Langford-type Difference Sets for Cycle Systems

T. Helms, H. Jordon, M. Murray, S. Zeppetello
Department of Mathematics, Illinois State University
Normal, IL 61790-4520

Abstract

A Langford-type m-tuple difference set of order t and defect d is a set of t m-tuples $\{(d_{i,1}, d_{i,2}, \ldots, d_{i,m}) \mid i = 1, 2, \ldots, t\}$ such that $d_{i,1} + d_{i,2} + \cdots + d_{i,m} = 0$ for $1 \leq i \leq t$ and $\{|d_{i,j}| \mid 1 \leq i \leq t, 1 \leq j \leq m\} = \{d, d+1, \ldots, d+mt-1\}$. In this paper, we give necessary and sufficient conditions on t and d for the existence of a Langford-type m-tuple difference set of order t and defect d when $m \equiv 0, 2 \pmod 4$. In the case that $m \equiv 1, 3 \pmod 4$, we provide sufficient conditions for the existence of a Langford-type m-tuple difference set of order t and defect d when d is at most about $t/2$. Using these results, we obtain cyclic m-cycle systems of the circulant graph $\langle\{d, d+1, \ldots, d+mt-1\}\rangle_n$ for all $n \geq 2(d+mt)-1$ with d and t satisfying certain conditions.

1 Introduction

For integers a and b, the notation $[a, b]$ denotes the set $\{a, a+1, \ldots, b\}$. A *Skolem sequence of order* t is a sequence $S = (s_1, s_2, \ldots, s_{2t})$ of $2t$ integers satisfying the conditions

(S1) for every $k \in [1, t]$ there exist exactly two elements $s_i, s_j \in S$ such that $s_i = s_j = k$;

(S2) if $s_i = s_j = k$ with $i < j$, then $j - i = k$.

A Skolem sequence of order t provides a partition of the set $[1, 3t]$ into t triples (a_i, b_i, c_i) such that $a_i + b_i = c_i$; for example, if $S = (s_1, s_2, \ldots, s_{2t})$ is a Skolem sequence of order t, then $\{(k, t+i, t+j) \mid 1 \leq k \leq t, s_i = s_j =$

k with $i < j\}$ is a partition of the set $[1, 3t]$ into t such triples. It is well-known that a Skolem sequence of order t exists if and only if $t \equiv 0, 1 \pmod 4$ [19].

A *Langford sequence of order t and defect d* is a sequence $L = (\ell_1, \ell_2, \ldots, \ell_{2t})$ of $2t$ integers satisfying the conditions

(L1) for every $k \in [d, d+t-1]$ there exists exactly two elements $\ell_i, \ell_j \in L$ such that $\ell_i = \ell_j = k$, and

(L2) if $\ell_i = \ell_j = k$ with $i < j$, then $j - i = k$.

Clearly, a Langford sequence with defect 1 is a Skolem sequence, and a Langford sequence of order t and defect d provides a partition of the set $[d, d+3t-1]$ into t triples (a_i, b_i, c_i) such that $a_i + b_i = c_i$. The following theorem gives necessary and sufficient conditions for the existence of Langford sequences.

Theorem 1.1 (Simpson [18]) *There exists a Langford sequence of order t and defect d if and only if*

(1) $t \geq 2d - 1$, and

(2) $t \equiv 0, 1 \pmod 4$ and d is odd, or $t \equiv 0, 3 \pmod 4$ and d is even.

Skolem sequences and their generalizations have been used widely in the construction of combinatorial designs and a survey on Skolem sequences can be found in [10]. In the literature, difference triples obtained from Skolem and Langford sequences are usually written (a, b, c) with $a+b = c$. However, the equivalent representation, with c replaced by $-c$ so that $a+b+c = 0$, is more convenient for the purpose of extending these ideas to m-tuples with $m > 3$. As such, the following definition was given in [3].

Definition 1.2 An m-tuple $(d_1, d_2, \ldots, d_m)$ is of *Skolem-type* if $d_1 + d_2 + \cdots + d_m = 0$. A set of t Skolem-type m-tuples $\{(d_{i,1}, d_{i,2}, \ldots, d_{i,m}) \mid i = 1, 2, \ldots, t\}$ such that $\{|d_{i,j}| \mid 1 \leq i \leq t, 1 \leq j \leq m\} = [1, mt]$ is called a *Skolem-type m-tuple difference set of order t.*

Necessary and sufficient conditions for the existence of Skolem-type m-tuple difference sets of order t were found in [3], where the following theorem was given.

Theorem 1.3 (Bryant, Gavlas, Ling [3]) *Let m and t be integers with $m \geq 3$ and $t \geq 1$. There exists a Skolem-type m-tuple difference set of order t if and only if $mt \equiv 0, 3 \pmod 4$.*

In this paper, we are interested in Langford-type m-tuple different sets and thus we make the following definition.

Definition 1.4 A *Langford-type m-tuple difference set of order t and defect d* is a set of t Skolem-type m-tuples $\{(d_{i,1}, d_{i,2}, \dots, d_{i,m}) \mid i = 1, 2, \dots, t\}$ such that $\{|d_{i,j}| \mid 1 \leq i \leq t, 1 \leq j \leq m\} = [d, d + mt - 1]$.

In Section 2, we show that a Langford-type m-tuple difference set of order t and defect d exists under the following conditions:

(1) for all positive integers t and d when $m \equiv 0 \pmod 4$;

(2) for all positive integers t and d with $t \equiv 0, 2 \pmod 4$ when $m \equiv 2 \pmod 4$;

(3) for all positive integers t and d with $2d - 1 \leq t$ and $t \equiv 0, 1 \pmod 4$ if d is odd or $t \equiv 0, 3 \pmod 4$ if d is even when $m \equiv 3 \pmod 4$; and

(4) for all positive integers t and d with $t \equiv 0, 1 \pmod 4$ and $d \leq \lfloor \frac{t}{2} \rfloor$ if d is even or $t \equiv 0, 3 \pmod 4$ and $d \leq \lfloor \frac{t-5}{2} \rfloor$ if d is odd when $m \equiv 1 \pmod 4$.

In Section 3, using the results of Section 2, we find cyclic m-cycle systems of the circulant graphs $\langle [d, d+mt-1] \rangle_n$ when d and t satisfy the conditions above and $n \geq 2(d + mt) - 1$.

2 Construction of Langford-type m-tuple Difference Sets

Before proving our main results, we will need the following lemma given in [3], and used in extending m-tuple difference sets of order t to $(m+4)$-tuple difference sets of order t. The proof of this lemma is included as we will refer to it in the proof of Theorem 3.4 in Section 3.

Lemma 2.1 (Bryant, Gavlas, Ling [3]) *Let n, r and t be positive integers. There exists a $t \times 4r$ matrix $Y(r, n, t) = [y_{i,j}]$ such that $\{|y_{i,j}| \mid 1 \leq i \leq t, 1 \leq j \leq 4r\} = [n + 1, n + 4rt]$, the sum of the entries in each row of $Y(r, n, t)$ is zero, and $|y_{i,1}| < |y_{i,2}| < \dots < |y_{i,4r}|$ for $i = 1, 2, \dots, t$.*

Proof. Let $Y'(r,n,t)$ be the matrix

$$\begin{bmatrix} 2t-1 & 2t & 4t-1 & 4t & & 4rt-1 & 4rt \\ 2t-3 & 2t-2 & 4t-3 & 4t-2 & & 4rt-3 & 4rt-2 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 3 & 4 & 2t+3 & 2t+4 & & (4r-2)t+3 & (4r-2)t+4 \\ 1 & 2 & 2t+1 & 2t+2 & & (4r-2)t+1 & (4r-2)t+2 \end{bmatrix} + N$$

where $N = [n]$ is the $t \times 4r$ matrix such that every entry is the positive integer n. Let Y be the matrix obtained from Y' by multiplying by -1 each entry in column j for all $j \equiv 2,3 \pmod 4$. It is straightforward to verify that Y has the required properties. ∎

The next lemma provides necessary conditions on the congruence class of mt modulo 4 for the existence of a Langford-type m-tuple difference set of order t and defect d.

Lemma 2.2 *Let m, d, and t be positive integers with $m \geq 3$. If a Langford-type m-tuple difference set of order t and defect d exists, then $mt \equiv 0,1 \pmod 4$ when d is even and $mt \equiv 0,3 \pmod 4$ when d is odd.*

Proof. Let m, d and t be positive integers with $m \geq 3$, and suppose a Langford-type m-tuple difference set $\{(d_{i,1}, d_{i,2}, \ldots, d_{i,m}) \mid 1 \leq i \leq t\}$ of order t and defect d exists. Since each m-tuple $(d_{i,1}, d_{i,2} \cdots d_{i,m})$ has the property that $d_{i,1} + d_{i,2} + \cdots + d_{i,m} = 0$, it follows that each m-tuple has an even number of odds and thus the set $[d, d+mt-1]$ must have an even number of odds. Hence, the sum $d + (d+1) + \cdots + (d+mt-1)$ must be even.

Suppose first that d is even. If $mt \equiv 2,3 \pmod 4$, then $d + (d+1) + \cdots + (d+mt-1)$ is odd and no Langford-type m-tuple difference set of order t and defect d exists. Thus, $mt \equiv 0,1 \pmod 4$.

Finally, suppose that d is odd. If $mt \equiv 1,2 \pmod 4$, then $d + (d+1) + \cdots + (d+mt-1)$ is odd and no Langford-type m-tuple difference set of order t and defect d exists. Thus, $mt \equiv 0,3 \pmod 4$. ∎

We now give sufficient conditions for the existence of Langford-type m-tuple difference sets of order t and defect d when $m \equiv 0 \pmod 4$ with $m \geq 4$ and d and t are positive integers.

Lemma 2.3 *Let m, d and t be positive integers such that $m \equiv 0 \pmod 4$ with $m \geq 4$. Then there exists a Langford-type m-tuple difference set of order t and defect d.*

Proof. Let m, d and t be positive integers such that $m \equiv 0 \pmod 4$ with $m \geq 4$. Let $X = Y(\frac{m}{4}, d-1, t) = [x_{ij}]$ be the $t \times m$ matrix given by Lemma

2.1. Clearly, the entries of X in absolute value are $d, d+1, \ldots, d+mt-1$ and for each $i = 1, 2, \ldots, t$, we have $\sum_{j=1}^{m} x_{ij} = 0$. Thus the t rows of X give a Langford-type m-tuple difference set of order t and defect d. ■

Observe that Lemmas 2.2 and 2.3 provide necessary and sufficient conditions on t and d for the existence of a Langford-type m-tuple difference set of order t and defect d when $m \equiv 0 \pmod 4$. We now give, in Lemma 2.4, sufficient conditions for the existence of Langford-type m-tuple difference sets of order t and defect d when $m \equiv 2 \pmod 4$ with $m \geq 6$ and $t \equiv 0, 2 \pmod 4$. Thus Lemmas 2.2 and 2.4 will give necessary and sufficient conditions on t and d for the existence of a Langford-type m-tuple difference set of order t and defect d when $m \equiv 2 \pmod 4$.

Lemma 2.4 *Let m, d and t be positive integers such that $m \equiv 2 \pmod 4$ with $m \geq 6$ and $t \equiv 0, 2 \pmod 4$. Then there exists a Langford-type m-tuple difference set of order t and defect d.*

Proof. Let m, d and t be positive integers such that $m \equiv 2 \pmod 4$ with $m \geq 6$ and $t \equiv 0, 2 \pmod 4$. Since t is even, $2t \equiv 0 \pmod 4$ so that the set $[d + 4t, d + 6t - 1]$ contains an even number of odds and an even number of evens. Thus, these $2t$ integers can be paired into sets $\{a_i, a_i + 2\}$ for $i = 1, 2, \ldots, t$.

Let $X = [x_{ij}]$ be the $t \times m$ matrix

$$X = \left[\begin{array}{cccccc|c} d & -(d+1) & d+2 & -(d+3) & -a_1 & a_1+2 & \\ d+4 & -(d+5) & d+6 & -(d+7) & -a_2 & a_2+2 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & Y \\ d+4t-4 & -(d+4t-3) & d+4t-2 & -(d+4t-1) & -a_t & a_t+2 & \end{array} \right]$$

where $Y = Y(\frac{m-6}{4}, d + 6t - 1, t)$ is the $t \times \frac{m-6}{4}$ matrix given by Lemma 2.1. Note that the entries of X in absolute values are $d, d+1, \ldots, d+mt-1$ and that for each $i = 1, 2, \ldots, t$, we have $\sum_{j=1}^{m} x_{ij} = 0$. Thus, the t rows of X give a Langford-type m-tuple difference set of order t and defect d. ■

We now give sufficient conditions for the existence of Langford-type m-tuple difference sets of order t and defect d when $m \equiv 3 \pmod 4$ with $m \geq 3$ and $2d - 1 \leq t$ with $t \equiv 0, 1 \pmod 4$ when d is odd, or $t \equiv 0, 3 \pmod 4$ when d is even.

Lemma 2.5 *Let m, d and t be positive integers such that $m \equiv 3 \pmod 4$ with $m \geq 3$, $2d - 1 \leq t$, and $t \equiv 0, 1 \pmod 4$ when d is odd, or $t \equiv$*

0, 3 (mod 4) *when d is even. Then there exists a Langford-type m-tuple difference set of order t and defect d.*

Proof. Let m, d and t be positive integers such that $m \equiv 3 \pmod 4$ with $m \geq 3$, $2d - 1 \leq t$, and $t \equiv 0, 1 \pmod 4$ when d is odd, or $t \equiv 0, 3 \pmod 4$ when d is even. By Theorem 1.1, there exists a Langford sequence of order t and defect d which will give a partition of the set $[d, d + 3t - 1]$ into t triples (a_i, b_i, c_i) with $a_i + b_i = c_i$ for $i = 1, 2, \ldots, t$.

Let $X = [x_{ij}]$ be the $t \times m$ matrix

$$X = \begin{bmatrix} a_1 & -c_1 & b_1 & \\ a_2 & -c_2 & b_2 & \\ \vdots & \vdots & \vdots & Y \\ a_t & -c_t & b_t & \end{bmatrix}$$

where $Y = Y(\frac{m-3}{4}, d + 3t - 1, t)$ is the $t \times \frac{m-3}{4}$ matrix given by Lemma 2.1. Note that the entries of X in absolute values are $d, d+1, \ldots, d+mt-1$ and that for each $i = 1, 2, \ldots, t$, we have $\sum_{j=1}^{m} x_{ij} = 0$. Thus, the t rows of X give a Langford-type m-tuple difference set of order t and defect d. ■

We now consider the case when $m \equiv 1 \pmod 4$ with $m \geq 5$. For $m \equiv 1 \pmod 4$, note that a Langford sequence of order t and defect $d - 1$ provides a Langford type m-tuple difference set of order t and defect d when $2(d-1) - 1 \leq t$, or $d \leq \frac{t+3}{2}$ as follows. Let d, t be positive integers such that $d \leq \frac{t+3}{2}$ and let $t \equiv 0, 1 \pmod 4$ if d is even, or $t \equiv 0, 3 \pmod 4$ if d is odd. By Theorem 1.1, there exists a partition of $[d - 1, d + 3t]$ into triples $\{a_i, b_i, c_i\}$ such that $a_i + b_i = c_i$ for $t = 1, 2, \ldots, t$. Let $X = [x_{ij}]$ be the $t \times m$ matrix

$$X = \begin{bmatrix} a_1+1 & -(c_1+1) & b_1+1 & d+3t+2 & -(d+3t+3) & \\ a_2+1 & -(c_2+1) & b_2+1 & d+3t+3 & -(d+3t+4) & \\ \vdots & \vdots & \vdots & \vdots & \vdots & Y \\ a_t+1 & -(c_t+1) & b_t+1 & d+5t & -(d+5t+1) & \end{bmatrix}$$

where $Y = Y(\frac{m-3}{4}, d+5t+1, t)$ is the $t \times \frac{m-5}{4}$ matrix given by Lemma 2.1. Note that the entries of X in absolute values are $d, d+1, \ldots, d+mt-1$ and that for each $i = 1, 2, \ldots, t$, we have $\sum_{j=1}^{m} x_{ij} = 0$. Thus, the t rows of X give a Langford-type m-tuple difference set of order t and defect d when $d \leq \frac{t+3}{2}$. However, for this construction, we must know the Langford sequence to proceed. In the proof below, we provide a direct construction of this result that does not depend on knowing a specific Langford sequence.

We begin by considering the case that d is even and show that a Langford-type m-tuple difference set of order t and defect d exists when $t \equiv 0, 1 \pmod 4$ with $d \le \lfloor \frac{t}{2} \rfloor$. Next, we consider the case that d is odd and show that a Langford-type m-tuple difference set of order t and defect d exists when $t \equiv 0, 3 \pmod 4$ with $d \le \lfloor \frac{t-5}{2} \rfloor$.

Lemma 2.6 *Let m, d and t be positive integers such that $m \equiv 1 \pmod 4$ with $m \ge 5$ and $t \equiv 0, 1 \pmod 4$ with $d \le \lfloor \frac{t}{2} \rfloor$ if d is even, or $t \equiv 0, 3 \pmod 4$ with $d \le \lfloor \frac{t-5}{2} \rfloor$ if d is odd. Then there exists a Langford-type m-tuple difference set of order t and defect d.*

Proof. Let m, d and t be positive integers such that $m \equiv 1 \pmod 4$ with $m \ge 5$, and $t \equiv 0, 1 \pmod 4$ with $d \le \lfloor \frac{t}{2} \rfloor$ if d is even, or $t \equiv 0, 3 \pmod 4$ with $d \le \lfloor \frac{t-5}{2} \rfloor$ if d is odd. The proof splits into two cases depending on the parity of d. For each case, we construct a $t \times m$ matrix $X = [x_{ij}]$ whose entries in absolute value are $d, d+1, \ldots, d+mt-1$ and, for each $i = 1, 2, \ldots, t$, we have $\sum_{j=1}^{m} x_{ij} = 0$. Then, the t rows of X will give a Langford-type m-tuple difference set of order t and defect d.

Case 1: *Suppose d is even.* Suppose first that $t \equiv 0 \pmod 4$ with $d \le \lfloor \frac{t}{2} \rfloor = \frac{t}{2}$. Consider the three intervals $[d+t+\frac{t}{2}, d+2t-1]$, $[d+2t+\frac{t}{2}, 3t-1]$, and $[4t, 4t+d-1]$ which contain $\frac{t}{2}$, $\frac{t}{2}-d$, and d integers respectively. Since $d \le \frac{t}{2}$, it is possible that $\frac{t}{2} - d = 0$ so that the second interval may be empty. Since the number of integers in each interval is even, these t integers can be paired into sets $\{a_i, a_i+1\}$ for each $i = 1, 2, \ldots, \frac{t}{2}$.

Let $X = [x_{ij}]$ be the $t \times m$ matrix

$$\begin{bmatrix}
d & -(d+t) & d+2t & 4t-1 & -(d+5t-1) & \\
d+1 & -(d+t+1) & d+2t+1 & 4t-3 & -(d+5t-2) & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \\
d+\frac{t}{2}-1 & -(d+t+\frac{t}{2}-1) & d+2t+\frac{t}{2}-1 & 3t+1 & -(d+4t+\frac{t}{2}) & \\
d+\frac{t}{2} & -a_1 & a_1+1 & 4t-2 & -(d+4t+\frac{t}{2}-1) & Y \\
d+\frac{t}{2}+1 & -a_2 & a_2+1 & 4t-4 & -(d+4t+\frac{t}{2}-2) & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \\
d+t-1 & -a_{t/2} & a_{t/2}+1 & 3t & -(d+4t) &
\end{bmatrix}$$

where $Y = Y(\frac{m-5}{4}, d+5t-1, t)$ is the $t \times \frac{m-5}{4}$ matrix given by Lemma 2.1. Note that the matrix X contains distinct entries since $d \le \frac{t}{2}$ implies $d + 2t + \frac{t}{2} - 1 \le 3t - 1$.

Now suppose that $t \equiv 1 \pmod 4$ with $d \le \lfloor \frac{t}{2} \rfloor = \frac{t-1}{2}$. Consider the intervals $[d+t+\frac{t-1}{2}, d+2t]$, $[d+2t+\frac{t-1}{2}+1, 3t-1]$, and $[4t, 4t+d-1]$ which contain $\frac{t+1}{2}+1$, $\frac{t+1}{2}-d-1$, and d integers respectively. Again, since $d \le \frac{t-1}{2}$, it is possible that the second interval may be empty. Since the number of integers in each interval is even, these $t+1$ integers can be paired into sets $\{a_i, a_i+1\}$ for $i = 1, 2, \ldots, \frac{t+1}{2}$.

Let $X = [x_{ij}]$ be the $t \times m$ matrix

$$\begin{bmatrix}
d & -(d+t) & d+2t+1 & 4t-2 & -(d+5t-1) & \\
d+1 & -(d+t+1) & d+2t+2 & 4t-4 & -(d+5t-2) & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \\
d+\frac{t-1}{2}-1 & -(d+t+\frac{t-1}{2}-1) & d+2t+\frac{t-1}{2} & 3t+1 & -(d+4t+\frac{t-1}{2}+1) & \\
d+\frac{t-1}{2} & -a_1 & a_1+1 & 4t-1 & -(d+4t+\frac{t-1}{2}) & Y \\
d+\frac{t-1}{2}+1 & -a_2 & a_2+1 & 4t-3 & -(d+4t+\frac{t-1}{2}-1) & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \\
d+t-1 & -a_{(t+1)/2} & a_{(t+1)/2}+1 & 3t & -(d+4t) &
\end{bmatrix}$$

where $Y = Y(\frac{m-5}{4}, d+5t-1, t)$ is the $t \times \frac{m-5}{4}$ matrix given by Lemma 2.1. Again, note that every entry in the matrix X is distinct since $d \le \frac{t-1}{2}$ implies $d + 2t + \frac{t-1}{2} \le 3t - 1$.

CASE 2: *Suppose d is odd.* Suppose first that $t \equiv 0 \pmod 4$ with $d \le \lfloor \frac{t-5}{2} \rfloor = \frac{t}{2} - 3$. Consider the intervals $[d+t+\frac{t}{2}, d+2t+1]$, $[d+2t+\frac{t}{2}+2, 3t-2]$, and $[4t-1, 4t+d-1]$ which contain $\frac{t}{2}+2$, $\frac{t}{2}-d-3$, and $d+1$ integers respectively. Since $d \le \frac{t}{2} - 3$, it is possible that $\frac{t}{2} - d - 3 = 0$ so that the second interval may be empty. Since the number of integers in each interval is even, these t integers can be paired into sets $\{a_i, a_i+1\}$ for $i = 1, 2, \ldots, \frac{t}{2}$.

Let $X = [x_{ij}]$ be the $t \times m$ matrix

$$\begin{bmatrix}
d & -(d+t) & d+2t+2 & 4t-3 & -(d+5t-1) & \\
d+1 & -(d+t+1) & d+2t+3 & 4t-5 & -(d+5t-2) & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \\
d+\frac{t}{2}-1 & -(d+t+\frac{t}{2}-1) & d+2t+\frac{t}{2}+1 & 3t-1 & -(d+4t+\frac{t}{2}) & \\
d+\frac{t}{2} & -a_1 & a_1+1 & 4t-2 & -(d+4t+\frac{t}{2}-1) & Y \\
d+\frac{t}{2}+1 & -a_2 & a_2+1 & 4t-4 & -(d+4t+\frac{t}{2}-2) & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \\
d+t-1 & -a_{t/2} & a_{t/2}+1 & 3t & -(d+4t) &
\end{bmatrix}$$

where $Y = Y(\frac{m-5}{4}, d+5t-1, t)$ is the $t \times \frac{m-5}{4}$ matrix given by Lemma 2.1. Note that the matrix X contains distinct entries since $d \le \frac{t}{2} - 3$ implies $d + 2t + \frac{t}{2} + 1 \le 3t - 2$.

Now suppose that $t \equiv 3 \pmod 4$ with $d \le \lfloor \frac{t-5}{2} \rfloor = \frac{t-5}{2}$. Consider the intervals $[d+t+\frac{t+1}{2}, d+2t]$, $[d+2t+\frac{t+1}{2}+1, 3t-2]$, and $[4t-1, 4t+d-1]$ which contain $\frac{t+1}{2}$, $\frac{t-1}{2} - d - 2$, and $d+1$ integers respectively. Again, since $d \le \frac{t-5}{2}$, it is possible that the second interval may be empty. Since the number of integers in each interval is even, these $t-1$ integers can be paired into sets $\{a_i, a_i+1\}$ for $i = 1, 2, \ldots, \frac{t-1}{2}$.

Let $X = [x_{ij}]$ be the $t \times m$ matrix

$$\left[\begin{array}{ccccc} d & -(d+t) & d+2t+1 & 4t-2 & -(d+5t-1) \\ d+1 & -(d+t+1) & d+2t+2 & 4t-4 & -(d+5t-2) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ d+\frac{t-1}{2} & -(d+t+\frac{t-1}{2}) & d+2t+\frac{t+1}{2} & 3t-1 & -(d+4t+\frac{t-1}{2}) \\ d+\frac{t-1}{2}+1 & -a_1 & a_1+1 & 4t-3 & -(d+4t+\frac{t-1}{2}-1) \\ d+\frac{t-1}{2}+2 & -a_2 & a_2+1 & 4t-5 & -(d+4t+\frac{t-1}{2}-2) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ d+t-1 & -a_{(t-1)/2} & a_{(t-1)/2}+1 & 3t & -(d+4t) \end{array} \quad Y \right]$$

where $Y = Y(\frac{m-5}{4}, d+5t-1, t)$ is the $t \times \frac{m-5}{4}$ matrix given by Lemma 2.1. Again, note that every entry in the matrix X is distinct since $d \leq \frac{t-5}{2}$ implies $d + 2t + \frac{t+1}{2} \leq 3t - 2$. ■

Lemmas 2.2, 2.5, and 2.6 do not provide necessary and sufficient conditions for the existence of Langford-type m-tuples of order t and defect d in the case that $m \equiv 1, 3 \pmod 4$; these lemmas provide Langford-type m-tuple difference sets of order t and defect d whenever d is at most about $\frac{t}{2}$. This condition is not necessary however as the following 3×5 array

$$X = \begin{bmatrix} 5 & -8 & 6 & 9 & -12 \\ 10 & -17 & 11 & 15 & -19 \\ 7 & -16 & 13 & 14 & -18 \end{bmatrix}$$

gives a Langford-type 5-tuple difference set of order 3 and defect 5.

3 Cyclic m-Cycle Systems of Some Specific Circulant Graphs

Let K_n denote the complete graph on n vertices and C_m denote the m-cycle $(v_1, v_2, \ldots, v_m)$. An *m-cycle system* of a graph G is a set C of m-cycles in G whose edges partition the edge set of G. Several obvious necessary conditions for an m-cycle system C of a graph G to exist are immediate: $3 \leq m \leq |V(G)|$, the degrees of the vertices of G must be even, and m must divide the number of edges in G. A survey on cycle systems is given in [7], and necessary and sufficient conditions for the existence of an m-cycle system of G in the cases $G = K_n$ and $G = K_n - I$ (the complete graph of order n with a 1-factor I removed) were given in [1, 17]. Such m-cycle systems exist if and only if $n \geq m$, every vertex of G has even degree, and m divides the number of edges in G.

Let ρ denote the permutation $(0, 1, \ldots, n-1)$, so $\langle\rho\rangle = \mathbb{Z}_n$, the additive group of integers modulo n. An m-cycle system C of a graph G with vertex set $\mathbb{Z}_n$ is *cyclic* if for every m-cycle $C = (v_1, v_2, \ldots, v_m)$ in $\mathcal{C}$, the m-cycle $\rho(C) = (\rho(v_1), \rho(v_2), \ldots, \rho(v_m))$ is also in $\mathcal{C}$. Finding necessary and sufficient conditions for cyclic m-cycle systems of a given graph G is an interesting problem and has attracted much attention (see, for example, [2, 3, 5, 6, 9, 11, 12, 15, 16]). The obvious necessary conditions for a cyclic m-cycle system of a graph G are the same as for an m-cycle system of G; that is, $3 \le m \le |V(G)|$, the degree of the vertices of G must be even, and m must divide the number of edges in G. However, these conditions are not sufficient. For example, it is not difficult to see that there is no cyclic 15-cycle system of K_{15}. Also, if p is an odd prime and $\alpha \ge 2$, then there is no cyclic p^α-cycle system of K_{p^α} [6].

The existence question for cyclic m-cycle systems of K_n has been completely settled in a few small cases, namely $m = 3$ [14], 5 and 7 [16]. For even m and $n \equiv 1 \pmod{2m}$, cyclic m-cycle systems of K_n are constructed for $m \equiv 0 \pmod 4$ in [12] and for $m \equiv 2 \pmod 4$ in [15]. Both of these cases are handled simultaneously in [9] as a consequence of a more general result. For odd m and $n \equiv 1 \pmod{2m}$, cyclic m-cycle systems of K_n are found using different methods in [2, 5, 11]. In [3], as a consequence of a more general result, cyclic m-cycle systems of K_n for all positive integers m and $n \equiv 1 \pmod{2m}$ with $n \ge m \ge 3$ are given. In [6], it is shown that a cyclic hamiltonian cycle system of K_n exists if and only if $n \ne 15$ and $n \notin \{p^\alpha \mid p \text{ is an odd prime and } \alpha \ge 2\}$. Thus, as a consequence of a result in [5], cyclic m-cycle systems of K_{2mk+m} exist for all $m \ne 15$ and $m \notin \{p^\alpha \mid p \text{ is an odd prime and } \alpha \ge 2\}$. In [20], the last remaining cases for cyclic m-cycle systems of K_{2mk+m} are settled, i.e., it is shown that, for $k \ge 1$, cyclic m-cycle systems of K_{2km+m} exist if $m = 15$ or $m \in \{p^\alpha \mid p \text{ is an odd prime and } \alpha \ge 2\}$. In [21], necessary and sufficient conditions for the existence of cyclic $2q$-cycle and m-cycle systems of the complete graph are given when q is an odd prime power and $3 \le m \le 32$. In [4], cycle systems with a sharply vertex-transitive automorphism group that is not necessarily cyclic are investigated. As a result, it is shown in [4] that no cyclic k-cycle system of K_n exist if $k < n < 2k$ with n odd and $\gcd(k, n)$ a prime power.

For $x \not\equiv 0 \pmod n$, the *modulo n length* of an integer x, denoted $|x|_n$, is defined to be the smallest positive integer y such that $x \equiv y \pmod n$ or $x \equiv -y \pmod n$. Note that for any integer $x \not\equiv 0 \pmod n$, it follows that $|x|_n \in [1, \lfloor \frac{n}{2} \rfloor]$. If L is a set of modulo n lengths, we define the *circulant graph* $\langle L \rangle_n$ to be the graph with vertex set $\mathbb{Z}_n$ and edge set $\{\{i,j\} \mid |i-j|_n \in L\}$. Observe that $K_n \cong \langle [1, \lfloor \frac{n}{2} \rfloor] \rangle_n$. An edge $\{i, j\}$ in a graph with vertex set $\mathbb{Z}_n$ is called an *edge of length* $|i-j|_n$. Notice that in order for a graph G to admit a cyclic m-cycle system, G must be a circulant graph; thus circulant graphs provide a natural setting in which to construct cyclic m-

cycle systems.

Let $n > 0$ be an integer and suppose there exists an ordered m-tuple $(d_1, d_2, \ldots, d_m)$ satisfying each of the following:

(i) d_i is an integer for $i = 1, 2, \ldots, m$;

(ii) $|d_i|_n \neq |d_j|_n$ for $1 \leq i < j \leq m$;

(iii) $d_1 + d_2 + \ldots + d_m \equiv 0 \pmod{n}$; and

(iv) $d_1 + d_2 + \ldots + d_r \not\equiv d_1 + d_2 + \ldots + d_s \pmod{n}$ for $1 \leq r < s \leq m$.

Since the ordering of the integers in the m-tuple is so important, we make the following definitions.

Definition 3.1 An m-tuple satisfying (i)-(iv) is called a *modulo n difference m-tuple,* it *corresponds* to the m-cycle $(0, d_1, d_1 + d_2, \ldots, d_1 + d_2 + \ldots + d_{m-1})$, and it *uses* edges of lengths $|d_1|_n, |d_2|_n, \ldots, |d_m|_n$. The m-cycle $C = (0, d_1, d_1 + d_2, \ldots, d_1 + d_2 + \ldots + d_{m-1})$ *generates* a cyclic m-cycle system of $\langle\{|d_1|, |d_2|, \ldots, |d_m|\}\rangle_n$ since $\{\rho^\alpha(C) \mid \alpha = 0, 1, \ldots, n-1\}$ is a cyclic m-cycle system of $\langle\{|d_1|, |d_2|, \ldots, |d_m|\}\rangle_n$.

Definition 3.2 A *modulo n m-cycle difference set of order t,* or an *m-cycle difference set of order t* when the value of n is understood, is a set consisting of t modulo n difference m-tuples that use edges of distinct lengths $\ell_1, \ell_2, \ldots, \ell_{tm}$. Since the m-cycles corresponding to the difference m-tuples generate a cyclic m-cycle system C of $\langle\{\ell_1, \ell_2, \ldots, \ell_{tm}\}\rangle_n$, we say the modulo n m-cycle difference set of order t *generates* C. If $\ell_1, \ell_2, \ldots, \ell_{tm}$ are consecutive integers starting with $\ell_1 = d$ for some positive integer d, then we have an *m-cycle difference set of order t and defect d.*

For $t \equiv 0, 1 \pmod{4}$, a Skolem sequence $S = (s_1, s_2, \ldots, s_{2t})$ of order t can be used to construct the 3-cycle difference set

$$\{(k, t + i, -(t + j)) \mid k = 1, 2, \ldots, t,\ s_i = s_j = k, i < j\}$$

which generates a cyclic 3-cycle system of K_{6t+1}. Notice that if $(d_1, d_2, \ldots, d_m)$ is a modulo n difference m-tuple with $d_1 + d_2 + \ldots + d_m = 0$, not just $d_1 + d_2 + \ldots + d_m \equiv 0 \pmod{n}$, then $(d_1, d_2, \ldots, d_m)$ is a modulo w difference m-tuple for all $w \geq \frac{M}{2} + 1$ where $M = |d_1| + |d_2| + \cdots + |d_m|$. All the difference triples obtained from Skolem sequences are of the form (d_1, d_2, d_3) with $d_1 + d_2 + d_3 = 0$. Thus Skolem sequences can be used to construct cyclic 3-cycle systems of $\langle[1, 3t]\rangle_n$ for all $n \geq 6t + 1$ and $t \equiv 0, 1 \pmod{4}$.

In a similar manner to which 3-cycle difference sets are constructed from Skolem sequences, a Langford sequence of order t and defect d can be used to construct a cyclic 3-cycle difference set that uses edges of lengths $d, d+1, d+2, \ldots, d+3t-1$ thereby giving a cyclic 3-cycle system of $\langle [d, d+3t-1] \rangle_n$ for each $n \geq 2d+6t+1$ when d and t satisfy the conditions of Theorem 1.1.

As a consequence of Theorem 1.3, the following result was given in [3] regarding cyclic m-cycle systems of specific circulant graphs.

Theorem 3.3 (Bryant, Gavlas, Ling [3]) *Let $t \geq 1$ and $m \geq 3$. Then for $mt \equiv 0, 3 \pmod 4$ and all $n \geq 2mt+1$, there exists a cyclic m-cycle system of $\langle [1, mt] \rangle_n$.*

Observe that the Langford-type m-tuples of order t and defect d constructed in the proofs of Lemmas 2.3, 2.4, 2.5, and 2.6 are not difference m-tuples since it is the case that $d_1+d_2+\ldots+d_r \equiv d_1+d_2+\ldots+d_s \pmod n$ for given values of n, r, and s in every one of the Langford-type m-tuples $(d_1, d_2, \ldots, d_m)$ constructed in these proofs. In fact, it is often the case that $d_1+d_2+\ldots+d_r = d_1+d_2+\ldots+d_s = 0$ for several values of r and s in a given m-tuple $(d_1, d_2, \ldots, d_m)$. For example, if $m \equiv 0 \pmod 4$, we have $d_1+d_2+\ldots+d_{4k} = d_1+d_2+\ldots+d_{4k+4} = 0$ for each positive integer k with $k < \frac{m}{4}$. However, as a consequence of these lemmas and after rearranging the entries in each Langford-type m-tuple, we have the following result regarding cyclic m-cycle systems of the circulant graph $\langle [d, d+mt-1] \rangle_n$ for all $n \geq 2(d+mt)-1$ with d and t satisfying the conditions below.

Theorem 3.4 *Let m, d and t be positive integers with $m \geq 3$ such that*

(1) $t \equiv 0, 2 \pmod 4$ when $m \equiv 2 \pmod 4$;

(2) $2d-1 \leq t$ and $t \equiv 0, 1 \pmod 4$ if d is odd, or $t \equiv 0, 3 \pmod 4$ if d is even when $m \equiv 3 \pmod 4$; and

(3) $t \equiv 0, 1 \pmod 4$ and $d \leq \lfloor \frac{t}{2} \rfloor$ if d is even, or $t \equiv 0, 3 \pmod 4$ and $d \leq \lfloor \frac{t-5}{2} \rfloor$ if d is odd when $m \equiv 1 \pmod 4$.

Then there exists a cyclic m-cycle system of the circulant graph $\langle [d, d+mt-1] \rangle_n$ for all $n \geq 2(d+mt)-1$.

Proof. Let m, d and t be positive integers with $m \geq 3$ such that

(1) $t \equiv 0, 2 \pmod 4$ when $m \equiv 2 \pmod 4$;

(2) $2d-1 \le t$ and $t \equiv 0,1 \pmod 4$ if d is odd, or $t \equiv 0,3 \pmod 4$ if d is even when $m \equiv 3 \pmod 4$; and

(3) $t \equiv 0,1 \pmod 4$ and $d \le \lfloor \frac{t}{2} \rfloor$ if d is even, or $t \equiv 0,3 \pmod 4$ and $d \le \lfloor \frac{t-5}{2} \rfloor$ if d is odd when $m \equiv 1 \pmod 4$.

By Lemmas 2.3, 2.4, 2.5, and 2.6, there exists a Langford-type m-tuple difference set of order t and defect d in each of these cases. Let $\{(x_{i,1}, x_{i,2}, x_{i,3}, x_{i,4}, \ldots, x_{i,m}) \mid i = 1, 2, \cdots, t\}$ denote the set of t Langford-type m-tuples given by the rows of the $t \times m$ matrix X in the proofs of Lemmas 2.3, 2.4, 2.5, and 2.6 respectively. Let n be a positive integer such that $n \ge 2(d+mt)-1$. Observe that each m-tuple is not a modulo n difference m-tuple since $x_{i,1} + x_{i,2} + \ldots + x_{i,r} \equiv x_{i,1} + x_{i,2} + \ldots + x_{i,s} \pmod n$ for (possibly) several values of r and s. Thus, we must rearrange the entries in each Langford-type m-tuple to obtain a modulo n difference m-tuple.

Suppose first $m \equiv 0 \pmod 4$. For $i = 1, 2, \ldots, t$, we have $|x_{i,1}| < |x_{i,2}| < \cdots < |x_{i,m}|$ and $x_{i,j} < 0$ only when $j \equiv 2,3 \pmod 4$. Hence the required set of t difference m-tuples is given by $\{(x_{i,1}, x_{i,3}, x_{i,5}, x_{i,7}, \ldots, x_{i,m-3}, x_{i,m-1}, x_{i,m-2}, x_{i,m-4}, x_{i,m-6}, \ldots, x_{i,6}, x_{i,4}, x_{i,2}, x_{i,m}) \mid i = 1, 2, \ldots, t\}$, giving an m-cycle difference set of order t and defect d in the case that $m \equiv 0 \pmod 4$.

Next suppose $m \equiv 2 \pmod 4$. For $i = 1, 2, \ldots, t$, we have $|x_{i,1}| < |x_{i,2}| < \cdots < |x_{i,m}|$, and $x_{i,j} < 0$ only when $j = 2$ and when $j \equiv 0,1 \pmod 4$ with $j \ge 4$. Hence, the required set of t difference m-tuples is given by $\{(x_{i,1}, x_{i,2}, x_{i,3}, x_{i,5}, x_{i,7} \ldots, x_{i,m-3}, x_{i,m-1}, x_{i,m-2}, x_{i,m-4}, x_{i,m-6}, \ldots, x_{i,6}, x_{i,4}, x_{i,m}) \mid i = 1, 2, \ldots, t\}$, giving an m-cycle difference set of order t and defect d in the case that $m \equiv 2 \pmod 4$ and $t \equiv 0,2 \pmod 4$.

Now suppose $m \equiv 3 \pmod 4$. For $i = 1, 2, \ldots, t$, we have $|x_{i,1}| < |x_{i,2}| < |x_{i,4}| < |x_{i,5}| < |x_{i,6}| < \cdots < |x_{i,m}|$, $|x_{i,3}| < |x_{i,5}|$, and $x_{i,j} < 0$ when $j \ge 2$ and $j \equiv 1,2 \pmod 4$. Hence, the required set of t difference m-tuples is given by $\{(x_{i,1}, x_{i,2}, x_{i,4}, x_{i,6}, x_{i,8}, \ldots, x_{i,m-3}, x_{i,m-1}, x_{i,m-2}, x_{i,m-4}, x_{i,m-6}, \ldots, x_{i,5}, x_{i,3}, x_{i,m}) \mid i = 1, 2, \ldots, t\}$, giving an m-cycle difference set of order t and defect d in the case that $m \equiv 3 \pmod 4$, $2d-1 \le t$ and $t \equiv 0,1 \pmod 4$ if d is odd, or $t \equiv 0,3 \pmod 4$ if d is even.

Finally, suppose $m \equiv 1 \pmod 4$. For $i = 1, 2, \ldots, t$, we have $|x_{i,1}| < |x_{i,2}| < |x_{i,3}| < |x_{i,5}| < |x_{i,6}| < \cdots < |x_{i,m}|$, $|x_{i,4}| < |x_{i,5}|$, and $x_{i,j} < 0$ when $j = 2$, $j = 5$ and when $j \equiv 0,3 \pmod 4$ with $j > 5$. Hence, the required set of t difference m-tuples is given by $\{(x_{i,1}, x_{i,2}, x_{i,3}, x_{i,6}, x_{i,8}, \ldots, x_{i,m-3}, x_{i,m-1}, x_{i,m-2}, x_{i,m-4}, x_{i,m-6}, \ldots, x_{i,5}, x_{i,4}, x_{i,m}) \mid i = 1, 2, \ldots, t\}$, giving an m-cycle difference set of order t and defect d in the case that $m \equiv 1 \pmod 4$, $t \equiv 0,1 \pmod 4$ and $d \le \lfloor \frac{t}{2} \rfloor$ if d is even, or $t \equiv 0,3 \pmod 4$ and $d \le \lfloor \frac{t-5}{2} \rfloor$ if d is odd. ■

In [8], a similar result is given regarding cyclic m-cycle systems of complete graphs where it is shown that if $m \geq 3$ is odd and $d \in [1, m]$ with $(m, d) \notin \{(3,3),(5,3)\}$, then there exists a cyclic m-cycle system of $\langle [d, d + mx - 1] \rangle_{2(d+mt)-1}$ for every positive integer t.

References

[1] B. Alspach and H. Gavlas, Cycle decompositions of K_n and $K_n - I$, *J. Combin. Theory Ser. B* **81** (2001), 77–99.

[2] A. Blinco, S. El Zanati and C. Vanden Eynden, On the cyclic decomposition of complete graphs into almost complete graphs, *Discrete Math.* **284** (2004), 71–81.

[3] D. Bryant, H. Gavlas, and A. C. H. Ling, Difference Sets for Cycle Systems, *Electron. J. Combin.* **10** (2003), #R38.

[4] M. Buratti, Cycle decompositions with a sharply vertex transitive automorphism group, *Le Matematiche* **59** (2004), 91–105.

[5] M. Buratti and A. Del Fra, Existence of cyclic k-cycle systems of the complete graph, *Discrete Math.* **261** (2003), 113–125.

[6] M. Buratti and A. Del Fra, Cyclic Hamiltonian cycle systems of the complete graph, *Discrete Math.* **279** (2004), 107–119.

[7] D. Bryant and C. A. Rodger, Cycle Decompositions, in the *CRC Handbook of Combinatorial Designs 2nd edition*, (eds. C.J. Colbourn and J.H. Dinitz), CRC Press, Boca Raton FL (2007), 373–382.

[8] S. El-Zanati, K. King, J. Mudrock, On the cyclic decomposition of circulant graphs into almost-bipartite graphs. *Australas. J. Combin.* **49** (2011), 61-76.

[9] S.I. El-Zanati, N. Punnim, C. Vanden Eynden, On the cyclic decomposition of complete graphs into bipartite graphs, *Austral. J. Combin.* **24** (2001), 209–219.

[10] N. Francetic, E. Mendelsohn, A Survey of Skolem-type sequences and Rosa's use of them, *Mathematica Slovaca.* **59** (2009), 39–76.

[11] H. Fu and S. Wu, Cyclically decomposing complete graphs into cycles, *Discrete Math.* **282** (2004), 267–273.

[12] A. Kotzig, On decompositions of the complete graph into $4k$-gons, *Mat.-Fyz. Cas.* **15** (1965), 227–233.

[13] E.S. O'Keefe, Verification of a conjecture of Th Skolem, *Math. Scand* **9** (1961), 80–82.

[14] R. Peltesohn, Eine Losung der beiden Heffterschen Differenzenprobleme, *Compos. Math.* **6** (1938), 251-257.

[15] A. Rosa, On cyclic decompositions of the complete graph into $(4m+2)$-gons, *Mat.-Fyz. Cas.* **16** (1966), 349–352.

[16] A. Rosa, On the cyclic decompositions of the complete graph into polygons with an odd number of edges, *Časopis Pěst. Math.* **91** (1966), 53–63.

[17] M. Šajna, Cycle Decompositions III: Complete graphs and fixed length cycles, *J. Combin. Des.* **10** (2002), 27–78.

[18] J.E. Simpson, Langford sequences: perfect and hooked, *Discrete Math.* **44** (1983), 97–104.

[19] Th. Skolem, On certain distributions of integers in pairs with given difference, *Math. Scand.* **5** (1957), 57–68.

[20] A. Vietri, Cyclic k-Cycle Systems of order $2kn + k$; a solution of the last open cases, *J. Combin. Des* **12** (2004), 299–310.

[21] S.-L. Wu and H.-L. Fu, Cyclic m-cycle systems with $m \leq 32$ or $m = 2q$ with q a prime power, *J. Combin. Des.* **14** (2006), 66–81.