Generalizations of certain binomial sums with generalized Fibonacci sequences

Zhizheng Zhang and Xiaoli Ye
Department of Mathematics, Luoyang Teachers' College,
Luoyang 471022, P. R. China
e-mail: zhzhzhang-yang@163.com

Abstract

The purpose of this note is to give two binomial sums with generalized Fibonacci sequences. These results generalize two binomial sums by Kilic and Ionascu in The Fibonacci Quarterly, 48.2(2010), 161-167.

2000 Mathematics Subject Classification: 11B39, 40A99

Keywords and phraseas: Generalized Fibonacci number, Generalized Lucas number, binomial sum

1 Introduction

Let p be a nonzero real number. The generalized Fibonacci and Lucas numbers are defined by

$$U_{n+2} = pU_{n+1} + U_n, \quad U_0 = 0, \quad U_1 = 1, \tag{1}$$

$$V_{n+2} = pV_{n+1} + V_n, \quad V_0 = 2, \quad U_1 = p, \tag{2}$$

respectively. If $\alpha = \left(p + \sqrt{p^2+4}\right)/2$ and $\beta = \left(p - \sqrt{p^2+4}\right)/2$, then the Binet's formulas are that

$$U_n = \frac{1}{\sqrt{p^2+4}}\left(\alpha^n - \beta^n\right), \qquad V_n = \alpha^n + \beta^n. \tag{3}$$

For $p = 1$, $\{U_n\}$ and $\{V_n\}$ are the well-known Fibonacci numbers F_n and Lucas numbers L_n, respectively.

In [1], the following sums were obtained.

$$\sum_{k=0}^{n}\binom{2n}{n+k}U_k^{2r} = \begin{cases} \frac{1}{(p^2+4)^r}\left(\binom{2r}{r}2^{2n-2} + \sum_{i=0}^{r-1}(-1)^{i(n+1)}\binom{2r}{i}V_{r-i}^{2n}\right), & if\ r\ even; \\ (p^2+4)^{n-r}\sum_{i=0}^{r-1}(-1)^{i(n+1)}\binom{2r}{i}U_{r-i}^{2n}, & if\ r\ odd; \end{cases}$$

and

$$\sum_{k=0}^{n}\binom{2n}{n+k}V_k^{2r} = \begin{cases} \binom{2r}{r}2^{2n-1} + 2^{2r-1}\binom{2n}{n} + \sum_{i=0}^{r-1}(-1)^{in}\binom{2r}{i}V_{r-i}^{2n}, & if\ r\ even; \\ 2^{2r-1}\binom{2n}{n} + (p^2+4)^n\left(\sum_{i=0}^{r-1}(-1)^{in}\binom{2r}{i}U_{r-i}^{2n}\right), & if\ r\ odd. \end{cases}$$

The purpose of this note is to extend these sums. We obtain that

Theorem 1.1 Let m, r and k be positive integers. Then

$$\sum_{k=0}^{n}\binom{2n}{n+k}U_{mk}^{2r} = \begin{cases} \frac{1}{(p^2+4)^r}\left\{\binom{2r}{r}2^{2n-1} + \sum_{i=0}^{r-1}(-1)^{i+imn}\binom{2r}{i}V_{m(r-i)}^{2n}\right\}, \\ \qquad if\ r\ even; \\ \frac{1}{(p^2+4)^r}\left\{-\binom{2r}{r}2^{2n-1} + \sum_{i=0}^{r-1}(-1)^{i}\binom{2r}{i}V_{m(r-i)}^{2n}\right\}, \\ \qquad if\ r\ odd\ and\ m\ even; \\ (p^2+4)^{n-r}\sum_{i=0}^{r-1}(-1)^{i+in}\binom{2r}{i}U_{m(r-i)}^{2n}, \\ \qquad if\ r\ odd\ and\ m\ odd. \end{cases} \tag{4}$$

Theorem 1.2 Let m, r and k be positive integers. Then

$$\sum_{k=0}^{n}\binom{2n}{n+k}V_{mk}^{2r}$$
$$= \begin{cases} \binom{2r}{r}2^{2n-1}+\binom{2n}{n}2^{2r-1}+\sum_{i=0}^{r-1}(-1)^{imn}\binom{2r}{i}V_{m(r-i)}^{2n}, & \text{if } r \text{ even;} \\ \binom{2r}{r}2^{2n-1}+\binom{2n}{n}2^{2r-1}+\sum_{i=0}^{r-1}\binom{2r}{i}V_{m(r-i)}^{2n}, & \text{if } r \text{ odd and } m \text{ even;} \\ \binom{2n}{n}2^{2r-1}+(p^2+4)^n\sum_{i=0}^{r-1}(-1)^{in}\binom{2r}{i}U_{m(r-i)}^{2n}, & \text{if } r \text{ odd and } m \text{ odd.} \end{cases} \tag{5}$$

2 The proof of the results

Let m be a positive integer. From (3) we have

$$\alpha^m = \frac{V_m+\sqrt{p^2+4}U_m}{2}, \quad \beta^m = \frac{V_m-\sqrt{p^2+4}U_m}{2}. \tag{6}$$

Then

$$U_{mn} = \frac{1}{\sqrt{p^2+4}}\left\{\left(\frac{V_m+\sqrt{p^2+4}U_m}{2}\right)^n - \left(\frac{V_m-\sqrt{p^2+4}U_m}{2}\right)^n\right\},$$

$$V_{mn} = \left(\frac{V_m+\sqrt{p^2+4}U_m}{2}\right)^n + \left(\frac{V_m-\sqrt{p^2+4}U_m}{2}\right)^n.$$

Define the function $f(n,a)$ of $a \in C\setminus\{0\}$ and $n \in N$ by

$$f(n,a) = \sum_{k=0}^{n}\binom{2n}{n+k}(a^k+a^{-k}), \tag{7}$$

Kilic and Ionascu [1] obtained that

$$f(n,a) = \frac{1}{a^n}(a+1)^{2n}+\binom{2n}{n}. \tag{8}$$

From

$$(1-1)^{2r} = 2\sum_{i=0}^{r-1}(-1)^i\binom{2r}{i} + (-1)^r\binom{2r}{r},$$

$$(1+1)^{2r} = 2\sum_{i=0}^{r-1}\binom{2r}{i} + \binom{2r}{r},$$

we have

$$2\sum_{i=0}^{r-1}(-1)^i\binom{2r}{i} + \binom{2r}{r}(-1)^r = 0, \tag{9}$$

$$2\sum_{i=0}^{r-1}\binom{2r}{i} + \binom{2r}{r} = 2^{2r}. \tag{10}$$

Let

$$A_m = \frac{V_m + \sqrt{p^2+4}U_m}{2}, \quad B_m = \frac{V_m - \sqrt{p^2+4}U_m}{2}.$$

Then

$$A_m^n + B_m^n = V_{mn}, \tag{11}$$

$$A_m^n - B_m^n = \sqrt{p^2+4}U_{mn}, \tag{12}$$

and

$$A_m B_m = (-1)^m. \tag{13}$$

The proof of Theorem 1.1: We have

$$\begin{aligned} & \sum_{k=0}^{n}\binom{2n}{n+k}U_{mk}^{2r} \\ \overset{(A)}{=} \quad & \frac{1}{(p^2+4)^r}\sum_{k=0}^{n}\binom{2n}{n+k}(A_m^k - B_m^k)^{2r} \end{aligned}$$

$$
\begin{aligned}
= \quad & \frac{1}{(p^2+4)^r}\sum_{k=0}^{n}\binom{2n}{n+k}\left\{\binom{2r}{r}(A_m^k)^r(-B_m^k)^r\right. \\
& \left.+\sum_{i=0}^{r-1}\binom{2r}{i}\left[(A_m^k)^{2r-i}(-B_m^k)^i + A_m^{ki}(-B_m^k)^{2r-i}\right]\right\} \\
\overset{(B)}{=} \quad & \frac{1}{(p^2+4)^r}\sum_{k=0}^{n}\binom{2n}{n+k}\left\{\binom{2r}{r}(-1)^r(-1)^{mkr}\right. \\
& \left.+\sum_{i=0}^{r-1}(-1)^i\binom{2r}{i}(-1)^{mik}\left[A_m^{2(r-i)k}+A_m^{-2(r-i)k}\right]\right\} \\
= \quad & \frac{1}{(p^2+4)^r}\left\{(-1)^r\binom{2r}{r}\sum_{k=0}^{n}\binom{2n}{n+k}(-1)^{mrk}+\sum_{i=0}^{r-1}(-1)^i\binom{2r}{i}\right. \\
& \left.\times\sum_{k=0}^{n}\binom{2n}{n+k}(-1)^{mik}\left[A_m^{2(r-i)k}+A_m^{-2(r-i)k}\right]\right\} \\
= \quad & \frac{1}{(p^2+4)^r}\left\{\frac{(-1)^r}{2}\binom{2r}{r}f(n,(-1)^{mr})\right. \\
& \left.+\sum_{i=0}^{r-1}(-1)^i\binom{2r}{i}f(n,(-1)^{mi}A_m^{2(r-i)})\right\} \\
\overset{(C)}{=} \quad & \frac{1}{(p^2+4)^r}\left\{\frac{(-1)^r}{2}\binom{2r}{r}\left((-1)^{mrn}((-1)^{mr}+1)^{2n}+\binom{2n}{n}\right)\right. \\
& +\sum_{i=0}^{r-1}(-1)^i\binom{2r}{i}\left(\frac{1}{(-1)^{imn}A_m^{2n(r-i)}}((-1)^{mi}A_m^{2(r-i)}+1)^{2n}\right. \\
& \left.\left.+\binom{2n}{n}\right)\right\} \\
= \quad & \frac{1}{(p^2+4)^r}\left\{\frac{(-1)^{r+rmn}}{2}\binom{2r}{r}((-1)^{mr}+1)^{2n}\right. \\
& +\sum_{i=0}^{r-1}(-1)^{i+imn}\binom{2r}{i}\left(A_m^{r-i}+(-1)^{mr}B_m^{r-i}\right)^{2n} \\
& \left.+\frac{1}{2}\binom{2n}{n}\left[2\sum_{i=0}^{r-1}(-1)^i\binom{2r}{i}+(-1)^r\binom{2r}{r}\right]\right\}
\end{aligned}
$$

$$\overset{(D)}{=} \frac{1}{(p^2+4)^r}\left\{\frac{(-1)^{r+rmn}}{2}\binom{2r}{r}((-1)^{mr}+1)^{2n} + \sum_{i=0}^{r-1}(-1)^{i+imn}\binom{2r}{i}\left(A_m^{r-i}+(-1)^{mr}B_m^{r-i}\right)^{2n}\right\}$$

$$= \begin{cases} \frac{1}{(p^2+4)^r}\left\{\binom{2r}{r}2^{2n-1}+\sum_{i=0}^{r-1}(-1)^{i+imn}\binom{2r}{i}V_{m(r-i)}^{2n}\right\}, & \textit{if } r \textit{ even};\\ \frac{1}{(p^2+4)^r}\left\{-\binom{2r}{r}2^{2n-1}+\sum_{i=0}^{r-1}(-1)^{i}\binom{2r}{i}V_{m(r-i)}^{2n}\right\}, & \textit{if } r \textit{ odd and } m \textit{ even};\\ (p^2+4)^{n-r}\sum_{i=0}^{r-1}(-1)^{i+in}\binom{2r}{i}U_{m(r-i)}^{2n}, & \textit{if } r \textit{ odd and } m \textit{ odd}; \end{cases}$$

where (A) follows from (12), (B) from (13), (C) from(7), (D) from (10). □

The proof of Theorem 1.2: We have

$$\sum_{k=0}^{n}\binom{2n}{n+k}V_{mk}^{2r}$$

$$\overset{(A)}{=} \sum_{k=0}^{n}\binom{2n}{n+k}(A_m^k+B_m^k)^{2r}$$

$$= \sum_{k=0}^{n}\binom{2n}{n+k}\left\{\binom{2r}{r}A_m^{rk}B_m^{rk}+\sum_{i=0}^{r-1}\binom{2r}{i} \times\left[A_m^{(2r-i)k}B_m^{ik}+A_m^{ik}B_m^{(2r-i)k}\right]\right\}$$

$$\overset{(B)}{=} \binom{2r}{r}\sum_{k=0}^{n}\binom{2n}{n+k}(-1)^{mrk}+\sum_{i=0}^{r-1}\binom{2r}{i}\sum_{k=0}^{n}\binom{2n}{n+k}(-1)^{imk} \times\left[A_m^{2(r-i)k}+A_m^{-2(r-i)k}\right]$$

$$= \frac{1}{2}\binom{2r}{r}f(n,(-1)^{mr})+\sum_{i=0}^{r-1}\binom{2r}{i}f(n,(-1)^{im}A_m^{2(r-i)})$$

$$\overset{(C)}{=} \frac{1}{2}\binom{2r}{r}\left[(-1)^{rmn}((-1)^{mr}+1)^{2n}+\binom{2n}{n}\right]$$

$$+\sum_{i=0}^{r-1}\binom{2r}{i}\left[\frac{1}{(-1)^{imn}A_m^{2n(r-i)}}((-1)^{im}A_m^{2(r-i)}+1)^{2n}+\binom{2n}{n}\right]$$

$$= \frac{(-1)^{rmn}}{2}\binom{2r}{r}((-1)^{mr}+1)^{2n}+\sum_{i=0}^{r-1}\binom{2r}{i}(-1)^{imn}$$

$$\times(A_m^{r-i}+(-1)^{mr}B_m^{r-i})^{2n}+\frac{1}{2}\binom{2n}{n}\left[\binom{2r}{r}+2\sum_{i=0}^{r-1}\binom{2r}{i}\right]$$

$$\overset{(D)}{=} \frac{(-1)^{rmn}}{2}\binom{2r}{r}((-1)^{mr}+1)^{2n}+\sum_{i=0}^{r-1}\binom{2r}{i}(-1)^{imn}$$

$$\times(A_m^{r-i}+(-1)^{mr}B_m^{r-i})^{2n}+\binom{2n}{n}2^{2r-1}$$

$$= \begin{cases} \binom{2r}{r}2^{2n-1}+\binom{2n}{n}2^{2r-1}+\sum_{i=0}^{r-1}(-1)^{imn}\binom{2r}{i}V_{m(r-i)}^{2n}, & if\ r\ even; \\ \binom{2r}{r}2^{2n-1}+\binom{2n}{n}2^{2r-1}+\sum_{i=0}^{r-1}\binom{2r}{i}V_{m(r-i)}^{2n}, & if\ r\ odd\ and\ m\ even; \\ \binom{2n}{n}2^{2r-1}+(p^2+4)^n\sum_{i=0}^{r-1}(-1)^{in}\binom{2r}{i}U_{m(r-i)}^{2n}, & if\ r\ odd\ and\ m\ odd; \end{cases}$$

where (A) follows from (11), (B) from (13), (C) from(7), (D) from (9). □

Acknowledgments

This research is supported by the National Natural Science Foundation of China (Grant No. 11071107).

References

[1] E. Kilic and E. J. Ionascu, Certain binomial sums with recursive coefficients, The Fibonacci Quarterly, 48.2(2010), 161-167.