. Therefore, C_3 and stars are graphs whose total detection numbers attain the upper bound (and the lower bound) in (5). On the other hand, $\det(P_7) = \det(C_6) = 3$ while $\text{td}(P_7) = \text{td}(C_6) = 6$. Therefore, the condition that $\det(G) = m$ is sufficient but not necessary in order to have $\text{td}(G) = m(\det(G) - 1)/2$ for a graph G.

3 The total detection numbers of complete graphs

In this section we study the total detection numbers of complete graphs. Since we have already seen that $\text{td}(K_3) = \text{td}(K_4) = 3$, we consider complete graphs of order at least 5. It has been proven that every complete graph of order at least 3 has detection number 3.

Theorem 3.1 [5] *For every integer $n \geq 3$,* $\det(K_n) = 3$.

3.1 The total detection numbers of complete graphs of order at most 6

Figure 5 shows detectable 3-labelings of K_n for $n = 5, 6$, where a dark solid edge and a dashed edge represent edges labeled 1 and 2, respectively, while the rest of the edges are labeled 0. Therefore, the labelings in Figure 5 show

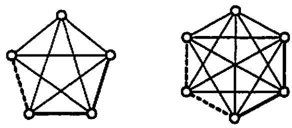

Figure 5: Detectable 3-labelings of K_5 and K_6

that $\text{td}(K_5) \leq 4$ and $\text{td}(K_6) \leq 6$. In order to determine the total detection numbers of K_n for $n = 5, 6$, we first consider an arbitrary detectable labeling $c : E(K_n) \to \{0, 1, \ldots, k-1\}$ of K_n using k colors ($k \geq 3$) and the resulting color classes $E_0, E_1, \ldots, E_{k-1}$. Let G_c be the subgraph induced by $E(K_n) - E_0$ and observe that

(a) the order of G_c is either n or $n-1$,

(b) every component in G_c contains at least three vertices, and

(c) G_c contains at most $k-1$ end-vertices.

Let m be the size of G_c. If G_c is connected, then $m \geq n-2$ by (a) and so

$$\begin{aligned} \text{val}(c) = \sum_{e \in E(G_c)} c(e) &\geq (1 + 2 + \cdots + (k-1)) + 1 \cdot (m - (k-1)) \\ &\geq 1 + m \geq n - 1. \end{aligned} \tag{6}$$

By (b), G_c must be connected for $n = 5$. Therefore, $\text{td}(K_5) \geq 4$ by (6), which in turn implies that $\text{td}(K_5) = 4$.

We next show that $\text{td}(K_6) \geq 6$. Assume, to the contrary, that there exists a detectable k-labeling $c' : E(K_6) \to \{0, 1, \ldots, k-1\}$ whose value is less than 6. Then $\text{val}(c') \geq 0 + 1 + \cdots + (k-1)$ and so $k = 3$. Thus, each edge in $G_{c'}$ is labeled either 1 or 2. If $G_{c'}$ is disconnected, then $G_{c'}$ contains a component isomorphic to K_3 by (b) and (c), say $C = (v_1, v_2, v_3, v_1)$ is a triangle in $G_{c'}$. Then we may assume, without loss of generality, that $c'(v_1v_2) = c'(v_2v_3)$. However then, $\text{code}_{c'}(v_1) = \text{code}_{c'}(v_3)$, which contradicts the fact that c' is a detectable labeling of K_6. Hence, $G_{c'}$ must be connected and $\text{val}(c') \geq 5$ by (6), that is, $\text{val}(c') = 5$. This in turn implies that $G_{c'}$ must be a tree of order 5 containing three edges labeled 1 and one edge labeled 2. Furthermore, $G_{c'}$ is a path by (c), say

$G_{c'} = P_5 = (v_1, v_2, v_3, v_4, v_5)$. Without loss of generality, we may further assume that $c'(v_1v_2) = c'(v_2v_3) = 1$. However then, either $\text{code}_{c'}(v_1) = \text{code}_{c'}(v_5) = (4,1,0)$ or $\text{code}_{c'}(v_2) = \text{code}_{c'}(v_3) = (3,2,0)$, neither of which is possible. Therefore, there is no such c' and $\text{val}(c) \geq 6$ for every detectable labeling c of K_6. We conclude that $\text{td}(K_6) \geq 6$, that is, $\text{td}(K_6) = 6$. We therefore have $\text{td}(K_3) = \text{td}(K_4) = 3$, $\text{td}(K_5) = 4$, and $\text{td}(K_6) = 6$. What can we say about $\text{td}(K_n)$ for $n \geq 7$ then?

3.2 An upper bound for $\text{td}(K_n)$

In this subsection we present an upper bound for $\text{td}(K_n)$, where $n \geq 7$, by constructing a detectable 3-labeling of K_n and finding its value.

For each positive integer ℓ, we construct a connected graph H_ℓ of order 2ℓ with

$$V(H_\ell) = \{u_1, u_2, \ldots, u_\ell\} \cup \{w_1, w_2, \ldots, w_\ell\}$$

such that $\deg u_i = \deg w_i = i$ for $1 \leq i \leq \ell$. For $\ell = 1, 2$, let $H_1 = (u_1, w_1)$ and $H_2 = (u_1, u_2, w_2, w_1)$ be paths of order 2 and 4, respectively. For $\ell \geq 3$, let

$$P = (u_1, u_3, u_4, \ldots, u_\ell, u_2, w_2, w_\ell, w_{\ell-1}, \ldots, w_3, w_1)$$

be a path of order 2ℓ. Then for each pair i, j of integers with $3 \leq i \leq \ell$ and $\ell + 3 - i \leq j \leq \ell$, adding the $i - 2$ edges u_iw_j to P results in H_ℓ. Observe that the size of H_ℓ is $\binom{\ell+1}{2}$.

Proposition 3.2 *For each integer $n \geq 7$,*

$$\text{td}(K_n) \leq \begin{cases} \frac{1}{8}(n^2 + 15) & \text{if } n \text{ is odd} \\ \frac{1}{8}(n^2 - 2n + 32) & \text{if } n \equiv 0 \pmod 4 \\ \frac{1}{8}(n^2 - 2n + 40) & \text{if } n \equiv 2 \pmod 4. \end{cases}$$

Proof. We consider four cases, according to the congruence classes modulo 4 to which n belongs.

Case 1. $n \equiv 1 \pmod 4$. Then $n = 4\ell + 5$ for some positive integer ℓ. Let $V(K_n) = U \cup W \cup \{v_0, v_1, v_2\}$, where $U = \{u_1, u_2, \ldots, u_{2\ell+1}\}$ and $W = \{w_1, w_2, \ldots, w_{2\ell+1}\}$. Let $c_1 : E(K_n) \to \{0, 1, 2\}$ be a 3-labeling of K_n with the color classes E_0, E_1, and E_2 such that (i) the subgraph induced by E_1 equals $H_{2\ell+1}$ whose vertex set is $U \cup W$ and (ii) $E_2 = \{w_1w_2, w_3v_1, v_1v_2\}$ if $\ell = 1$ while $E_2 = \{w_1w_2, w_{2\ell+1}v_1, v_1v_2\} \cup \{w_iw_{i+\ell-1} : 3 \leq i \leq \ell + 1\}$ if $\ell \geq 2$. Then c_1 is a detectable labeling of K_n and $\text{val}(c_1) = \binom{2\ell+2}{2} + 2(\ell + 2) = 2\ell^2 + 5\ell + 5 = (n^2 + 15)/8$.

Case 2. $n \equiv 3 \pmod 4$. Then $n = 4\ell + 3$ for some positive integer ℓ. Let $V(K_n) = U \cup W \cup \{v_0, v_1, v_2\}$, where $U = \{u_1, u_2, \ldots, u_{2\ell}\}$ and $W = \{w_1, w_2, \ldots, w_{2\ell}\}$. Let $c_2 : E(K_n) \to \{0, 1, 2\}$ be a 3-labeling of K_n with the color classes E_0, E_1, and E_2 such that (i) the subgraph induced by E_1 equals $H_{2\ell}$ whose vertex set is $U \cup W$ with an additional pendant edge $w_{2\ell}v_1$ and (ii) $E_2 = \{v_1v_2, v_1w_1\}$ if $\ell = 1$ while $E_2 = \{v_1v_2, v_1w_1\} \cup \{w_iw_{i+\ell-1} : 2 \le i \le \ell\}$ if $\ell \ge 2$. Then c_2 is a detectable labeling of K_n and $\text{val}(c_2) = \binom{2\ell+1}{2} + 1 + 2(\ell+1) = 2\ell^2 + 3\ell + 3 = (n^2 + 15)/8$.

Case 3. $n \equiv 0 \pmod 4$. Then obtain K_n from K_{n-1} constructed as described in Case 2 by adding a new vertex v_3. Observe that $c_3 : E(K_n) \to \{0, 1, 2\}$ such that (i) $c_3(v_2v_3) = 2$ while $c_3(v_3v) = 0$ for each $v \in V(K_n) - \{v_2, v_3\}$ and (ii) $c_3(e) = c_2(e)$ for each $e \in E(K_n - v_3)$ is a detectable 3-labeling of K_n whose value is $\left((n-1)^2 + 15\right)/8 + 2 = (n^2 - 2n + 32)/8$.

Case 4. $n \equiv 2 \pmod 4$. Then obtain K_n from K_{n-1} constructed as described in Case 1 by adding a new vertex v_3. Then $c_4 : E(K_n) \to \{0, 1, 2\}$ defined by

$$c_4(e) = \begin{cases} 0 & \text{if } e = v_3v \text{ and } v \neq v_2, v_3 \\ 1 & \text{if } e = w_{2\ell+1}v_1 \\ 2 & \text{if } e \in \{w_2v_1, v_2v_3\} \\ c_1(e) & \text{otherwise} \end{cases}$$

is a detectable 3-labeling of K_n with $\text{val}(c) = \left((n-1)^2 + 15\right)/8 + 3 = (n^2 - 2n + 40)/8$. ■

3.3 Detectable 3- and 4-labelings of complete graphs and their values

In this subsection we provide a lower bound for val(c), where c is a detectable k-labeling of K_n using the colors $0, 1, \ldots, k-1$ for $k \in \{3, 4\}$. In order to do this, we introduce some additional notation. For a given connected graph G of order $n \ge 3$, let $c : E(G) \to \{0, 1, \ldots, k-1\}$ be a detectable k-labeling of G, which may or may not be a minimum detectable labeling of G. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $\text{code}_c(v_i) = (a_{0,i}, a_{1,i}, \cdots, a_{k-1,i})$ for $1 \le i \le n$. Then define a vertex labeling $c^* : V(G) \to \mathbb{N}$ by $c^*(v_i) = \sum_{j=0}^{k-1} ja_{j,i}$. For example, a detectable 3-labeling c of K_{10} and the induced vertex labeling c^* are shown in Figure 6. (Again, a dark solid edge and a dashed edge represent edges labeled 1 and 2, respectively, while the rest of the edges are labled 0.)

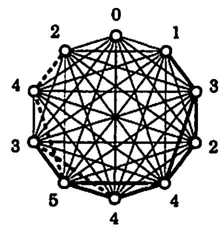

Figure 6: A detectable 3-labeling of K_{10}

Observe that

$$\mathrm{val}(c) = \tfrac{1}{2}\sum_{i=1}^{n} c^*(v_i). \tag{7}$$

Furthermore,

$$m_j = \tfrac{1}{2}\sum_{i=1}^{n} a_{j,i}$$

is the number of edges in G that are assigned the color j for $0 \le j \le k-1$. Therefore, $\sum_{i=1}^{n} a_{j,i}$ must be a nonnegative even integer for each j.

Now suppose that $c : E(K_n) \to \{0,1,2\}$ is a detectable 3-labeling of the complete graph of order $n \ge 3$. Then for each vertex $v \in V(K_n)$,

$$\begin{array}{lll} c^*(v) = 0 & \text{if and only if} & \mathrm{code}_c(v) = (n-1,0,0) \\ c^*(v) = 1 & \text{if and only if} & \mathrm{code}_c(v) = (n-2,1,0) \\ c^*(v) = 2 & \text{if and only if} & \mathrm{code}_c(v) \in \{(n-2,0,1),(n-3,2,0)\}. \end{array}$$

In general, it is straightforward to verify that there can be at most $\lfloor i/2 \rfloor + 1$ vertices v for which $c^*(v) = i$.

Proposition 3.3 *Let $c : E(K_n) \to \{0,1,2\}$ be a detectable 3-labeling of K_n, where $n \ge 5$. Let $p = \lceil \sqrt{n} \rceil$. Then*

$$\mathrm{val}(c) \ge \begin{cases} \frac{1}{2}\left(n(2p-3) - \frac{1}{6}p(p-1)(4p-5)\right) & \text{if } (p-1)^2 < n \le p(p-1) \\ \frac{1}{2}\left(n(2p-2) - \frac{1}{6}p(p-1)(4p+1)\right) & \text{if } p(p-1) < n \le p^2. \end{cases}$$

Proof. Since $p = \lceil \sqrt{n} \rceil$, it follows that $(p-1)^2 < n \le p^2$. We consider the following two cases.

Case 1. $(p-1)^2 < n \le p(p-1)$. Then $n = (p-1)^2 + q$, where

$1 \le q \le p-1$. Observe then that

$$\sum_{v \in V(K_n)} c^*(v) \ge 0+1+2+2+3+3+4+4+4+5+5+5+\cdots+ \underbrace{(2p-6)+\cdots+(2p-6)}_{p-2}+\underbrace{(2p-5)+\cdots+(2p-5)}_{p-2}+ \underbrace{(2p-4)+\cdots+(2p-4)}_{p-1}+\underbrace{(2p-3)+\cdots+(2p-3)}_{q}$$
$$= \sum_{i=1}^{p-2} i(4i-3)+(p-1)(2p-4)+\left(n-(p-1)^2\right)(2p-3)$$
$$= n(2p-3)-\tfrac{1}{6}p(p-1)(4p-5).$$

Then the result follows by (7).

Case 2. $p(p-1) < n \le p^2$. Then $n = p(p-1)+q$, where $1 \le q \le p$. Using the result obtained in Case 1, observe that

$$\sum_{v \in V(K_n)} c^*(v) \ge \left(p(p-1)(2p-3)-\tfrac{1}{6}p(p-1)(4p-5)\right)+q(2p-2)$$
$$= n(2p-2)-\tfrac{1}{6}p(p-1)(4p+1).$$

Then the result follows again by (7). ■

The following are consequences of Propositions 3.2 and 3.3:

- There exists a detectable 3-labeling of K_7 whose value equals 8 while there is no detectable 3-labeling of K_7 whose value is less than 8.
- There exists a detectable 3-labeling of K_8 whose value equals 10 while there is no detectable 3-labeling of K_8 whose value is less than 10.

Note that Proposition 3.3 *does not* necessarily give us a lower bound for $\mathrm{td}(K_n)$. It turns out that $\mathrm{td}(K_7) = 8$ while $\mathrm{td}(K_8) = 9$. The detectable 4-labelings of K_n for $8 \le n \le 10$ in Figure 7(a), where edges labeled 0 are omitted, show that $\mathrm{td}(K_8) \le 9$, $\mathrm{td}(K_9) \le 11$, and $\mathrm{td}(K_{10}) \le 13$. In fact, more can be said:

- If c is a detectable k-labeling of K_7 such that $\mathrm{val}(c) = \mathrm{td}(K_7)$, then $k \in \{3,4\}$.
- For $8 \le n \le 10$, if c is a detectable k-labeling of K_n such that $\mathrm{val}(c) = \mathrm{td}(K_n)$, then $k = 4$.

To see these, let us suppose that $c : E(K_n) \to \{0,1,\ldots,k-1\}$ is a detectable

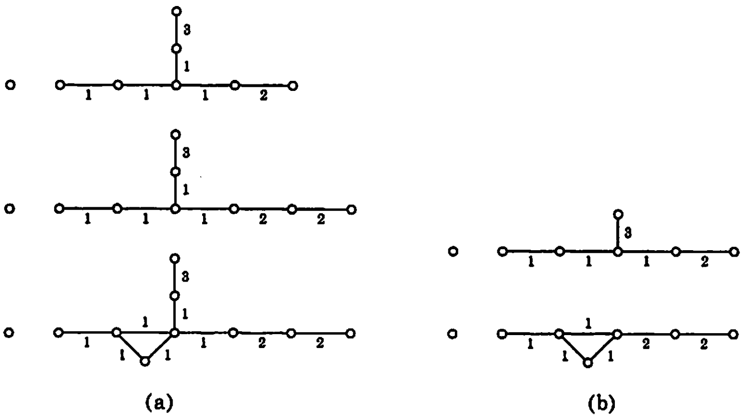

Figure 7: Detectable labelings of K_n $(7 \le n \le 10)$

k-labeling of K_n, where $k \ge 4$. Then for each $v \in V(K_n)$,

$$\begin{array}{lll} c^*(v) = 0 & \text{if and only if} & \text{code}_c(v) = (n-1, 0, 0, 0, \cdots) \\ c^*(v) = 1 & \text{if and only if} & \text{code}_c(v) = (n-2, 1, 0, 0, \cdots) \\ c^*(v) = 2 & \text{if and only if} & \text{code}_c(v) \in \{(n-2, 0, 1, 0, \cdots), (n-3, 2, 0, 0, \cdots)\} \\ c^*(v) = 3 & \text{if and only if} & \text{code}_c(v) \in \left\{ \begin{array}{l} (n-2, 0, 0, 1, \cdots), (n-3, 1, 1, 0, \cdots), \\ (n-4, 3, 0, 0, \cdots) \end{array} \right\} \end{array}$$

where the entries in "$\cdots$" are all zeros if $k \ge 5$. Also, if $c(xy) = k - 1$, then we may assume that $c^*(x) \ge k - 1$ and $c^*(y) \ge k$. Therefore, if c is a detectable k-labeling of K_7 with $k \ge 4$, then

$$\sum_{v \in V(K_7)} c^*(v) \ge 0 + 1 + 2 + 2 + 3 + (k-1) + k = 2k + 7$$

and so $\text{val}(c) \ge k + 4$ by (7). Thus, $\text{val}(c) \ge 8$ and $\text{val}(c) = 8$ only if $k \in \{3, 4\}$. Figure 7(b) shows detectable 3- and 4-labelings of K_7 whose values are both 8. Thus, $\text{td}(K_7) = 8$. Also, for K_8,

$$\sum_{v \in V(K_8)} c^*(v) \ge 0 + 1 + 2 + 2 + 3 + 3 + (k-1) + k = 2k + 10$$

and so $\text{val}(c) \ge k + 5 \ge 9$ and $\text{val}(c) = 9$ only if $k = 4$. One can similarly verify that

$$\text{val}(c) \ge \begin{cases} 12 & \text{if } k = 3 \\ k + 7 & \text{if } k \ge 4 \end{cases}$$

for every detectable k-labeling c of K_9 and

$$\text{val}(c) \geq \begin{cases} 14 & \text{if } k = 3 \\ k+9 & \text{if } k \geq 4 \end{cases}$$

for every detectable k-labeling c of K_{10}. Hence, $\text{td}(K_7) = 8$, $\text{td}(K_8) = 9$, $\text{td}(K_9) = 11$, and $\text{td}(K_{10}) = 13$.

Let us next consider the values of detectable 4-labelings of complete graphs in general. For a fixed integer $n \geq 3$, let $c : E(K_n) \to \{0,1,2,3\}$ be a detectable 4-labeling of K_n. Then there are at most $\lfloor i(i+6)/12 \rfloor + 1$ vertices v for which $c^*(v) = i$. Consider a function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = (x+1)(12x^2+9x+2)/2$. Since $f(0) = 1$ and $f'(x) > 0$ on $[0, \infty)$, there exists a unique positive integer p such that $f(p-1) < n \leq f(p)$. The following is a result parallel to Proposition 3.3, providing us with a lower bound for $\text{val}(c)$, where c is now a detectable 4-labeling of K_n using the four colors 0, 1, 2, and 3. The proof is omitted since it is done in the same manner as in the proof of Proposition 3.3.

Proposition 3.4 *Let $c : E(K_n) \to \{0,1,2,3\}$ be a detectable 4-labeling of K_n, where $n \geq 5$. Let p be the integer such that $f(p-1) < n \leq f(p)$. Then*

$$\text{val}(c) \geq \begin{cases} \frac{1}{2}\left(n(6p-5) - p^2(3p-2)^2\right) \\ \qquad \text{if } \frac{1}{2}p(12p^2-15p+5) < n \leq \frac{1}{2}p(12p^2-9p+1) \\ \frac{1}{2}\left(n(6p-4) - \frac{1}{2}p(3p-1)(6p^2-2p-1)\right) \\ \qquad \text{if } \frac{1}{2}p(12p^2-9p+1) < n \leq \frac{1}{2}p(12p^2-3p-1) \\ \frac{1}{2}\left(n(6p-3) - p^2(9p^2-2)\right) \\ \qquad \text{if } \frac{1}{2}p(12p^2-3p-1) < n \leq \frac{1}{2}p(12p^2+3p-1) \\ \frac{1}{2}\left(n(6p-2) - \frac{1}{2}p(3p+1)(6p^2+2p-1)\right) \\ \qquad \text{if } \frac{1}{2}p(12p^2+3p-1) < n \leq \frac{1}{2}p(12p^2+9p+1) \\ \frac{1}{2}\left(n(6p-1) - p^2(3p+2)^2\right) \\ \qquad \text{if } \frac{1}{2}p(12p^2+9p+1) < n \leq \frac{1}{2}p(12p^2+15p+5) \\ \frac{1}{2}\left(n(6p) - \frac{1}{2}p(p+1)(18p^2+18p+5)\right) \\ \qquad \text{if } \frac{1}{2}p(12p^2+15p+5) < n \leq \frac{1}{2}(p+1)(12p^2+9p+2). \end{cases}$$

Table 1 shows the total detection numbers of complete graphs of small order. Note that for each $n \in \{11, 12, \ldots, 15\}$, there is a detectable 5-labeling of K_n whose value equals $\text{td}(K_n)$. Furthermore, for $n = 14$, it turns out that $c : E(K_{14}) \to \{0, 1, \ldots, k-1\}$ is a detectable k-labeling of K_{14} with $\text{val}(c) = \text{td}(K_{14}) = 22$ only if $k = 5$.

n	3	4	5	6	7	8	9	10	11	12	13	14	15
$\text{td}(K_n)$	3	3	4	6	8	9	11	13	16	18	20	22	25

Table 1: $\text{td}(K_n)$ for $3 \le n \le 15$

Problem 3.5 *Let $n \ge 3$ be an integer. For a fixed integer $k \ge 3$, let $c: E(K_n) \to \{0, 1, \ldots, k-1\}$ be a detectable k-labeling of K_n. Find a lower bound for* $\text{val}(c)$ *in terms of k and n.*

4 The total detection numbers of cycles

In this section we study the total detection numbers of cycles in general. First we state an observation.

Observation 4.1 *If G is a connected graph of order at least* 3 *containing a connected regular spanning subgraph H, then* $\text{td}(G) \le \text{td}(H)$.

Proof. Let c be a detectable labeling of H such that $\text{val}(c) = \text{td}(H)$. Define a labeling c' of G by $c'(e) = c(e)$ if $e \in E(H)$ and $c'(e) = 0$ otherwise. Then c' is a detectable labeling of G and $\text{val}(c') = \text{val}(c) = \text{td}(H)$. Therefore, $\text{td}(G) \le \text{val}(c') = \text{td}(H)$. ■

By Observation 4.1, it follows that $\text{td}(G) \le \text{td}(C_n)$ if G is a Hamiltonian graph of order $n \ge 3$. In particular, $\text{td}(K_n) \le \text{td}(C_n)$ for each $n \ge 3$. Furthermore, we already know that a strict inequality is possible since $\text{td}(K_8) = 9$ while $\text{td}(C_8) = 11$.

4.1 The total detection numbers of cycles of order at most 8

We have already seen in Section 2 that $\text{td}(C_3) = \text{td}(C_4) = 3$, $\text{td}(C_5) = 4$, $\text{td}(C_6) = 6$, and $\text{td}(C_8) = 11$. The labelings shown in Figure 8 are detectable labelings of C_n $(3 \le n \le 8)$ and so $\text{td}(C_7) \le 9$. We show that equality holds. Let $c : E(C_7) \to \{0, 1, \ldots, k-1\}$ be a detectable k-labeling of C_7 such that $\text{val}(c) = \text{td}(C_7)$, where then $k \ge \det(C_7) = 4$ (by Theorem 2.1). First observe that $k = 4$ by (4) since $\text{val}(c) = \text{td}(C_7) \le 9$. We show that c cannot assign the same color to three edges in C_7. Let $C_7 = (v_1, v_2, \ldots, v_7, v_1)$ and assume, to the contrary, that there are three edges that are assigned the same color by c. By Observation 2.2 we may assume that $c(v_1v_2) = c(v_2v_3) = c(v_5v_6) = A$. Since v_2 is incident with two

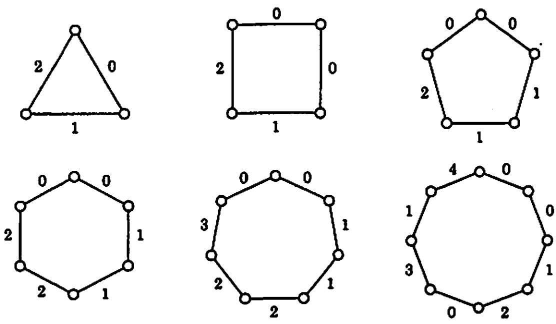

Figure 8: Detectable labelings of C_n $(3 \leq n \leq 8)$

edges labeled A, no other edge can be assigned the color A. However then, this implies that at least two of the four vertices v_1, v_3, v_5, and v_6 have the same code, which contradicts the assumption that c is a detectable labeling of C_7. Therefore, as claimed, c assigns the same color to at most two edges and so $\text{val}(c) \geq 0+0+1+1+2+2+3 = 9$, that is, $\text{td}(C_7) \geq 9$. We therefore have $\text{td}(C_3) = \text{td}(C_4) = 3$, $\text{td}(C_5) = 4$, $\text{td}(C_6) = 6$, $\text{td}(C_7) = 9$, $\text{td}(C_8) = 11$.

4.2 An upper bound for $\text{td}(C_n)$

In this subsection we present an upper bound for $\text{td}(C_n)$, where $n \geq 9$. Figure 9 shows that $\text{td}(C_9) \leq 12$.

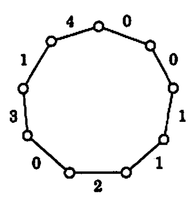

Figure 9: A detectable labeling of C_9

In order to present an upper bound for the total detection numbers of cycles of order greater than 9, we introduce the following terminology. For an arbitrary labeling c of a cycle $C_n = (v_1, v_2, \ldots, v_n, v_1)$ of order $n \geq 3$, define the *label sequence* as

$$c(v_1v_2), c(v_2v_3), \ldots, c(v_{n-1}v_n), c(v_nv_1).$$

Proposition 4.2 *For each integer $n \geq 10$,*

$$\mathrm{td}(C_n) \leq \begin{cases} \frac{1}{18}(2n^2+9n-18) & \text{if } n \equiv 0 \pmod 6 \\ \frac{1}{18}(2n^2+5n+11) & \text{if } n \equiv 1 \pmod 6 \\ \frac{1}{18}(2n^2+7n-22) & \text{if } n \equiv 2 \pmod 6 \\ \frac{1}{18}(2n^2+9n-45) & \text{if } n \equiv 3 \pmod 6 \\ \frac{1}{18}(2n^2+5n+2) & \text{if } n \equiv 4 \pmod 6 \\ \frac{1}{18}(2n^2+7n-13) & \text{if } n \equiv 5 \pmod 6. \end{cases}$$

Proof. We first define some sequences of nonnegative integers. For each positive integer ℓ, let $s(\ell)$ be the sequence of length 6ℓ given by

$$s(\ell) : \begin{cases} 0,0,1,1,2,2 & \text{if } \ell = 1 \\ 0,0,1,1,2,2,0,3,3,1,4,4,\ldots,0,2\ell-1,2\ell-1,1,2\ell,2\ell & \text{if } \ell \geq 2. \end{cases}$$

Furthermore, let $s_1(\ell), s_2(\ell), \ldots, s_5(\ell)$ be the following sequences:

$$\begin{aligned} s_1(\ell) &: \ 0,2\ell-1,1,2\ell,2,2\ell-1,2\ell \\ s_2(\ell) &: \ 0,2\ell-1,2\ell-1,1,2\ell,2,2\ell-1,2\ell \\ s_3(\ell) &: \ 2,2\ell-1,2\ell \\ s_4(\ell) &: \ 0,2\ell+1,1,2\ell+2 \\ s_5(\ell) &: \ 0,2\ell+1,2\ell+1,1,2\ell+2. \end{aligned}$$

Now let $p = \lfloor n/6 \rfloor$ and consider the following three cases.

Case 1. $n \equiv 0 \pmod 6$. Then observe that $c : E(C_n) \to \{0,1,\ldots,2p\}$ whose label sequence equals $s(p)$ is a detectable $(2p+1)$-labeling of C_n with $\mathrm{val}(c) = 4p^2+3p-1 = (2n^2+9n-18)/18$. Thus, $\mathrm{td}(C_n) \leq (2n^2+9n-18)/18$.

Case 2. $n \equiv i \pmod 6$, *where* $i \in \{1,2\}$. If $n \equiv 1 \pmod 6$, then the sequence $s(p-1)$ followed by $s_1(p)$ induces a detectable $(2p+1)$-labeling of C_n whose value equals $4p^2+3p+1 = (2n^2+5n+11)/18$. For $n \equiv 2 \pmod 6$, similarly, consider the sequence $s(p-1)$ followed by $s_2(p)$.

Case 3. $n \equiv i \pmod 6$, *where* $i \in \{3,4,5\}$. Considering the sequence $s(p)$ followed by $s_i(p)$, we obtain the desired result. ∎

4.3 A lower bound for td(C_n)

In this subsection we use a technique similar to that used in Subsection 3.3 to present a lower bound for $\mathrm{td}(C_n)$ by considering a lower bound for $\mathrm{val}(c)$, where $k \geq \det(C_n)$ and c is a detectable k-labeling of C_n using the colors $0,1,\ldots,k-1$.

Suppose that $c: E(C_n) \to \{0, 1, \ldots, k-1\}$ is a detectable k-labeling of C_n ($n \geq 3$). Then $0 \leq c^*(v) \leq 2(k-1)$ for each $v \in V(C_n)$. Furthermore,

- if $0 \leq i \leq k-1$, then there are at most $\lfloor i/2 \rfloor + 1$ vertices v with $c^*(v) = i$;
- if $k \leq i \leq 2(k-1)$, then there are at most $k - \lceil i/2 \rceil$ vertices v with $c^*(v) = i$;
- there are two vertices x and y with $c^*(x) \geq k-1$ and $c^*(y) \geq k$.

Therefore, we obtain the following, whose proof will be similar to that of Proposition 3.3 and therefore is omitted.

Proposition 4.3 *For a fixed integer $n \geq 3$, let $c: E(C_n) \to \{0, 1, \ldots, k-1\}$ be a detectable k-labeling of C_n.*
If $n \leq \lceil k/2 \rceil (\lfloor k/2 \rfloor + 1)$, then let $p = \lceil \sqrt{n-2} \rceil$. Then

$$\mathrm{val}(c) \geq \begin{cases} k + \frac{1}{2}\left(n(2p-3) - \frac{1}{6}(4p-5)(p^2-p+6)\right) & \text{if } (p-1)^2 < n-2 \leq p(p-1) \\ k + \frac{1}{2}\left(n(2p-2) - \frac{1}{6}(p-1)(4p^2+p+24) - 1\right) & \text{if } p(p-1) < n-2 \leq p^2. \end{cases}$$

If $\lceil k/2 \rceil (\lfloor k/2 \rfloor + 1) < n \leq k(k+1)/2$, then let $n' = k(k+1)/2 - n$. If $n' \neq 0$, then let $p = \lceil \sqrt{n'} \rceil$. Then

$$\mathrm{val}(c) \geq \begin{cases} \frac{1}{2}\left(n(2k-2p+1) - \frac{1}{2}k(k+1)(k-2p+2) - \frac{1}{6}p(p-1)(4p-5)\right) & \text{if } (p-1)^2 < n' \leq p(p-1) \\ \frac{1}{2}\left(n(2k-2p) - \frac{1}{2}k(k+1)(k-2p+1) - \frac{1}{6}p(p-1)(4p+1)\right) & \text{if } p(p-1) < n' \leq p^2 \\ \frac{1}{2}n(k-1) & \text{if } n' = 0. \end{cases} \tag{8}$$

For $n \geq 5$, we see from Proposition 4.3 that $\mathrm{td}(C_n)$ is bounded below by the expression in (8) with k being the largest positive integer such that $\lceil k/2 \rceil (\lfloor k/2 \rfloor + 1) < n$. We therefore obtain the following result, which gives us a lower bound of the total detection numbers of cycles of order $n \geq 5$.

Theorem 4.4 *For each integer $n \geq 5$, let $p = \lceil \sqrt{n} \rceil$. Then*

$$\mathrm{td}(C_n) \geq \begin{cases} \frac{1}{2}\left(n(2p-3) - \frac{1}{6}p(p-1)(4p-5)\right) & \text{if } (p-1)^2 < n \leq p(p-1)-2 \\ \frac{1}{2}\left(n(2p-2) - \frac{1}{6}(4p^3-3p^2-p-6)\right) & \text{if } n \in \{p(p-1)-1, p(p-1)\} \\ \frac{1}{2}\left(n(2p-2) - \frac{1}{6}p(p-1)(4p+1)\right) & \text{if } p(p-1) < n \leq p^2-2 \\ \frac{1}{2}\left(n(2p-1) - \frac{1}{6}(p-1)(4p^2+7p+6)\right) & \text{if } n \in \{p^2-1, p^2\}. \end{cases}$$

By Theorems 4.2 and 4.4, we now see that $\mathrm{td}(C_9) = 12$, $\mathrm{td}(C_{10}) = 14$, and $\mathrm{td}(C_{11}) = 17$.

Table 2 shows the detection numbers and total detection numbers of cycles of small order. For $n \in \{8, 15, 16\}$, note that there is no minimum detectable labeling c for which $\mathrm{val}(c) = \mathrm{td}(C_n)$.

n	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$\det(C_n)$	3	3	3	3	4	4	5	5	5	5	5	5	5	6
$\mathrm{td}(C_n)$	3	3	4	6	9	11	12	14	17	21	23	26	29	33

Table 2: $\det(C_n)$ and $\mathrm{td}(C_n)$ for $3 \leq n \leq 16$

5 Other related concepts and open problems

In [5] (and in (2)), it is shown that if G is a connected graph of order $n \geq 4$, then $2 \leq \det(G) \leq n-1$. Furthermore, the following realization result is presented.

Theorem 5.1 [5] *A pair n, k of positive integers is realizable as the order and detection number of some nontrivial connected graph if and only if* (i) $n = k = 3$ *or* (ii) $n \geq 3$ *and* $2 \leq k \leq n-1$.

We have seen in Section 2 that if G is a connected graph of order 3, then $\mathrm{td}(G) \in \{1, 3\}$. Furthermore, if G is a connected graph of order $n \geq 4$, then it is stated in (3) that $1 \leq \mathrm{td}(G) \leq \binom{n-1}{2}$. This gives rise to the following realization problem. We say that a pair (n, k) of positive integers is *realizable* if there exists a nontrivial connected graph G of order n with $\mathrm{td}(G) = k$. Therefore, if (n, k) is a realizable pair, then either (i) $(n, k) \in \{(3,1), (3,3)\}$ or (ii) $n \geq 4$ and $1 \leq k \leq \binom{n-1}{2}$. We have already seen in Figure 3 that

each of $(3, k)$ and $(4, k)$ is realizable if and only if $1 \leq k \leq 3$ and $k \neq 2$.

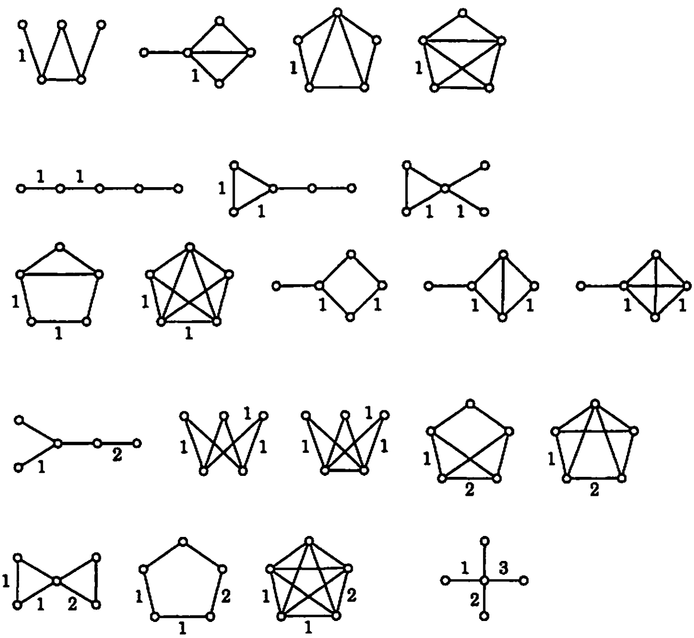

Figure 10: Detectable labelings of connected graphs of order 5

Figure 10 shows all connected graphs G of order 5 and detectable labelings c with $\text{val}(c) = \text{td}(G)$, where the unlabeled edges are assinged 0. Therefore,

$$(5, k) \text{ is realizable if and only if } 1 \le k \le 6 \text{ and } k \ne 5.$$

Similarly, by inspecting all connected graphs of order 6 (see [7] pp.218–224), we see that

$$(6, k) \text{ is realizable if and only if } 1 \le k \le 10 \text{ and } k \notin \{7, 8, 9\}.$$

What can we say if $n \ge 7$ then?

Problem 5.2 *Which pair (n, k) of positive integers with $n \ge 7$ and $1 \le k \le \binom{n-1}{2}$ is realizable?*

There is another interesting concept. In Sections 2 and 4, we considered the total detection numbers of cycles C_n of small order ($3 \le n \le 16$). For $n = 8$, recall that $\det(C_8) = 4$ while there is no detectable 4-labeling c of C_8 whose value equals $\text{td}(C_8) = 11$. We further showed that $c : E(C_8) \to$

$\{0, 1, \ldots, k-1\}$ is a detectable k-labeling of C_8 with val$(c) = 11$ only when $k = 5$. Similarly, it turns out that there is a detectable k-labeling of C_{16} whose value equals $\mathrm{td}(C_{16}) = 33$ only if $k = 7 = \det(C_{16})+1$. Furthermore, there is a detectable 7-labeling of C_{15} such that val$(c) = \mathrm{td}(C_{15}) = 29$ while there is no detectable labeling c of C_{15} with val$(c) = 29$ if c uses less than 7 colors, although $\det(C_{15}) = 5$.

This gives rise to the following graphical parameter and problem. For a connected graph G of size at least 2, let $f(G)$ be the smallest integer k such that there exists a detectable k-labeling $c : E(G) \to \{0, 1, \ldots, k-1\}$ with val$(c) = \mathrm{td}(G)$. Therefore, $\det(G) \leq f(G)$. We have seen some graphs G for which $\det(G) = f(G)$ and some graphs H for which $\det(H) < f(H)$. For $3 \leq n \leq 16$, for example,

$$f(C_n) = \begin{cases} \det(C_n) + 1 & \text{if } n \in \{8, 16\} \\ \det(C_n) + 2 & \text{if } n = 15 \\ \det(C_n) & \text{otherwise.} \end{cases}$$

Problem 5.3 *Study $f(G)$ for some well-known classes G of graphs.*

Problem 5.4 *For each positive integer ℓ, is there a connected graph G such that $f(G) = \det(G) + \ell$?*

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