Mutation of Vertex-Magic Regular Graphs

J. S. Kimberley and J. A. MacDougall
School of Mathematical and Physical Sciences
The University of Newcastle, NSW 2308 Australia
{jason.kimberley, jim.macdougall}@newcastle.edu.au

Abstract

A mutation of a vertex-magic total labeling of a graph G is a swap of some set of edges incident on one vertex of G with some a set of edges incident with another vertex where the labels on the two sets have the same sum. Mutation has previously been seen to be a useful method for producing new labelings from old. In this paper we study mutations which mutate labelings of regular graphs into labelings of other regular graphs. We present results of extensive computations which confirm how prolific this procedure is. These computations add weight to MacDougall's conjecture that all non-trivial regular graphs are vertex-magic.

In memory of Jean Kimberley, 27th May 1921 - 20th March 2011.

1 Introduction

A *vertex-magic total labeling* (VMTL) on a graph with v vertices and e edges is a one-to-one mapping from the vertices and edges onto the integers $1, 2, \cdots, v + e$ so that the sum of the label on a vertex and the labels of its incident edges is constant, independent of the choice of vertex. The 2nd author has conjectured [5] that, with the exception of K_2 and $2K_3$, all regular graphs have at least one VMTL.

In [2] Gray and MacDougall developed the method of *mutation* for converting a VMTL of one graph into a different VMTL for the same graph, or a VMTL for a different graph of the same order and size. This process

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exchanges a set of s edges incident with one vertex for a set of t edges incident with another vertex and having the same sum. The vertex labels are unchanged and the new labeling has the same magic constant. If we begin with a VMTL of some *regular* graph and exchange equal numbers of edges ($s = t$), then our mutation produces a VMTL for the same graph or some other *regular* graph. Readers are referred to [2] for further definitions and details. In that paper we showed that mutation is an fruitful method for generating VMTLs for regular graphs, for example, generating VMTLs for all order 8 cubic graphs by mutation from a single starter VMTL. In this paper we present the results of some more extensive computational work which illustrates dramatically just how powerful a method mutation is.

After mutation, of course, the graph is still regular, and the labeling is subject to being mutated into yet another labeling for the same graph or a different graph. The process can be continued as long as one wants. The holy grail of mutating labelings would be to find a positive answer to the following question:

Problem. *For the set of regular graphs of given order v, size e and each feasible magic constant k, does there exist a VMTL of one of the graphs which after repeated mutation yields VMTLs for all of the others?*

If so, the proof of the conjecture would reduce to finding an appropriate labeling for just one graph for each constant. We will see very quickly that the answer is negative even for $v = 6$, the simplest non-trivial case. However, we have found the answer to be "yes" for many other cases. We point out that graphs with different sets of vertex labels may have the same magic constant but, of course, cannot mutate into each other.

VMTLs were introduced in [6], where it is shown that the range of feasible values for the magic constant k for any labeling λ of an r-regular graph $G(V, E)$ is determined by

$$vr^2 + 2(v+1)(r+1) \leq 4k \leq vr^2 + 2(v+1)(r+1) + 2vr$$

and the labeling is *strong* when the v largest labels are on the vertices, in which case $k = \frac{1}{4}(vr^2 + 2(v+1)(r+1))$. Strong labelings are of special importance because of Gray's constructions: in [3] and [4], it is shown how to begin with a strong VMTL on a graph and adjoin an arbitrary 2-factor to produce another strong VMTL for a graph of larger size. This is a powerful method of creating a myriad of VMTLs for regular graphs. Readers are referred to those papers for details. Another reason that strong labelings are of interest is that for this magic constant, they all share the same set of vertex labels. Thus it is conceivable they might *all* mutate into each other. We note that strong labelings can not exist for odd-regular graphs of order congruent to 2 (mod 4).

It was shown in [6] that VMTLs for regular graphs come in dual pairs: if λ is a VMTL for G, then $\lambda' = e + v + 1 - \lambda$ is also a VMTL for G. It is easy to check that mutation respects this duality in the following sense:

Lemma 1. *Any (n, n)-mutation of a labeling of a regular graph corresponds to a (n, n)-mutation of the dual labeling.*

Suppose G is a 2-regular graph. There are two edges meeting each vertex, and since the vertex labels are all distinct, the sum of the labels on the pair of edges meeting one vertex will be different from the sum of the labels meeting any other vertex. Thus no $(2, 2)$-mutation will ever be possible. So the first meaningful graphs to examine are the 3-regular graphs. Similar reasoning to the above shows that for cubic graphs we can not have $(3, 3)$-mutations, only potentially $(2, 2)$-mutations. More generally, for r-regular graphs (r, r)-mutations cannot exist.

2 Mutation Classes and the Metagraph

As mentioned above, a VMTL λ of some graph of order v can only mutate into labelings for graphs having the same vertex label set. If we then begin with a particular labeling, the set of all labelings obtainable from it by any sequence of (n, n)-mutations (for some fixed n) will be called the *(n, n)-mutation class* of the labeling. For a different value of n, this would be a different set of labelings. The set of all labelings reachable from λ by any sequence of mutations as n varies will be called the *mutation class* of λ. For some quartic graphs, for example it may be that a labeling λ_1 mutates into a labeling λ_2 by a sequence of both $(2, 2)$- and $(3, 3)$-mutations but not just by $(2, 2)$-mutations alone or $(3, 3)$-mutations alone.

For a each vertex set V (of r-regular graphs with order v) it is helpful to visualise a *metagraph* whose vertices are all the labelings (of all the graphs) for that vertex set, where two labelings λ_1 and λ_2 are adjacent if and only if there is an (n, n)-mutation between λ_1 and λ_2 for some n. Thus each mutation class of labelings is a connected component of the metagraph of V. Typically we find many mutation classes for any given V, so the metagraph has many components.

In view of the above we can rephrase the *Problem* stated in the Introduction as follows: "For each feasible magic constant, is there a corresponding vertex set whose metagraph contains at least one connected component containing a labeling for each regular graph of that order?"

3 Cubic Graphs of Orders 4 and 6

As mentioned above, we are considering only (2, 2)-mutations. The cubic graph of smallest order is K_4 and it is easy to find all possible VMTLs. These are counted in Table 1, where $\#\mathcal{V}$ is the number of different vertex sets corresponding to the given magic constant, and $\#\mathcal{L}$ is the number of labelings. Many of these vertex sets admit no VMTLs and more interestingly, three of the feasible values of the magic constant are not achieved by any VMTL. This is one of the few known graphs where VMTLs do not exist for some feasible magic constant. The duality mentioned above explains the symmetry of the table. Pictured in Table 2 are the 7 VMTLs for $k = 20$ and 21; the other 7 are the duals of these.

Table 1: The number of VMTLs of K_4

k	$\#\mathcal{V}$	$\#\mathcal{L}$
19	1	0
20	5	2
21	13	5
22	18	0
23	13	5
24	5	2
25	1	0
Σ	56	14

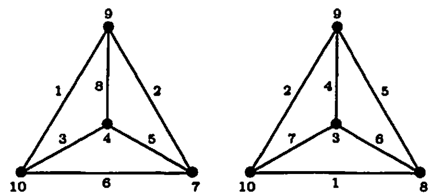
Our first observation is a negative one. An examination of each of these labelings finds all the vertex sets distinct. Thus there are no candidates for mutation among the labelings of K_4 (and so each metagraph is a single vertex).

The 2 cubic graphs of order 6 are the prism and the complete bipartite graph, $K_{3,3}$. These are designated respectively as C2 and C3 in Read & Wilson's *An Atlas of Graphs* [7], and we will refer to them and to other graphs by their numbers in the *Atlas*. We have enumerated the vertex sets and labelings for each magic constant for each graph and the counts are shown in Table 3.

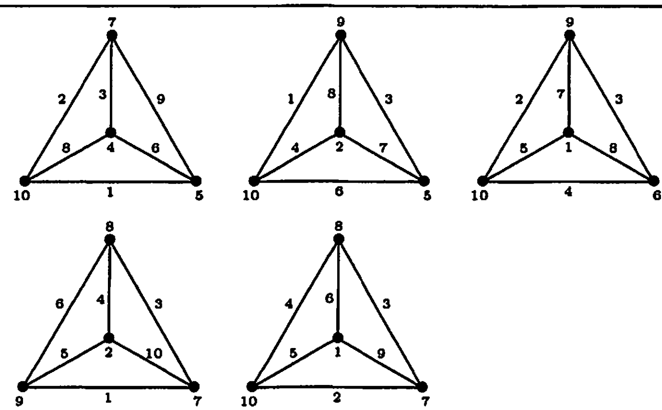
What about mutations of C2 and C3? Again we have a somewhat negative result; namely, that any (2, 2)-mutation of a labeling on $K_{3,3}$ can yield only another labeling of $K_{3,3}$. This is a special case of a more general result that follows. Readers are encouraged to look at the definition of *mutation* in [2].

Table 2: The VMTLs of K_4 with $k = 20$ and 21

$k = 20$



$k = 21$



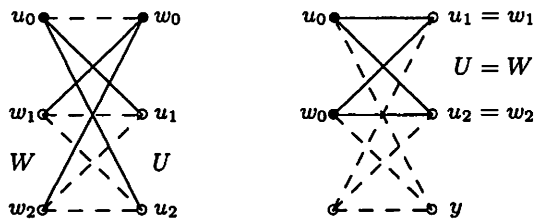
Theorem 2. *Every $(r-1, r-1)$-mutation of a VMTL of $K_{r,r}$ is a VMTL of $K_{r,r}$.*

Proof. We let U be the set of neighbours of u_0 and W the set of neighbours of w_0. Now the pair of mutation vertices u_0, w_0 are either adjacent or not.

Firstly, suppose they are adjacent (left diagram below). Now, since $w_0 \notin U$ and $u_0 \notin W$, the edge $\{u_0, w_0\}$ is not a mutation edge. Hence the mutation just swaps the labels of u_0 and w_0.

Secondly, suppose u_0, w_0 are not adjacent (right diagram below). From the definition of mutation, the only vertex adjacent to u_0 that is not in U is also not in W. So $U = W$ and hence the mutation swaps the labels of

u_0 and w_0 and swaps the labels of edges $\{u_0, y\}$ and $\{w_0, y\}$.



□

Corollary 3. *There are no mutations between* C2 *and* C3.

Even though C2 and C3 only mutate into themselves, we would like to know the number and sizes of the mutation classes for each graph. Table 3 shows the number of mutation classes for each magic constant. As noted above, strong labelings will not exist. Here $\#\mathcal{M}$ is the number of mutation classes and Σ is the total.

Table 3: Counts of VMTLs of the order 6 cubic graphs.

		$\#\mathcal{L}$			$\#\mathcal{M}$		
k	$\#\mathcal{V}$	C2	C3	Σ	C2	C3	Σ
28	3	99	35	134	55	10	65
29	26	242	70	312	183	43	226
30	91	597	477	1074	391	115	506
31	182	851	250	1101	624	143	767
32	227	999	882	1881	669	198	867
33	182	851	250	1101	624	143	767
34	91	597	477	1074	391	115	506
35	26	242	70	312	183	43	226
36	3	99	35	134	55	10	65
Σ	831	4577	2546	7123	3175	820	3995

For $k = 28$ we find there are 3 possible vertex sets, each having its own collection of mutation classes. Surprisingly almost all these classes are small. For C2, the 12 singleton classes for the vertex set $\{7, 11, 12, 13, 14, 15\}$ are shown in Table 16 in the Appendix - these labelings cannot be mutated at all. The remaining 34 VMTLs of C2 with that vertex set fall into 8 mutation classes.

4 Cubic Graphs of Order 8

In the previous section, we appeared to be off to a disappointing start in our quest. However, these negative results seem to be just a case of the "law of small numbers" and things get much better as the order increases. There are six order 8 cubic graphs, five of them connected. Each one has large numbers of VMTLs for every feasible magic constant, and the counts are shown in Table 4.

Table 4: Counts of VMTLs of order 8 cubic graphs

		$\#\mathcal{L}$						
k	$\#\mathcal{V}$	2C1	C4	C5	C6	C7	C8	Σ
36	1	34	23364	8108	5399	6524	3048	46477
37	22	203	70041	24606	16249	19795	6997	137891
38	179	904	247360	77241	64232	56159	30788	476684
39	738	1907	511157	175042	127277	133496	50765	999644
40	1870	3394	1046242	353620	256993	260525	117101	2037875
41	3184	4305	1326810	456243	320644	343855	132358	2584215
42	3788	7678	1717020	511830	488847	350886	258575	3334836
43	3184	4305	1326810	456243	320644	343855	132358	2584215
44	1870	3394	1046242	353620	256993	260525	117101	2037875
45	738	1907	511157	175042	127277	133496	50765	999644
46	179	904	247360	77241	64232	56159	30788	476684
47	22	203	70041	24606	16249	19795	6997	137891
48	1	34	23364	8108	5399	6524	3048	46477
Σ	15776	29172	8166968	2701550	2070435	1991594	940689	15900408

In [2], it was shown that one *strong* labeling (i.e. $k = 36$), which was called the *seed*, could be repeatedly mutated to yield at least one strong labeling for each of the other graphs. We wondered whether this seed would mutate into *all* the strong labelings for all the graphs. The answer is almost 'yes'. Remarkably, of the 46477 strong labelings, all but 33 are in the same mutation class. Of the remaining 33 labelings, 31 are immutable singletons, and there is one pair. The classes are tabulated in Table 5, where for example the 4th row indicates there are 14 singleton mutation classes which all occur in the graph C5. Some of these are illustrated in Table 6 and Table 7.

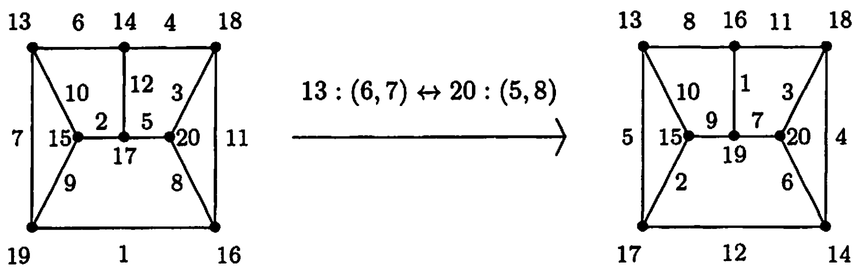
We were interested to discover how many generations of mutations were necessary for that particular seed to generate all 46444 mutations in its class. Table 8 shows the number of new labelings produced at each generation.

What if we ask the same question for any one of the other magic constants - can we find a seed labeling that mutates into labelings for all the other graphs? Here we have success - an example for one vertex set from

Table 5: Isomorphism types in mutation classes of strong VMTLs of order 8 cubic graphs.

2C1	C4	C5	C6	C7	C8	Σ	$\#\mathcal{M}$
34	23360	8094	5399	6514	3043	46444	1
	1					1	2
	2					2	1
		1				1	14
				1		1	10
					1	1	5
34	23364	8108	5399	6524	3048	46477	33

Table 6: The mutation class with two strong VMTLs of C4



each magic constant is given in Table 17 in the Appendix, where the size of the mutation class is also given. This provides us with the following theorem:

Theorem 4. *For each feasible magic constant, there is a mutation class containing labelings for all order 8 cubic graphs.*

It doesn't seem to be an accident that almost all strong labelings are in the same mutation class. For the constant $k = 37$ we chose one of the label sets, namely $\{5, 14, 15, 16, 17, 18, 19, 20\}$ and discovered a similar result. Table 9 shows one very large set and a number of singletons and other small sets.

Table 7: The pair of immutable strong VMTLs of C4.

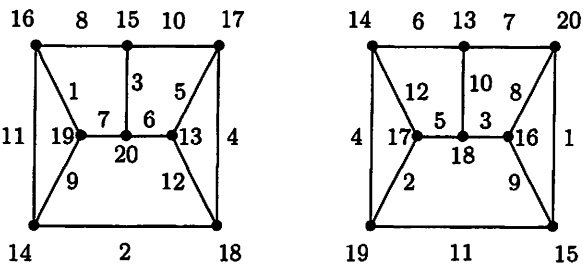


Table 8: Mutation generations from Gray's strong C8 seed.

Gen	2C1	C4	C5	C6	C7	C8	Σ
0	0	0	0	0	0	1	1
1	0	2	0	0	0	8	10
2	0	11	0	5	2	18	36
3	2	42	9	13	0	40	106
4	0	153	36	55	13	70	327
5	4	471	106	121	71	129	902
6	1	1262	352	348	229	191	2383
7	7	3113	860	803	646	387	5816
8	8	5973	1929	1525	1472	694	11601
9	6	7510	2690	1704	2175	877	14962
10	6	4198	1777	741	1493	531	8746
11	0	598	310	83	381	93	1465
12	0	24	25	1	30	4	84
13	0	3	0	0	2	0	5
14	0	0	0	0	0	0	0
Σ	34	23360	8094	5399	6514	3042	46444

5 Cubic Graphs of Larger Order

There are 21 cubic graphs of order 10, two of which are disconnected. As expected, they possess vast numbers of labelings. The range of feasible magic constants is $k \in \{45 \dots 59\}$ and since $r \equiv 2 \pmod 4$, no strong labelings can exist. For order 10 even the number of label sets for each k is large, as shown in Table 10.

Table 9: Mutation classes of VMTLs of order 8 cubic graphs with vertex labels $\{5, 14, 15, 16, 17, 18, 19, 20\}$.

2C1	C4	C5	C6	C7	C8	Σ	#$\mathcal{M}$
15	5738	2050	1504	1396	303	11006	1
	1					1	8
	2					2	1
	1	1				2	1
		1				1	3
		2				2	1
		1		1		2	3
		1		2		3	1
				1		1	17
				2		2	2
				3		3	1
				4		4	1
15	5749	2060	1504	1429	303	11060	40

Can we find appropriate seeds that mutate into labelings for every graph? Our computer searches again produced success, and we have the following positive answer to our *Problem*.

Theorem 5. *For each feasible magic constant, there is a* $(2,2)$*-mutation class containing labelings for all 21 order 10 cubic graphs.*

There are 94 cubic graphs of order 12, of which 9 are disconnected. The number of labelings involved becomes inconveniently large for exhaustive enumeration. However, because the number is so large, randomised searching is fairly efficient at finding good sequences of mutations. The range of feasible magic constants is $k \in \{53 \dots 71\}$. A result similar to the previous theorem holds for order $v = 12$:

Theorem 6. *For each feasible magic constant, there is a* $(2,2)$*-mutation class containing labelings for all 94 order 12 cubic graphs.*

Curiously, we noticed that none of the labelings of any of the graphs that we considered so far (up to $v = 12$) occurred when the vertex set consisted of all odd numbers (or dually, all even numbers). We are so far unable to prove this is true for larger v, but present it as a conjecture.

Conjecture 7. *No cubic graphs have a VMTL with every vertex having an odd label, nor with every vertex having an even label.*

Table 10: Number of vertex sets for order 10 cubic graphs

k	$\#\mathcal{V}$
45	7
46	164
47	1 358
48	6 272
49	18 854
50	39 803
51	61 481
52	70 922
53	61 481
54	39 803
55	18 854
56	6 272
57	1 358
58	164
59	7
Σ	326 800

We have not explored cubic graphs with more than 12 vertices in enough detail to draw any conclusions.

6 Quartic Graphs

As noted above, $(4,4)$-mutations are not possible for 4-regular graphs, but we will expect to find both $(2,2)$- and $(3,3)$-mutations. The unique quartic graph of smallest order is Q1= K_5. Table 11 shows the number of VMTLs for each magic constant. This case is already interesting. Considering the strong labelings, there are four $(2,2)$-mutation classes: 2 singleton classes, and 2 large classes of equal size 91. So the metagraph has 4 components. The singletons are shown in Table 12; however, notice that there is a relation between them. Swapping the labels of parallel edges in the left VMTL produces the right VMTL; the internal edges have their labels decreased by 5, and the external edges have their labels increased by 5; so the edge sums at each vertex remain constant. This is virtually a swap of the inner 5-cycle and the outer 5-cycle. Examining the 2 large components reveals the other extreme: each contains one labeling of meta-degree 8. These are shown in Table 13; each yields 8 distinct labelings via single mutations. Further,

Table 11: The number of VMTLs of Q1= K_5 for each magic constant.

k	$\#\mathcal{V}$	$\#\mathcal{L}$	$\#\mathcal{M}$
35	1	184	4
36	7	408	30
37	30	1 172	144
38	72	1 859	352
39	121	3 310	511
40	141	3 240	606
41	121	3 310	511
42	72	1 859	352
43	30	1 172	144
44	7	408	30
45	1	184	4
Σ	603	17 106	2 688

MAGMA [1] reveals that the components are isomorphic which surely indicates the presence of some kind of duality among the labelings. Whether this is part of some more general phenomenon would be an interesting question to explore further.

We saw earlier that there were no $(2, 2)$-mutations of the complete graph K_4. In that case, the reason was that all VMTLs had distinct sets of vertex labels. We found that there were no $(3, 3)$-mutations of K_5; indeed, the following general result is true:

Lemma 8. *There are no $(r-1, r-1)$-mutations of K_{r+1}.*

Proof. For, if we wish to swap $r-1$ edges incident on vertex v_1 with $r-1$ edges incident on vertex v_2, we find that the remaining edge incident with v_1 is the same as the remaining edge incident with v_2. Thus the sum of all edge weights is the same on both vertices, which is impossible in a VMTL. □

Are $(r-2, r-2)$-mutations of K_{r+1} possible? We have seen above that the answer is yes for $r = 4$. For each feasible magic constant there exist VMTLs of Q1= K_5 that are $(2, 2)$-mutatable. We conjecture that the answer will continue to be yes for all $r > 4$ and all k, but do not have a proof yet.

The unique order 6 quartic graph Q2 is the octahedron. Table 14 shows the large number of VMTLs Q2 possesses. Here $\mathcal{M}_{(2,2)}$ is the number of

Table 12: The pair of immutable strong VMTLs of Q1= K_5.

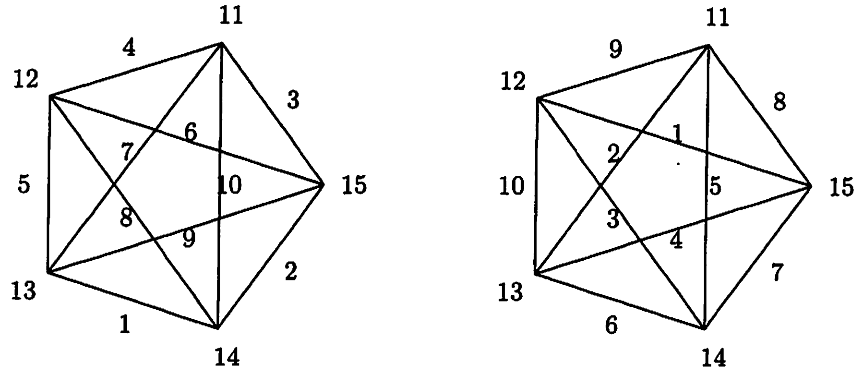
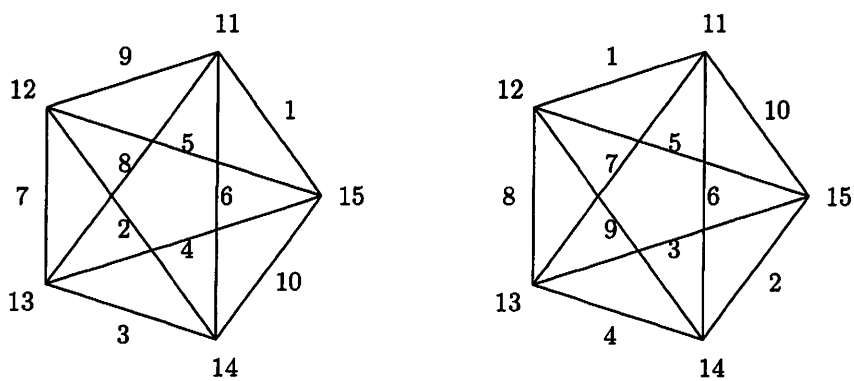


Table 13: The strong VMTLs of Q1= K_5 having degree 8 in the metagraph.



$(2,2)$-mutation classes and $\mathcal{M}_{(3,3)}$ is the number of $(3,3)$-mutation classes. Remarkably, for each appropriate magic constant, we can find VMTLs that are both $(2,2)$- and $(3,3)$-mutatable, $(3,3)$-mutatable but not $(2,2)$-mutatable, $(2,2)$-mutatable but not $(3,3)$-mutatable, and neither.

There are 2 order 7 quartics, Q3 and Q4 in the *Atlas*, where Q3 is the circulant graph $C_7(1,2)$. As will be expected by now, there are vast numbers of VMTLs for each magic constant (see Table 15). For each magic constant, we find a $(2,2)$-mutation between Q3 and Q4. An example of

Table 14: For each magic constant, the number $\#\mathcal{L}$ of VMTLs of Q2, and number of mutation classes $\#\mathcal{M}$.

k	$\#\mathcal{V}$	$\#\mathcal{L}$	$\#\mathcal{M}$	$\#\mathcal{M}_{(2,2)}$	$\#\mathcal{M}_{(3,3)}$
42	3	13 377	75	238	8 998
43	26	50 431	827	2 285	34 796
44	106	133 884	3 120	7 800	94 818
45	271	262 575	7 691	17 252	192 770
46	488	386 146	14 346	30 960	289 448
47	648	468 896	19 006	40 626	352 637
48	648	468 896	19 006	40 626	352 637
49	488	386 146	14 346	30 960	289 448
50	271	262 575	7 691	17 252	192 770
51	106	133 884	3 120	7 800	94 818
52	26	50 431	827	2 285	34 796
53	3	13 377	75	238	8 998
Σ	3 084	2 630 618	90 130	198 322	1 946 934

each is shown in Table 18 in the Appendix.

When we consider $(3,3)$-mutations it is rather surprising that we find a situation somewhat similar to that described for the 2 cubics of order 6: there are no $(3,3)$-mutations between the 2 graphs. It will be worth trying to determine whether these 2 cases are examples from some infinite families of such failures.

Lemma 9. *All* $(3,3)$*-mutations of VMTLs of Q3 lead to another VMTL of Q3.*

Proof. As in Theorem 2 and Lemma 8, this is shown *a priori* without considering the labels. □

At order 8 there are 6 quartic graphs to consider, all of which have very large numbers of labelings. Recall that there are no strong labelings of even regular even order graphs. There are 5 vertex label sets with the smallest possible magic constant 55; choosing $V = \{13, 18, \ldots, 24\}$ we can easily find mutation classes containing all graphs, as desired. Table 19 in the Appendix shows a suitable sequence of mutations that uses both $(2,2)$- and $(3,3)$-mutations.

Theorem 10. *(i) For each magic constant there is a (2,2)-mutation class*

Table 15: Counts of VMTLs of order 7 quartic graphs.

k	$\#\mathcal{V}$	Q3	Q4	Σ
48	1	813 878	254 983	1 068 861
49	15	3 395 810	1 053 290	4 449 100
50	105	12 521 284	3 911 522	16 432 806
51	406	32 606 713	9 903 602	42 510 315
52	1 069	63 912 832	19 345 425	83 258 257
53	2 043	99 831 438	30 491 450	130 322 888
54	2 983	129 964 295	39 813 149	169 777 444
55	3 370	145 105 764	43 926 894	189 032 658
56	2 983	129 964 295	39 813 149	169 777 444
57	2 043	99 831 438	30 491 450	130 322 888
58	1 069	63 912 832	19 345 425	83 258 257
59	406	32 606 713	9 903 602	42 510 315
60	105	12 521 284	3 911 522	16 432 806
61	15	3 395 810	1 053 290	4 449 100
62	1	813 878	254 983	1 068 861
Σ	16 614	831 198 264	253 473 736	1 084 672 000

containing all 6 order 8 quartic graphs. (ii) For each magic constant there is a (3,3)-mutation class containing all 5 order 8 quartic graphs except $K_{4,4}$.

Theorem 11. (i) *For each magic constant there is a (2,2)-mutation class containing all 16 order 9 quartic graphs.* (ii) *For each magic constant there is a (3,3)-mutation class containing all 16 order 9 quartic graphs.*

7 Algorithms

The algorithms used to generate the data presented in this paper were implemented in MAGMA [1] and run at The University of Newcastle [8].

Our method of finding the VMTLs of all r-regular graphs on v vertices involves the following. As $2e = rv$, the total number of labels is $t = v + e = \frac{v}{2}(r+2)$. The total label set is $T = \{1, \ldots, t\}$, with the vertex label set $V \subset T$ and the edge label set $E = T \backslash V$. Summing the vertex and incident edges over all the vertices, we have:

$$\begin{aligned} kv &= \sum V + 2\sum E \\ &= 2\sum T - \sum V \\ &= t(t+1) - \sum V. \end{aligned}$$

When k is an integer, we say that V is a feasible vertex label set for magic constant k. Let the family of all feasible vertex labels sets be denoted $\mathcal{V}$, and those with magic constant k be denoted $\mathcal{V}_k$. For fixed r and v the feasibility of V only depends on its sum.

Fix a vertex label set V. For each vertex label $i \in V$, we pre-calculate the family of all size r sets E_i of edge labels with $\sum E_i = k - i$.

We represent a VMTL as a sequence L of adjacency sets: L(i) is the set of the labels of the objects adjacent to i; if i is a vertex we have a set of r adjacent edges; if i is an edge we have a set of a two adjacent vertices.

To find VMTLs we perform a depth first search in a tree; each vertex (other than those at depth v) in the search tree is labeled with an element of V that is not affixed to any of its ancestors; the downward edges of a vertex x labeled i are labeled with those elements of E_i that are compatible with the edge labels on the path from the root to x; compatibility requires each edge label to be used at most twice (once for each end of an edge) and requires that no parallel edges would be built; each vertex at depth v defines a VMTL specified by the edges label on the path from the root.

The underlying graph of each VMTL can be tested to determine its isomorphism class; we can either count cycles in the graph, or use `IsIsomorphic` in MAGMA.

When searching for the sequences of mutations required for the proofs of Theorems 5 and 6, it is more efficient to use a randomised search. For each magic constant k, the algorithm randomly chooses a vertex set, then randomly chooses a potential seed and mutates it as follows. Put the seed in a queue; from the head of the queue find all the mutants; if the underlying graph of a mutant is not isomorphic to one already processed, then add the mutant to the tail of the queue (we also add a random selection of those mutants whose type we have seen before, with probability $\frac{1}{8}$); continue until we have found all the required graphs, or we have exhausted the queue and need to try another potential seed.

References

[1] Wieb Bosma, John Cannon, and Catherine Playoust, *The Magma algebra system. I. The user language*, J. Symbolic Comput. **24** (1997), no. 3-4, 235–265, Computational algebra and number theory (London, 1993). (`http://magma.maths.usyd.edu.au/`)

[2] I. D. Gray & J. A. MacDougall, Vertex-magic Labelings: Mutations, *Austral. J. Combin.* **45** (2009), 189–206.

[3] I. D. Gray, Vertex-magic total labelings of regular graphs. *SIAM J. Discrete Math.* **21** (2007), no. 1, 170–177.

[4] I. D. Gray and J. A. MacDougall, Vertex-magic total labelings of regular graphs II, *Discrete Math.* **309** (2009), 5986–5999.

[5] J. A. MacDougall, Vertex-magic Labeling of Regular Graphs, lecture July, 2002, DIMACS Connect Institute.

[6] J. A. MacDougall, Mirka Miller, Slamin, W. D. Wallis, Vertex-magic Total Labelings of Graphs, *Utilitas Math* **61** (2002), 3–21.

[7] R. C. Read & R. J. Wilson, *An Atlas of Graphs*, Oxford 1998.

[8] The University of Newcastle High Performance Computing Facility, `http://hdl.handle.net/1959.13/45300`

Appendix

Table 16: The immutable VMTLs of C2 with vertex set $\{7, 11, 12, 13, 14, 15\}$

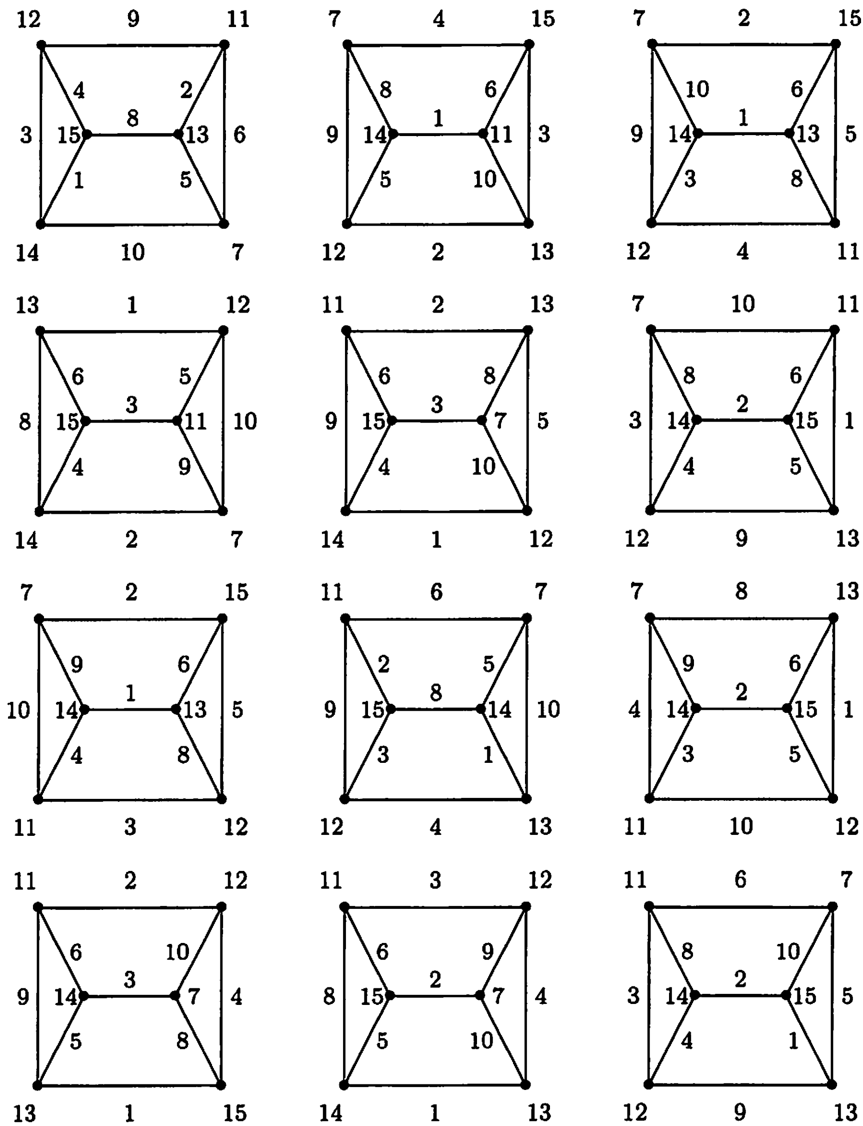


Table 17: For each magic constant k, a seed VMTL of $C1 \cup C1$ that mutates into VMTLs for each of the other order 8 cubic graphs

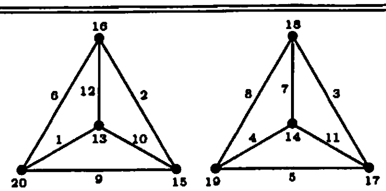
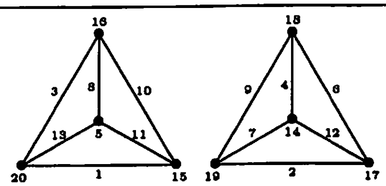
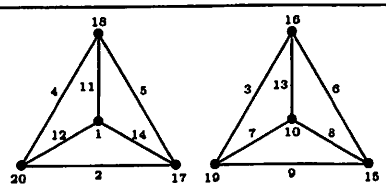
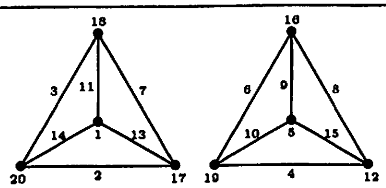
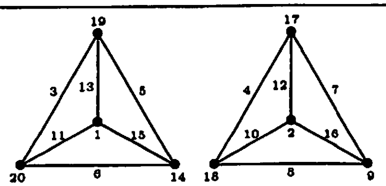
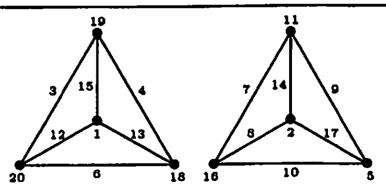
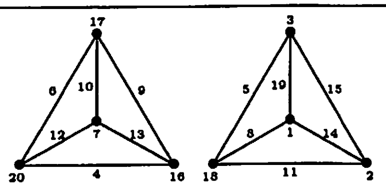
k	$C1 \cup C1$ seed	Class size
36		46444
37		11006
38		271
39		233
40		234
41		57
42		54

Table 18: For magic constant k, a $(2,2)$-mutation from Q3 to Q4.

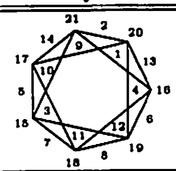
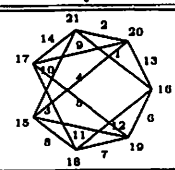
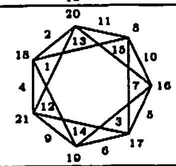
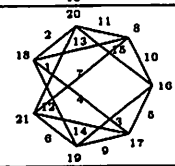
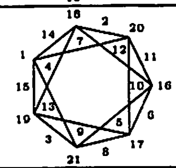
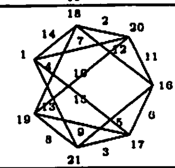
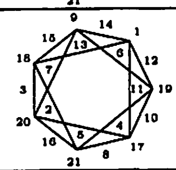
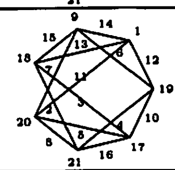
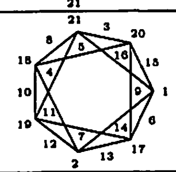
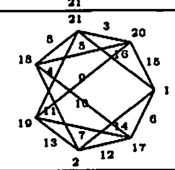
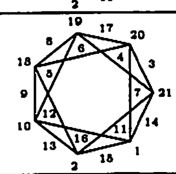
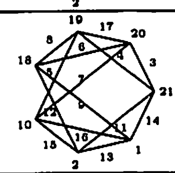
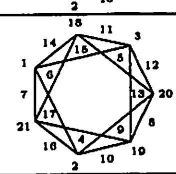
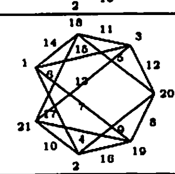
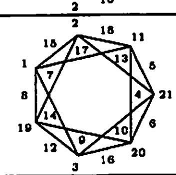
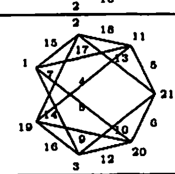
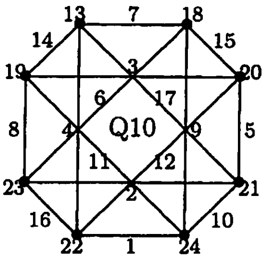
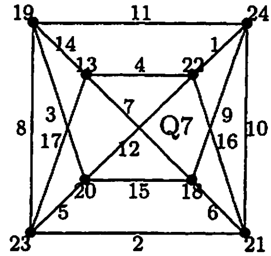
k	Q3	Mutation	Q4
48		15 : (5, 7) 19 : (4, 8)	
49		17 : (6, 7) 21 : (4, 9)	
50		17 : (8, 10) 19 : (3, 15)	
51		17 : (8, 11) 20 : (3, 16)	
52		17 : (9, 13) 19 : (10, 12)	
53		1 : (7, 15) 10 : (9, 13)	
54		19 : (10, 13) 21 : (7, 16)	
55		19 : (8, 12) 20 : (4, 16)	

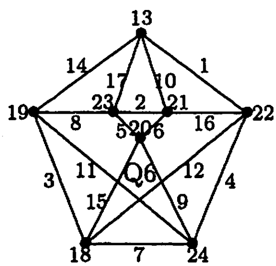
Table 19: Mutations among VMTLs, with vertex label set $\{13, 18, \ldots, 24\}$, of all 6 order 8 quartic graphs.



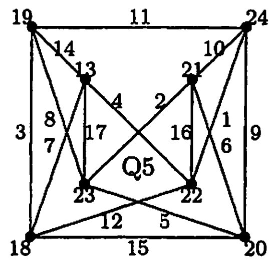
$21 : (5, 17)$
$23 : (6, 16)$ →



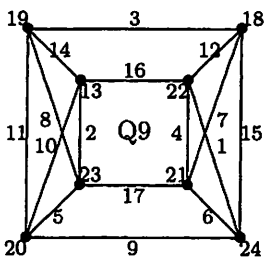
$18 : (6, 9)$
$20 : (3, 12)$
↓



← $13 : (4, 7)$
$24 : (1, 10)$



$24 : (10, 11)$
$20 : (6, 15)$
↓



← $13 : (4, 7, 17)$
$21 : (2, 10, 16)$