On Balance Index Sets of Generalized Book and Ear Expansion Graphs

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Abstract

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$, and let $\mathbb{Z}_2 = \{0,1\}$. A labeling $f : V(G) \to \mathbb{Z}_2$ induces a partial edge labeling $f^* : E(G) \to \mathbb{Z}_2$ defined by $f^*(uv) = f(u)$ if and only if $f(u) = f(v)$. For $i \in \mathbb{Z}_2$, let $v_f(i) = |\{v \in V(G) : f(v) = i\}|$ and $e_f(i) = |\{e \in E(G) : f^*(e) = i\}|$. A labeling f is called a friendly labeling if $|v_f(0) - v_f(1)| \le 1$. The BI(G), the balance index set of G, is defined as $\{|e_f(0) - e_f(1)| :$ the vertex labeling f is friendly.$\}$. This paper focuses on the balance index sets of generalized book and ear expansion graphs.

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1 Introduction

In [8], Lee, Liu and Tan considered a labeling problem in graph theory. Let G be a graph with a vertex set $V(G)$ and an edge set $E(G)$. A vertex labeling of a graph G is a mapping f from $V(G)$ into $\{0,1\}$. For each vertex labeling f of G, we can define a partial edge labeling f^* of G as follows: for each edge uv in E, define

$$f^*(uv) = \begin{cases} 0 & \text{if } f(u) = f(v) = 0, \\ 1 & \text{if } f(u) = f(v) = 1. \end{cases}$$

Note that if $f(u) \neq f(v)$, then the edge uv is not labeled by f^*. We shall refer f^* as the ***induced partial function*** of f. For $i = 0, 1$, let $v_f(i)$ denote the number of vertices of G that are labeled by i under the mapping f. Similarly, let $e_f(i)$ denote the numbers of edges of G that are labeled by i under the induced partial function f^*. In other words, for $i = 0, 1$,

$$\begin{aligned} v_f(i) &= |\{u \in V(G) : f(u) = i\}|, \text{ and,} \\ e_f(i) &= |\{uv \in E(G) : f^*(uv) = i\}|. \end{aligned}$$

Definition 1. A vertex labeling f of a graph G is said to be ***friendly*** if $|v_f(0) - v_f(1)| \leq 1$, and ***balanced*** if both $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$.

It is clear that not all the friendly labelings are balanced.

In [6], A.N.T. Lee, Lee and Ng introduced the following notion as an extension of their study of the balanced graphs.

Definition 2. The ***balance index set*** of a graph G is defined as

$$\mathrm{BI}(G) = \{|e_f(0) - e_f(1)| : \text{the vertex labeling } f \text{ is friendly}\}.$$

Example 1. Figure 1 shows a graph G with $\mathrm{BI}(G) = \{0, 1, 2\}$. □

In [12], Lee, Wang and Wen found the balance index set of cycles.

Proposition 1.1. *Let $\bigcup_{i=1}^{k} C^i_{n_i}$ be a finite disjoint union of k cycles, where $C^i_{n_i}$ is the cycle of order n_i for all $1 \leq i \leq k$. The balance index set is*

$$BI(\bigcup_{i=1}^{k} C^i_{n_i}) = \begin{cases} \{0\} & \text{if } \sum n_i \text{ is even, and,} \\ \{1\} & \text{if } \sum n_i \text{ is odd.} \end{cases}$$

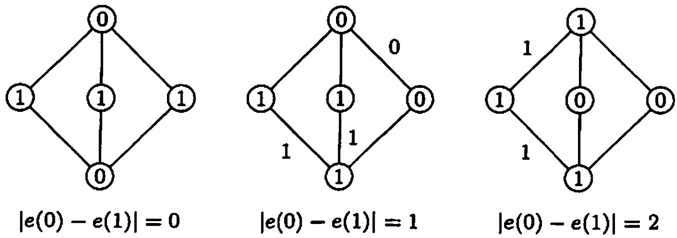

Figure 1: The friendly labelings of a graph G with $\mathrm{BI}(G) = \{0, 1, 2\}$.

We note here that not every graph has a balance index set consisting of an arithmetic progression.

Example 2. The graph $\Phi(1, 3, 1, 1)$ is composed of $C_4(3)$ with a pendant edge appended to each of x_1, x_3 and x_4, and three pendant edges appended to x_2. Figure 2 shows that $\mathrm{BI}(\Phi(1, 3, 1, 1)) = \{0, 1, 2, 3, 4, 6\}$. Note that 5 is missing from the balance index set. □

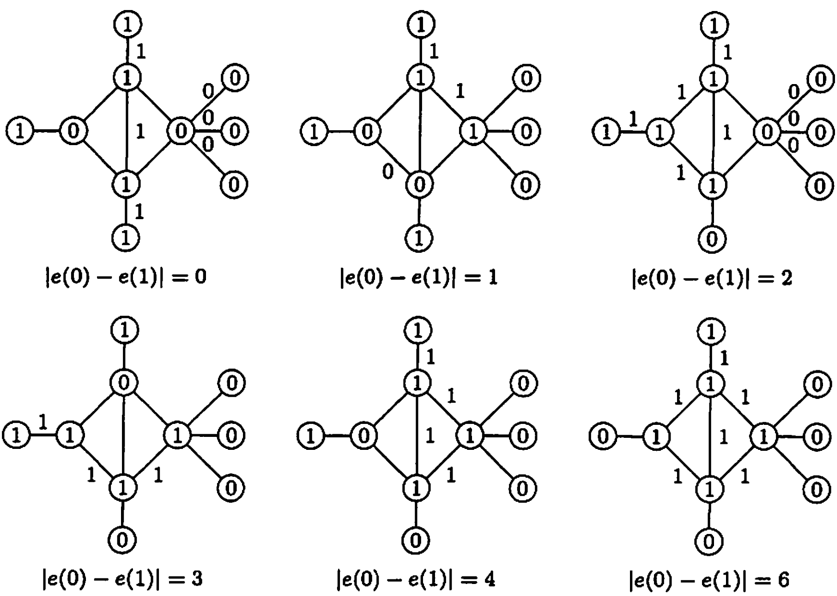

Figure 2: The six friendly labelings of $\Phi(1, 3, 1, 1)$.

Some balanced graphs are considered in [1, 2, 5, 8, 13]. In general, it is difficult to determine the balance index set of a given graph. Most of existing research on this problem have focused on some special families of graphs with simple structures, see [3, 4, 6, 9, 10, 12].

The balance index sets of the graph which are formed by the amalgamation of complete graphs, stars, and generalized theta graphs are studied in [3, 4]. In this paper, we complete the study of the balance index sets of generalized book and generalized ear expansion graphs.

2 General Theory of the balanced index sets

For a graph with a vertex labeling f, we denote $e_f(\times)$ to be the number of the unlabeled edges.

For brevity, when the context is clear, we will simply write $v(0)$, $v(1)$, $e(0)$, $e(1)$ and $e(\times)$ without any subscript.

In [14], an algebraic approach to solving balance index sets was discovered. In [11], this approach was adapted into the following powerful equalities:

Proposition 2.1. *For any vertex labeling f, let $V(0)$ and $V(1)$ be the subsets of $V(G)$ which contain the vertexes labeled 0 and 1, respectively. We have the following equalities:*

1. $2e(0) + e(\times) = \sum_{v \in V(0)} \deg v;$

2. $2e(1) + e(\times) = \sum_{v \in V(1)} \deg v;$

3. $2|E(G)| = \sum_{v \in V(G)} \deg v = \sum_{v \in V(0)} \deg v + \sum_{v \in V(1)} \deg v.$

Proposition 2.2. *For any friendly vertex labeling f, the balance index is*

$$e(0) - e(1) = \frac{1}{2}\left(\sum_{v \in V(0)} \deg v - \sum_{v \in V(1)} \deg v\right).$$

Note here that Proposition 2.2 gives the balance index for a given friendly labeling before applying the absolute value function.

Lemma 2.3. *Let G be a graph with $2n$ vertices and H be a graph with $2n + 1$ vertices, where n is a positive integer. Let their degree sequences be*

$2 \le g_1 \le g_2 \le \ldots \le g_n \le g_{n+1} \le \ldots \le g_{2n}$ *and* $2 \le h_1 \le h_2 \le \ldots \le h_n \le h_{n+1} \le h_{n+2} \le \ldots \le h_{2n+1}$, *respectively. If*

$$g_i = h_i = 2, \text{ for } i = 1, 2, \ldots, n$$

and

$$g_i = h_{i+1}, \text{ for } i = n+1, n+2, \ldots, 2n$$

with $h_{n+1} = 2$, *then, for a vertex labeling* f, *a balance index of* H *can be found by adding one or subtracting one from a balance index of* G.

Proof. For any friendly labeling, by Proposition 2.2, its balance index is

$$\frac{1}{2}\left(\sum_{v \in v(0)} \deg v - \sum_{v \in v(1)} \deg v\right).$$

Since all the degrees are the same except H has one extra order 2 in the middle of the degree sequence, obviously, the difference between two balance indexes is ± 1 depends on the label of the vertex whose degree is h_{n+1}. □

This lemma leads us to focus on the balance index sets of the even-number-of-vertices graphs first. Then, the balance index sets of the odd-number-of-vertices graphs just follows as:

Corollary 2.4. *Let* G *be a graph with* $2n$ *vertices and* H *be a graph with* $2n+1$ *vertices, where* n *is a positive integer. Let their degree sequences be* $2 \le g_1 \le g_2 \le \ldots \le g_n \le g_{n+1} \le \ldots \le g_{2n}$ *and* $2 \le h_1 \le h_2 \le \ldots \le h_n \le h_{n+1} \le h_{n+2} \le \ldots \le h_{2n+1}$, *respectively. If*

$$g_i = h_i = 2, \text{ for } i = 1, 2, \ldots, n$$

and

$$g_i = h_{i+1}, \text{ for } i = n+1, n+2, \ldots, 2n$$

with $h_{n+1} = 2$, *then,*

$$BI(H) = BI(G) \pm 1,$$

where $BI(G) \pm 1$ *is defined as*

$$\left\{ |BI \pm 1| \;\middle|\; \begin{array}{c} BI \text{ is a balance index, } e_f(0) - e_f(1), \text{ of } G \\ \text{before applying absolute value function.} \end{array} \right\}.$$

3 On Balance Index Sets of Generalized Book Graphs

A generalized book graph has two inner vertices which form the "spine" of the book. Note that in this paper, these vertices will be called v_1 and v_2. The "pages" are cycles which are attached to these inner vertices.

For $k \geq 1$, the notation $\mathrm{GB}(k; n_1, n_2, \ldots, n_k)$ refers to a graph with k cycles with vertices $n_1, n_2, \ldots, n_k$ respectively. However, we note here that the position of the cycles is irrelevant to the balance index set. We also note that the total vertex count of a generalized book graph G is $|V(G)| = n_1 + n_2 + ... + n_k - 2(k-1)$.

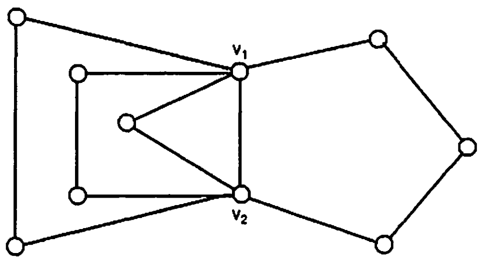

Figure 3: GB(4; 3, 4, 4, 5)

Theorem 3.1. *For $GB(k; n_1, n_2, \ldots, n_k)$, where the total number of vertices is even, the balance index set is*

$$BI(GB(k; n_1, n_2, \ldots, n_k)) = \{0, k-1\}.$$

Proof. Let G be a generalized book graph, $\mathrm{GB}(k; n_1, n_2, \ldots, n_k)$, with even total vertex count $|V(G)| = n_1 + n_2 + ... + n_k - 2(k-1) = 2M$. Therefore, for a friendly labeling, there are M 0-vertices and M 1-vertices.

Our inner vertices v_1 and v_2 have degree $k+1$ while all other vertices in the graph have degree 2. We consider the case where both inner vertices are labeled 0. By Proposition 2.2, it follows that

$$\begin{aligned} e(0) - e(1) &= \frac{1}{2}\left(\sum_{v \in v(0)} \deg v - \sum_{v \in v(1)} \deg v\right) \\ &= \frac{1}{2}(2(k+1) + 2(M-2) - 2M) \\ &= k - 1 \end{aligned}$$

By similar computations, we get the following table covering all possible cases of inner vertex labelings.

$f(v_1)$	$f(v_2)$	Balance Index
0	0	$k-1$
0	1	0
1	1	$k-1$

Therefore, the set of balance indices is $\{0, k-1\}$ for graphs with an even number of vertices with $k \geq 1$. □

By Lemma 2.3, we can easily extend the result to the generalized book graphs with odd number of vertices.

Corollary 3.2. *For $GB(k; n_1, n_2, \ldots, n_k)$, where the total number of vertices is odd, the balance index set is*

$$\{1, |k-2|, k\}.$$

4 On Balance Index Sets of Generalized Ear Expansion Graphs

A generalized ear expansion graph has an inner cycle, C_n, with n vertices and edges. We will refer to these vertices as $v_1, v_2, \ldots, v_n$ and edges as $e_1, e_2, \ldots, e_n$ with the convention that e_1 is located between v_1 and v_2, etc. Outer cycles may be attached to any of the edges of the inner cycle. An outer cycle should share no edges with another cycle besides the one edge it shares with the inner cycle.

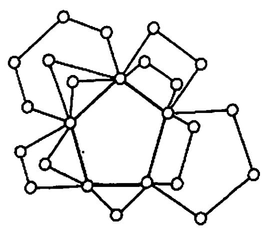

Figure 4: GE(C_5; [3, 3, 6], [3, 4], [3], [4, 5], [4, 4])

We use GE(C_n; $[p_1^1, p_2^1, \ldots, p_{k_1}^1], [p_1^2, p_2^2, \ldots, p_{k_2}^2], \ldots, [p_1^n, p_2^n, \ldots, p_{k_n}^n]$) to denote an ear expansion graph with inner cycle C_n, where $k_i \geq 1$ for all $i = 1, 2, \ldots, n$ and $p_i^j \geq 3$ for all i, j. Each set of brackets contains the cycles which are attached to a particular edge. In general, $[p_1^i, p_2^i, \ldots, p_{k_i}^i]$ refers to the collection of vertices of k_i cycles which share the edge e_i in the inner cycle. Note that the order of the bracketed collections may effect the balance index set of the graph. We also note that the total vertex count of a generalized ear expansion graph G is

$$|V(G)| = p_1^1+p_2^1+\ldots+p_{k_1}^1+p_1^2+\ldots+p_{k_2}^2+\ldots+p_{k_n}^n-2(k_1+k_2+\ldots+k_n)+n$$

Note here that, for convenience, all the subscripts of v and e are modulo n. Therefore, $v_{n+1} = v_1$, $e_{n+2} = e_2$, and so on.

Let G be a generalized ear expansion graph $\mathrm{GE}(C_n; [p_1^1, p_2^1, \ldots, p_{k_1}^1], [p_1^2, p_2^2, \ldots, p_{k_2}^2], \ldots, [p_1^n, p_2^n, \ldots, p_{k_n}^n])$ with even number of vertices. For a friendly labeling, we have M vertices labeled 0 and M vertices labeled 1.

For any inner vertex v_i of the inner cycle, the degree is the sum of the edges of the inner cycle adjacent to v_i (which is always 2) with the number of cycles attached to each inner cycle edge adjacent to v_i. In particular,

$$\deg v_i = 2 + k_{i-1} + k_i$$

Also, every vertex which is not in the inner cycle has degree of exactly 2. If there are q inner 0-vertices, then there are $M - q$ outer 0-vertices. We also have $n - q$ inner 1-vertices and $M - (n - q)$ outer 1-vertices. By Proposition 2.2, some of the 0 and 1 outer vertices will cancel each other's degree contribution to the balance index. Without loss of generality, we have $2(M - q) - 2(M - (n - q)) = 2(n - q) - 2q$ as the degree contribution leftover from the outer vertices. The q inner 0-vertices provide a degree of at least 2 and the $n - q$ inner 1-vertices subtract a degree of at least 2. Therefore, two degrees of each inner vertex can be used to effectively cancel out any contribution from outer vertices. This leaves us with remaining contributions from inner vertices only. In particular, any inner vertex v_i could contribute $k_{i-1} + k_i$ to the balance index before dividing by two.

If the vertex v_i is labeled 0, then, according to Proposition 2.2 and the above discussion, it contributes $\frac{1}{2}(k_{i-1} + k_i)$ to the balance index. Similarly, if the vertex v_i is labeled 1, then it contributes $-\frac{1}{2}(k_{i-1} + k_i)$.

For any value k_i associated to the edge e_i, only two adjacent vertices v_i and v_{i+1} can contribute it into the balance index. When both v_i and v_{i+1} are labeled 0, we can see the contribution is $k_{i-1} + 2k_i + k_{i+1}$. Thus, the value k_i survives in the balance index. Similarly, the balance index contains the following term:

1. k_i if v_i and v_{i+1} are both labeled 0.

2. $-k_i$ if v_i and v_{i+1} are both labeled 1.

3. no k_i if v_i and v_{i+1} are differently.

Example 3. For an ear expansion graph GE(C_5; [3, 3, 6], [3, 4], [3], [4, 5], [4, 4]) with totally 24 vertices, if you label v_1, v_2, v_3 by 0 and v_4, v_5 by 1, then the balance index is

$$\begin{aligned} & \frac{1}{2}\left((k_5+k_1)+(k_1+k_2)+(k_2+k_3)-(k_3+k_4)-(k_4+k_5)\right) \\ = \; & k_1+k_2-k_4 \end{aligned}$$

□

We can permute the labels of the inner cycle vertices. It can generates other balance indexes in the same pattern. See the following example.

Example 4. From the previous example, we have a labeling with the balance index $k_1 + k_2 - k_4$. If we permute the labels, for example, we label v_2, v_3, v_4 by 0 and v_5, v_1 by 1, then we have a new labeling with the balance index $k_2 + k_3 - k_5$. The balance index permutes too. □

Example 5. Here we completely list all combinations in a generalized ear expansion graph with inner cycle C_3 and even number of vertices.

v_1	v_2	v_3	Balance Index
0	0	0	$k_1+k_2+k_3$
0	0	1	k_1
0	1	0	k_3
1	0	0	k_2
0	1	1	$\|-k_2\|$
1	0	1	$\|-k_3\|$
1	1	0	$\|-k_1\|$
1	1	1	$\|-k_1-k_2-k_3\|$

Thus, the balance index set is $\{k_1, k_2, k_3, k_1 + k_2 + k_3\}$. □

We note that by changing any single vertex's labeling in the inner cycle, the resulting change in the number of k terms in the corresponding balance index is reduced by 0 or 2. Also, if n is even or odd, we can get a balance index of $k_1 + k_2 + ... + k_n$ by labeling all inner vertices 0. Therefore, if n is even, each labeling of the inner vertices will return an even number of k terms for its balance index and if n is odd, an odd number of terms.

Now, we can write down an algorithm to produce the balance index set of a generalized ear expansion graph.

Theorem 4.1. *Let $n \geq 3$ be a positive integer. Consider a generalized ear expansion graph, GE(C_n; $[p_1^1, p_2^1, ..., p_{k_1}^1]$, $[p_1^2, p_2^2, ..., p_{k_2}^2]$, ..., $[p_1^n, p_2^n, ..., p_{k_n}^n]$) with* *even number of vertices. The following algorithm can produce its balance index set.*

1. *The maximal balance index is* $\sum_{i=1}^{n} k_i$.

2. *Label inner vertices by* 0 *and* 1 *where* v_1 *is labeled* 1 *and* v_n *is labeled* 0 *with the number of* 0*'s is greater or equal to the number of* 1*'s.*

3. *Determine if* k_i *shows up in the pattern by pattern contains*

   (a) k_i *if* v_i *and* v_{i+1} *are both labeled* 0.

   (b) $-k_i$ *if* v_i *and* v_{i+1} *are both labeled* 1.

   (c) *no* k_i *if* v_i *and* v_{i+1} *are differently.*

4. *Rotate the pattern to get all balance indexes which look alike.*

5. *Repeat steps 2-4 until we run through all possible combinations of the inner vertices labeling.*

6. *Collect balance indexes generated by step 2-5 to the balance index set.*

Proof. When every inner vertex is labeled 0, every k_i shows up. Thus, according to the previous discussion, the maximal balance index is $\sum_{i=1}^{n} k_i$.

Step 2 restricts v_1 to be labeled 1 and v_n to be labeled 0 to avoid redundancy. Obviously, by step 3, rotating covers all possible labeling of the inner vertices.

By the symmetry of 0 and 1 in the definition of the balance index set, after taking absolute value, labeling the inner vertices with more 1's than 0 only gives redundant balance indexes.

Since we run through all combinations of the inner vertices labeling, the collection is the balance index set we are looking for. □

Example 6. For an ear expansion graph $GE(C_4; [3,3,6],[3,4],[3],[4,4])$, we have $k_1 = 3$, $k_2 = 2$, $k_3 = 1$ and $k_4 = 2$ with 18 vertices , there are four patterns for the balance indexes: $\{k_1+k_2+k_3+k_4\}$, $\{k_2+k_3\}$, $\{k_3-k_1\}$, $\{0\}$. After permutation, the balance index set is

$$\begin{aligned} \mathrm{BI} &= \{k_1+k_2+k_3+k_4, k_2+k_3, k_3+k_4, k_4+k_1, k_1+k_2, \\ &\quad |k_3-k_1|, |k_4-k_2|, |k_1-k_3|, |k_2-k_3|, 0\}. \end{aligned}$$

□

Example 7. For an ear expansion graph GE(C_5; [3, 3, 6], [3, 4], [3], [4, 5], [4, 4]), we have $k_1 = 3$, $k_2 = 2$, $k_3 = 1$, $k_4 = 2$ and $k_5 = 2$ with 24 vertices, there are five patterns for the balance indexes: $\{k_1 + k_2 + k_3 + k_4 + k_5\}$, $\{k_2 + k_3 + k_4\}$, $\{k_3 + k_4 - k_1\}$, k_4, k_2. After permutation, the balance index set is

$$\begin{aligned} \text{BI} = \{ & k_1 + k_2 + k_3 + k_4 + k_5, k_2 + k_3 + k_4, k_3 + k_4 + k_5, k_4 + k_5 + k_1, \\ & k_5 + k_1 + k_2, k_1 + k_2 + k_3, |k_3 + k_4 - k_1|, |k_4 + k_5 - k_2|, \\ & |k_5 + k_1 - k_3|, |k_1 + k_2 - k_4|, |k_2 + k_3 - k_5|, k_1, k_2, k_3, k_4, k_5\}. \end{aligned}$$

□

For any generalized ear expansion graph G with odd number of vertices, we can turn it into a generalized ear expansion graph H with even number of vertices by removing an outer order 2 vertex with one of its adjacent edge. This means that we can compare G to a similar graph H with $2M$ vertices with one less vertex in one of its outer cycles. (Note that this does not affect any k_i values). We can find the balance index set of a graph with $2M$ vertices by the algorithm in Theorem 4.1. Then, by Corollary 2.4, we have $\text{BI}(H) = \text{BI}(G) \pm 1$.

Example 8. For an ear expansion graph GE(C_4; [3, 3, 6], [3, 5], [3], [4, 4]), we have $k_1 = 3$, $k_2 = 2$, $k_3 = 1$ and $k_4 = 2$ with 19 vertices, after removing an outer order 2 vertex with one of its adjacent edge, it becomes an ear expansion graph GE(C_4; [3, 3, 6], [3, 4], [3], [4, 4]). Thus, the balance index set is

$$\begin{aligned} \text{BI} = \{ & k_1 + k_2 + k_3 + k_4 \pm 1, k_2 + k_3 \pm 1, k_3 + k_4 \pm 1, k_4 + k_1 \pm 1, \\ & k_1 + k_2 \pm 1, |k_3 - k_1 + 1|, |k_4 - k_2 + 1|, |k_1 - k_3 + 1|, |k_2 - k_3 + 1|, \\ & |k_3 - k_1 - 1|, |k_4 - k_2 - 1|, |k_1 - k_3 - 1|, |k_2 - k_3 - 1|, 1\}. \end{aligned}$$

□

Example 9. For an ear expansion graph GE(C_5; [3, 3, 6], [3, 4], [3], [4, 6], [4, 4]), we have $k_1 = 3$, $k_2 = 2$, $k_3 = 1$, $k_4 = 2$ and $k_5 = 2$ with 25 vertices, after removing an outer order 2 vertex with one of its adjacent edge, it becomes an ear expansion graph GE(C_5; [3, 3, 6], [3, 4], [3], [4, 5], [4, 4]). Thus, the balance index set is

$$\begin{aligned}\mathrm{BI} = \{&k_1+k_2+k_3+k_4+k_5 \pm 1, k_2+k_3+k_4 \pm 1, k_3+k_4+k_5 \pm 1,\\ &k_4+k_5+k_1 \pm 1, k_5+k_1+k_2 \pm 1, k_1+k_2+k_3 \pm 1,\\ &|k_3+k_4-k_1+1|, |k_4+k_5-k_2+1|, |k_5+k_1-k_3+1|,\\ &|k_1+k_2-k_4+1|, |k_2+k_3-k_5+1|, |k_3+k_4-k_1-1|,\\ &|k_4+k_5-k_2-1|, |k_5+k_1-k_3-1|, |k_1+k_2-k_4-1|,\\ &|k_2+k_3-k_5-1|, k_1 \pm 1, k_2 \pm 1, k_3 \pm 1, k_4 \pm 1, k_5 \pm 1\}.\end{aligned}$$

□

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