On efficient dominating sets in simplicial graphs

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Abstract. Determining whether or not a graph has an efficient dominating set (equivalently, a perfect code) is an NP-complete problem. Here we present a polynomial time algorithm to decide if a given simplicial graph has an efficient dominating set. However, the efficient domination number decision problem is NP-complete for simplicial graphs.

Keywords: Dominating sets; Efficient domination number; Simplicial graphs; Computational complexity.

1 Introduction

The open neighborhood of a vertex v in a graph G is the set of vertices adjacent to v, $N(v) = \{w \in V(G) : vw \in E(G)\}$. The closed neighborhood of v is $N[v] = N(v) \cup \{v\}$. Vertex v is said to dominate each vertex in $N[v]$, and $D \subseteq V(G)$ is a *dominating set* if $\bigcup_{v \in D} N[v] = V(G)$. If D satisfies $|N[v] \cap D| = 1$ for every $v \in V(G)$, then D is called a *perfect code* in Biggs [6] or an *efficient dominating set* (EDS) in Bange, Barkaukas, (Host) and Slater [1, 2, 3, 4]. Note that if D is an EDS then for every pair of distinct vertices $u, v \in D$ we must have the distance $d(u, v) \geq 3$. Thus, $D \subseteq V(G)$ is an EDS for G if and only if D is a dominating set and a packing. Efficient domination and related influence parameters are studied in Sinko and Slater [14, 15], for example, for chessboard graphs.

Theorem 1 *[1, 2] If G has an efficient dominating set, then the cardinality of any efficient dominating set equals the domination number $\gamma(G)$. In particular, all efficient dominating sets have the same cardinality.*

EFFICIENT DOMINATING SET(EDS)

Instance: A graph $G = (V, E)$

Question: Does G have an efficient dominating set?

The EDS decision Problem is computationally difficult.

Theorem 2 *EDS is NP-complete*

(a)[8]for planar graphs of maximum degree three,

(b)[13]for bipartite graphs, and

(c)[13]for chordal graphs.

A vertex v in $V(G)$ is *simplicial* if any two vertices in $N(v)$ are adjacent, that is, if $< N[v] >$, the subgraph generated by $N[v]$, is a clique. A graph G is *simplicial* if every vertex in G is simplicial or it has a simplicial vertex as a neighbor. Simplicial graphs were introduced by Cheston, Hare, Hedetniemi and Laskar in [7]. A graph is *well-covered* if every maximal independent set of vertices is also maximum. A graph is Z_m-well-covered, if for any two maximal independent sets of vertices I and $J \in V(G)$, $|I| \equiv |J| \pmod{m}$. Simplicial well-covered graphs and simplicial Z_m-well-covered graphs were characterized, respectively, in [11] and [5]. Some properties of simplicial graphs are also given in [12]. The clique $< N[v] >$ of a simplicial vertex v is called a *simplex* of G.

We note that every split graph is a simplicial graph. A split graph G has $V(G) = V_1 \dot{\cup} V_2$ where the subgraph induced by V_1 is complete and V_2 is an independent set. Every vertex in V_2 is simplicial, and, if $v \in V_1$ with $N(v) \cap V_2 = \emptyset$, then v is simplicial. We also note that every graph H is the induced subgraph of a simplicial graph G. One can let G be the corona $H \circ K_1$, formed by attaching an endpoint to each $v \in V(H)$.

Here we give a characterization of simplicial graphs that have an EDS that provides a polynomial time recognition algorithm.

2 Efficient domination in simplicial graphs

Let $V(G) = \{v_1, v_2, ..., v_n\}$. One can easily identify the simplicial vertices and verify that G is simplicial in polynomial time. We assume that $v_1, v_2, ..., v_t$ are the simplicial vertices with simplicies $S_1, S_2, ..., S_t$ with $v_i \in S_i$, for $1 \leq i \leq t$. If v_i and v_j are *identical twins* (that is $N[v_i] = N[v_j]$), then G has an EDS if and only if $G - \{v_j\}$ does. So, we can assume that G does not have any identical twins.

The following is an obvious sufficient condition for G to have an EDS.

Proposition 1 *Let G be a simplicial graph. If each vertex in $V(G)$ belongs to exactly one simplex, then the set of simplicial vertices in G is an EDS.*

This condition is not necessary for G to have an EDS, as shown by graph G_1 in Figure 1. There are four simplicial vertices v_1, v_2, v_3, v_4, and v is adjacent to two of them ($v \in S_1$ and $v \in S_2$), but $\{v, v_3, v_4\}$ is an EDS.

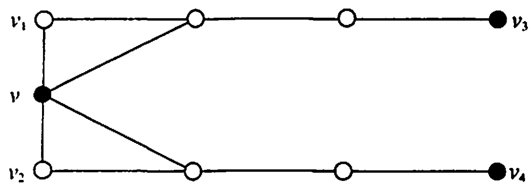

Figure 1: A graph G_1 with EDS $\{v, v_3, v_4\}$.

Let $S = \{v_1, v_2, ..., v_t\}$ be the set of simplicial vertices (with $t \geq 2$). The second neighborhood of a vertex u is the set of vertices at distance two from u, $N^2(u) = \{x \in V(G) : d(u,x) = 2\}$. The second closed neighborhood is $N^2[u] = N^2(u) \cup N[u]$.

Proposition 2 *If $u \in S$ is simplicial with $N^2(u) \cap S = \emptyset$ and $D \subseteq V(G)$ is an EDS, then $u \in D$.*

Proof: Assume $u \notin D$, and let $v \in N(u) \cap D$. Because u is simplicial, $N[u] \subseteq N[v]$. Because we have assumed there are no identical twins, there is a vertex $w \in N(v) \cap N^2(u)$. Since $w \notin S$ by assumption, there exists $x \in N(w) \cap S$. But now $v \in D$ implies that $N[x] \cap D = \emptyset$, a contradiction. □

Proposition 3 *For each simplicial vertex u and EDS D, we have $N^2(u) \cap D = \emptyset$. That is, each vertex at distance two from a simplicial vertex can not be in any EDS.*

Proof: Let u be a simplicial vertex and $w \in N^2(u)$. Then $N[u] \subseteq N^2[w]$. Hence, if $w \in D$ where D is a packing, then D does not dominate u. □

Let $S_2 = \{w \in V(G) : N^2(w) \cap S \neq \emptyset\}$. If D is an EDS of G then, by Proposition 3, $D \cap S_2 = \emptyset$.

For each vertex v let $f(v) = |N(v) \cap S|$ be the number of simplicial vertices adjacent to v.

Theorem 3 *Let G be an identical twin-free simplicial graph with simplicial vertex $u \in S$, and assume that $N^2(u) \cap S \neq \emptyset$ (that is, $u \in S_2$). Let $X = \{x \in N(u) : f(x) \geq f(v), \forall v \in N(u)\}$. If G has an EDS, then there is a unique vertex $x \in X$ with $x \notin S_2$ and x is in every EDS.*

Proof: Because G is identical twin-free, the set $S = \{v_1, v_2, ..., v_t\}$ of simplicial vertices is independent. Because $u \in S_2$ we have $(\{u\} \cup N^2(u)) \cap S \subseteq S_2$. Hence , any EDS D satisfies $|D \cap N(u)| = 1$. If $w \in N(u)$ and $w \notin X$, then $w \in S_2$ and so $w \notin D$ for any EDS D. It follows that, assuming G has an EDS, we have $D \cap (X - S_2) \neq \emptyset$.

Let $x \in X$ with $x \notin S_2$. To see that $|X - S_2| = 1$ (that is, $X - S_2 = \{x\}$) it will be shown that $x \neq x_2 \in X - S_2$ would imply that x and x_2 are identical twins, a contradiction. Assume $f(x) = f(x_2) = d \geq 2$. We can assume that $N(x) \cap S = \{u = v_1, v_2, ..., v_d\}$. If some $v_j \in N(x_2) \cap S$ with $d + 1 \leq j \leq t$, then $x \in S_2$, a contradiction. Hence, $N(x) \cap S = N(x_2) \cap S$. Suppose $y \in N(x_2)$ with $y \notin S$. Let s be a simplicial vertex in $N(y)$. Then $d(x_2, s) \leq 2$ and $x_2 \notin S_2$ implies that $x_2 s \in E(G)$. Now $s \in N(x_2) \cap S = N(x) \cap S$ implies that $xs \in E(G)$. A similar argument for any $y \in N(x) - S$ shows that $N[x] = N[x_2]$, a contradiction, completing the proof of the theorem. □

We now have a polynomial algorithm to determine if a simplicial graph G has an EDS. As noted, we can begin by deleting one of any pair of identical twins until G is identical twin-free. Form a dominating set D as follows. For each simplicial vertex s, if $s \notin S_2$ then put s in D, and if $s \in S_2$ then for the vertex x identified in Theorem 3, put $x \in D$ (while if no such x exists then G does not have an EDS). Then G has an EDS if and only if D is also packing.

3 Efficient Domination Number

Bange, Barkauskas and Slater [1, 2] considered determining how many vertices could be dominated given that no vertex gets dominated more than once. This has come to be called the *efficient domination number* of graph G, denoted by $F(G)$. Considering how close a graph is to having an

EDS/perfect code, Grinstead and Slater [9, 10] defined the *influence* of a vertex set $S \subseteq V(G)$ to be $I(S) = \sum_{s \in S}(1 + deg(s))$, the total amount of domination being done by S. Then $F(G) = max\{I(S) : S$ is a packing in $G\}$. (A related parameter is the redundance $R(G) = min\{I(S) : S$ is a dominating set for $G\}$.)

Theorem 2 states that the decision problem of deciding if $F(G) = |V(G)|$ is NP-complete. A linear time algorithm for determining $F(G)$ for generalized series-parallel graphs appears in Grinstead and Slater [10]. We have presented a polynomial time algorithm to see if $F(G) = |V(G)|$ for simplicial graphs in Section 2. Next we show that deciding if $F(G) \geq K$ is NP-complete even when G is restricted to be simplicial.

EFFICIENT DOMINATION NUMBER (EDN)

Instance: Graph=(V,E), positive integer $K \leq |V(G)|$.

Question: Is $F(G) \geq K$?

Theorem 4 *EDN is NP-complete even when G is restricted to be simplicial.*

Proof: We show an easy reduction from the known NP-complete problem of deciding if the packing number of a graph H satisfies $\rho(H) \geq L$. The reduction is facilitated by the fact that vertices of degree one are always simplicial.

Given a graph H and positive integer $L \leq |V(H)|$, let $n = |V(H)|$. Construct graph G by adding $2n+1$ endpoints attached to each $v \in V(H)$, so $|V(G)| = (2n+2)n$. It is straightforward to verify that $\rho(H) \geq L$ if and only if $F(G) \geq K = (2n+2)L$. □

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