Group Divisible Designs with Two Association Classes

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Abstract

We investigate group divisible designs with two association classes (known as GDDS, GADs or PBIBDs) with block size 3 and unequal size groups. We completely determine the necessary and sufficient conditions for groups with size vector $(n, 1)$ for any $n \geq 3$, and $(n, 2, 1)$ for $n \in \{2, 3, ..., 6\}$. We also have some general results for (n_1, n_2, n_3).

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1 Introduction

A *group divisible design* GDD($v = v_1+v_2+\cdots+v_g, g, k, \lambda_1, \lambda_2$) is a collection of k-subsets (called blocks) of a v-set of elements, where the v-set is divided into g groups of sizes $v_1, v_2, \ldots, v_g$; each pair of elements from the same group occurs in exactly λ_1 blocks; and each pair of elements from different groups occurs in exactly λ_2 blocks. Pairs of symbols occurring in the same group are known to statisticians as *first associates*, and pairs occurring in different groups are called *second associates.*

It is useful to describe GDDs graphically. Let λK_n denote the graph on n vertices in which each pair of vertices is joined by λ edges. Let G_1 and G_2 be graphs. The graph $G_1 \vee_\lambda G_2$ is formed from the union of G_1 and G_2 by joining each vertex in G_1 to each vertex in G_2 with λ edges. If $\lambda = 1$ then we simply write $G_1 \vee G_2$. A G-decomposition of a graph H is a partition of the edges of H such that each element of the partition induces a copy of G. Hence a GDD($v = m + n$, $2, 3, \lambda_1, \lambda_2$) is equivalent to a K_3-decomposition of $\lambda_1 K_m \vee_{\lambda_2} \lambda_1 K_n$. In this graph theoretic setting, edges joining vertices (symbols) in the same group are referred to as *pure* edges, whereas edges joining vertices in different groups are called *mixed* edges.

In general, if the number of groups is less than the block size, or if the groups are of unequal size, then the construction of such GDDs is notoriously difficult. For $k = 3$ this existence problem was completely solved [6, 7] in the case where all groups have the same size.

The existence of such GDDs has been of interest over the years, going back to at least the work of Bose and Shimamoto in 1952 who began classifying such designs [2]. More recently, much work has been done on the existence of such designs when $\lambda_1 = 0$ (see [3] for a summary), and the designs here are called partially balanced incomplete block designs (PBIBDs) of group divisible type in [3]. In [10] they are called group association designs. The case of group divisible designs with two associate classes, with $k = 3$, GDD ($v = m + n, 2, 3, \lambda_1, \lambda_2$), has been considered by El-Zanati, Punnim and Rodger [5], Pabhapote and Punnim [13], and others.

In Section 2 we completely solve the problem for $k = 3$ of determining all pairs of integers (n, λ) in which a GDD($v = 1 + n, 2, 3, 1, \lambda$) exists. The results have circulated in preprint form but have not been published. In fact, these results and techniques have been used by other papers [5] , [13] and [10], and have also been cited by them. The paper [13] found all triples of integers (m, n, λ) in which a GDD($n = m + n, 2, 3, \lambda, 1$) exists, and in [10], the cases for group sizes $(n, 1, 1)$ and $(n, 1, 1, 1)$ were dealt with.

In Section 3 we introduce new necessary conditions for three groups of unequal sizes including congruence restrictions on the indices. In Section 4 we apply these necessary conditions to three groups with sizes $(n, 2, 1)$. Then, using these conditions, we completely solve the $(n, 2, 1)$ cases for $n \in \{2, ..., 6\}$.

1.1 Special notation

The following notation for sets of triples will be used throughout the paper for our constructions.

1. Let $T = \{x, y, z\}$ be a triple and $a \notin T$. We use $a * T$ for the three triples $\{a, x, y\}, \{a, x, z\}, \{a, y, z\}$. If $\mathcal{T}$ is a set of triples, then $a * \mathcal{T}$ is defined as $\{a * T : T \in \mathcal{T}\}$.
2. Let $e = uv$ be an edge of a graph G. We use $a + e$ for the triple $\{a, u, v\}$. If X is a set of edges of a graph G, then $a + X$ is defined as $\{a + e : e \in X\}$.
3. By $\{a, b, c\} \times j$ we mean use j copies of the block $\{a, b, c\}$.
4. In later sections, the block size will always be three, and we abbreviate the notation when the block size is three: if the number of groups is two, we use the notation GDD$(v_1, v_2; \lambda_1, \lambda_2)$ and when the number of groups is three we use the notation GDD$(v_1, v_2, v_3; \lambda_1, \lambda_2)$.

1.2 Remarks on triple systems

A BIBD(v, b, r, k, λ) is a set S of v elements together with a collection of b k-subsets of S, called blocks, where each point occurs in r blocks and each pair of distinct elements occurs in exactly λ blocks (see [12]). In a BIBD(v, b, r, k, λ) the parameters must satisfy the necessary conditions: $vr = bk$ and $\lambda(v - 1) = r(k - 1)$. Hence a BIBD$(v, b, r, k, \lambda)$ is usually written as BIBD(v, k, λ). When $k = 3$, a BIBD is often called a triple system. We will review some known results concerning BIBDs with block size three, or triple systems [12], [4], and apply them in Sections 2 and 3.

2 GDD$(1, v; 1, \lambda)$

We will give in this section necessary and sufficient conditions for the existence of GDD$(1, v; 1, \lambda)$. First we obtain necessary conditions. It is clear by definition of $K_1 \vee_\lambda K_v$ that the graph has order $v+1$ and size $\binom{v}{2}+\lambda v$. Furthermore, it has one vertex of degree λv and v vertices of degree $\lambda+v-1$. Thus the existence of a K_3-decomposition of $K_1 \vee_\lambda K_v$ implies $2 \mid \lambda v$, $2 \mid (\lambda+v-1)$ and $3 \mid \binom{v}{2}+\lambda v$. Thus:

Theorem 1 *The necessary conditions for the existence of a $GDD(1,v;1,\lambda)$ are*

1. $2 \mid (v-1-\lambda)$,
2. $6 \mid v(v-1-\lambda)$, and
3. $\frac{\lambda v}{2} \leq \binom{v}{2}$.

Let v be a positive integer $v \geq 3$. We define the *spectrum* of λ, denoted $S_{1,v}$ to be:

$$S_{1,v} = \{\lambda : \text{ a GDD}(1,v;1,\lambda) \text{ exists}\}.$$

Theorem 2 *The necessary conditions for the existence of $GDD(1,v;1,\lambda)$ imply the following.*

1. $S_{1,v} \subseteq \{1,3,5,\ldots,v-1\}$ if $v \equiv 0 \pmod 6$.
2. $S_{1,v} \subseteq \{6,12,18,\ldots,v-1\}$ if $v \equiv 1 \pmod 6$.
3. $S_{1,v} \subseteq \{1,7,13,\ldots,v-1\}$ if $v \equiv 2 \pmod 6$.
4. $S_{1,v} \subseteq \{2,4,6,\ldots,v-1\}$ if $v \equiv 3 \pmod 6$.
5. $S_{1,v} \subseteq \{3,9,15,\ldots,v-1\}$ if $v \equiv 4 \pmod 6$.
6. $S_{1,v} \subseteq \{4,10,16,\ldots,v-1\}$ if $v \equiv 5 \pmod 6$.

Thus we can conclude the section with the following theorem and remarks.

Theorem 3 *Let $v \geq 3$ be any integer. Then $\lambda \in S_{1,v}$ if and only if $2 \mid (v-1-\lambda)$ and $6 \mid v(v-1-\lambda)$.*

Detailed notes on a proof of these results have been circulated for some time. Here we merely indicate a new method of proof contributed by an annonymous referee.

Note that a GDD$(1, v; 1, \lambda)$ is equivalent to a decomposition of the graph $K_1 \vee K_v$ into edge-disjoint copies of K_3. By counting, there must be $\frac{\lambda v}{2}$ blocks containing the element from the singleton group and the remaining $\frac{v(v-1-\lambda)}{6}$ blocks with all elements from the group containing v elements. It is known that there exists a partial triple system on v points with $\frac{v(v-1-\lambda)}{6}$ blocks. Using such a partial triple system, one can apply a method given in Andersen, Hilton and Mendelsohn [1] to obtain an equitable partial triple system. Each of the remaining pairs or edges from K_v can be used with the element from the singleton group to create the remaining $\frac{\lambda v}{2}$ blocks.

This proof has the advantage of showing the existence but of course requires understanding of the techniques in [1].

In [9], the problem of existence of minimal enclosings for triple systems with $1 \le \lambda \le 6$ and any v, i.e., an inclusion of BIBD$(v, 3, \lambda)$ into BIBD$(v+1, 3, \lambda+m)$ for minimal positive m. Note that the problem under consideration in this note is not the same as enclosing considered in [9]

3 GDDs With Three Groups of Unequal Size

In this section we consider the problem of determining necessary conditions for the existence of GDDs with three groups of unequal size and prove that the usual conditions are not sufficient for the cases we consider. The three groups will be $G_n = \{1, 2, ..., n\}$, $G_2 = \{a, b\}$, and $G_1 = \{z\}$ with sizes, respectively of n, 2, and 1. We begin with an infinite family of examples.

Example 1 Let $n = 3t$. We give a family of GDD$(n, 2, 1; 2n + 2, 2)$, where $G_n = \{1, 2, ..., n\}$, $G_2 = \{a, b\}$ and $G_3 = \{z\}$ are the three groups. We suppose there exists a BIBD$(n, 3, \mu)$ which has (at least) one parallel clsss C. Then use the following blocks for the GDD. Use $z * C$, that is, for each block $\{c, d, e\}$ in C, form the three blocks $z * \{c, d, e\}$. In this way point z meets each point of G_n twice. Use two copies of block $\{a, b, j\}$ for each $j \in G_n$. and two copies of block $\{a, b, z\}$. It follows that $\lambda_2 = 2$. Points a, b of G_2 already meet in $2n+2$ blocks, and so we require $\mu = 2n+2$. It follows that $\lambda_1 = 2n + 2$. The parameter n may be taken to be $6s + 3$ for $s \ge 0$ or $6s$ for $s \ge 1$, the since resolvable BIBDs are known to exist for $\lambda = 2$ and such n [see Section 7.4 of [3]; if $n = 6$, a resolvable BIBD$(6, 3, 4)$ exists]. It is especially noteworthy that, if $n = 3u$, then u and λ_1 may increase arbitrarily while the second index stays fixed at 2. This may be contrasted with the results in the next sections where n is small and $\lambda_2 > \lambda_1$.

3.1 Necessary Conditions for the Three Group Case

Necessary conditions on the existence of a GDD($n_1, n_2, n_3, \lambda_1, \lambda_2$) can be obtained from a graph theoretic point of view as before. The existence of a GDD(n_1, n_2, n_3; λ_1, λ_2) is easily seen to be equivalent to the existence of a K_3-decomposition of $(\lambda_1 K_{n_1} \vee_{\lambda_2} \lambda_1 K_{n_2}) \vee_{\lambda_2} \lambda_1 K_{n_3}$, from here on designated simply as $\lambda_1 K_{n_1} \vee_{\lambda_2} \lambda_1 K_{n_2} \vee_{\lambda_2} \lambda_1 K_{n_3}$ by associativity of joins and folds. The graph $\lambda_1 K_{n_1} \vee_{\lambda_2} \lambda_1 K_{n_2} \vee_{\lambda_2} \lambda_1 K_{n_3}$ is of order $n_1+n_2+n_3$ and size $\lambda_1[\binom{n_1}{2}+\binom{n_2}{2}+\binom{n_3}{2}]+\lambda_2(n_1n_2+n_2n_3+n_3n_1)$. It contains n_1 vertices of degree $\lambda_1(n_1-1)+\lambda_2(n_2+n_3)$, n_2 vertices of degree $\lambda_1(n_2-1)+\lambda_2(n_1+n_3)$, and n_3 vertices of degree $\lambda_1(n_3-1)+\lambda_2(n_1+n_2)$ Thus the existence of a K_3-decomposition of $\lambda_1 K_{n_1} \vee_{\lambda_2} \lambda_1 K_{n_2} \vee_{\lambda_2} \lambda_1 K_{n_3}$ implies:

Lemma 1 *For any GDD(n_1, n_2, n_3; λ_1, λ_2), it is necessary that:*

1. $3 \mid \lambda_1[\binom{n_1}{2}+\binom{n_2}{2}+\binom{n_3}{2}]+\lambda_2(n_1n_2+n_2n_3+n_3n_1)$, *and*
2. $2 \mid \lambda_1(n_1-1)+\lambda_2(n_2+n_3)$, $2 \mid \lambda_1(n_2-1)+\lambda_2(n_1+n_3)$ *and* $2 \mid \lambda_1(n_3-1)+\lambda_2(n_1+n_2)$.
3. $b = \frac{1}{6}\left(\lambda_1(n_1^2+n_2^2+n_3^2-n_1-n_2-n_3)+2\lambda_2(n_1n_2+n_1n_3+n_2n_3)\right)$.

The dividends divided by 2 in item 2 are the replication numbers of an element of each group: r_1, r_2 and r_3 respectively. Hence the total number of blocks can be readily computed: $b = \frac{1}{3}(r_1n_1+r_2n_2+r_3n_3)$, which in turn leads to the equation in (3).

3.2 GDD(n,2,1; λ_1, λ_2)

Now we continue to investigate all triples of integers $(\lambda_1, n, \lambda_2)$ in which a GDD($n, 2, 1; \lambda_1, \lambda_2$) exists, where $\lambda_i \geq 1$. First we specialize the formulas of the previous section to our situation: $n_1 = n$, $n_2 = 2$ and $n_3 = 1$, involving the sets $G_1 = \{1, 2, \ldots, n\}$, $G_2 = \{a, b\}$, and $G_3 = \{z\}$ respectively. After some simplification, we obtain:

1. $\lambda_1(n(n-1)+2)+\lambda_2 \equiv 0 \pmod 3$
2. $\lambda_1(n-1)+\lambda_2 \equiv 0 \pmod 2$, $\lambda_1+\lambda_2(n+1) \equiv 0 \pmod 2$, and $\lambda_2 n \equiv 0 \pmod 2$.
3. $b = \frac{1}{6}\left(\lambda_1(n^2-n+2)+2\lambda_2(3n+2)\right)$.

It is convenient in what follows to have available the replication numbers r_1, r_2, and r_3, for their respective groups. These are $r_1 = [\lambda_1(n-1)+3\lambda_2]/2$, $r_2 = [\lambda_1 + \lambda_2(n+1)]/2$, and $r_3 = (n+2)\lambda_2/2$.

Using these formulas, the first condition in Theorem 4 can be shown to be $(3n-4)\lambda_2 \leq (n^2 - n + 2)\lambda_1$, or for large n, when $\lambda_2 > \lambda_1$, it follows that $\lambda_2 \lneqq (n\lambda_1)/3$.

Two rather hidden necessary conditions require a close consideration of the blocks containing the elements of the small groups.

Theorem 4 *For any GDD*$(n,2,1;\lambda_1,\lambda_2)$ *with* β *blocks, (1) it is necessary that* $r_2-(\lambda_1+\lambda_2) \leq \beta-(r_3+r_2-\lambda_2)$ *or equivalently,* $2r_2+r_3 \leq \beta+\lambda_1+2\lambda_2$, *and (2) it is necessary that* $(n+1)\lambda_2 \geq \lambda_1$.

The argument for (2) is attractive: point a appears in blocks to create $(n+1)\lambda_2$ pairs with points from the two other groups. But point a appears in λ_1 blocks with b which create only λ_1 of these pairs. Thus, $(n+1)\lambda_2 - \lambda_1 \geq 0$ and the result follows. Next, for item (1), we define δ_1 to be the total number of blocks less the number of blocks with z, and also less the number of blocks with a but without z. We also define δ_2 to be the number of blocks with b but without a and without z. From these definitions, $\delta_2 \leq \delta_1$.Now, there are r_3 blocks containing z, there are r_2 blocks with a, and there are λ_2 blocks with both. By inclusion-exclusion, it is easy to see that $\delta_1 \leq \beta-(r_3+r_2-\lambda_2)$. Since there are r_2 blocks with b and, as the set $\{a,b\}$ is contained in exactly λ_1 blocks, and as the set $\{z,b\}$ is contained in λ_2 blocks, the number $r_2-(\lambda_1+\lambda_2)$ is a lower bound for δ_2. We have shown $r_2-(\lambda_1+\lambda_2) \leq \delta_2 \leq \delta_1 \leq \beta-(r_3+r_2-\lambda_2)$. The result follows.

3.3 Congruence Restrictions on the Indices

We consider the congruences itemized in Section 3.2. First consider $\lambda_1(n(n-1)+2)+\lambda_2 \equiv 0 \pmod 3$. If $n = 1+6t$, then $2\lambda_1+\lambda_2 \equiv 0 \pmod 3$. With these values, the congruence in (2) above implies λ_2 is even, and then the second implies λ_1 is also even. But, as both indices are even, from (1) again, $2\lambda_1+\lambda_2 \equiv 0 \pmod 3$ implies $\lambda_1 \equiv \lambda_2 \pmod 6$. Other cases require similar computations. The arguments are similar and we omit them.

Theorem 5 *If a GDD*$(n,2,1;\lambda_1,\lambda_2)$ *exists, it is necesary that: (1) the indices are both even or else both odd; (2) if* n *is odd, the indices must both be even; (3) the indices and* n *must satisfy the entries (mod* 6*) in the table below.*

n (mod 6)	λ_1	λ_2	λ_1	λ_2	n (mod 6)	λ_1	λ_2	λ_1	λ_2
0	0	0	1	1	3	0	0	-	-
0	2	2	3	3	3	2	2	-	-
0	4	4	5	5	3	4	4	-	-
1	0	0	-	-	4	0	0	1	1
1	2	2	-	-	4	2	2	3	3
1	4	4	-	-	4	4	4	5	5
2	0	0	3	3	5	0	0	-	-
2	2	4	1	5	5	2	4	-	-
2	4	2	5	1	5	4	2	-	-

We close this section with a new general construction which realizes the extreme case in Theorem 4(2).

Theorem 6 *There exists a GDD$(n, 2, 1; 6w, 6)$ for every $n \geq 3$ and for $2 \leq w \leq n+1$.*

Use six copies of block $\{a, b, j\}$ for $j \in \{1, 2, 3, ..., n, z\}$. Use $n+1$ copies of a BIBD$(n, 3, 6)$. WLOG, we may assume existence of a 3-resolution class C, and we take $C = \{\{1, 2, 3\}, \{2, 3, 4\}, \{3, 4, 5\}, ..., \{n-1, n, 1\}, \{n, 1, 2\}\}$. Use blocks $z*C$. Counting pairs shows $(\lambda_1, \lambda_2) = (6(n+1), 6)$.This construction can be modified to reduce λ_1 by six and maintain $\lambda_2 = 6$. Use only n copies of the BIBD$(n, 3, 6)$ and delete one block, say $\{p, q, r\}$, from the retained BIBDs. Now delete the six blocks $\{a, b, j\} \times 2$ for $j \in \{p, q, r\}$ and. replace them with the six blocks $a*\{p, q, r\}$ and $b*\{p, q, r\}$. Now $(\lambda_1, \lambda_2) = (6n, 6)$. Continue in this way, reducing λ_1 by six at each stage ending with $\lambda_1 = 12$.

4 Restricting the Large Group to Small Numbers

The groups in this section have sizes $(n, 2, 1)$ for $n \in \{2, ..., 6\}$. For each case, we state a necessary condition, and then we show in each case that the necessary conditions we have found are sufficient for their existence.

4.1 Groups With Sizes (2,2,1)

We begin with a couple of non-existence results although the congruences in Theorem 5 are satisfied..

Example 2 There does not exist a GDD$(2,2,1;1,5)$. If there were such a design, it would have 14 triples and 10 of them would have z as a vertex. The remaining triples are subsets of $\{1,2,a,b\}$ which would mean $\lambda_1 \geq 2$, a contradiction.

Example 3 There does not exist a GDD$(2,2,1;5,1)$. There are six blocks for the design. The pair {1,2} requires 5 blocks, and the pair {a,b} requires 5 blocks. There are not enough blocks.

We now arrive at a couple of positive results.

Example 4 There exists a GDD$(2,2,1;5,7)$. The blocks are: $z12$, zab, $z1a$, $z1b$, $z2a$, $z2b$, $a12$, $b12$, $1ab$, $2ab$, bz, abz, $ab1$, $ab2$, $12z$, $12z$, $12a$, $12b$. We note that we now have 7 pairs of ab and 12, as well as 5 "mixed" pairs from different groups as required.

Example 5 There exists a GDD$(2,2,1;7,5)$. We will need 22 triples. The degree of z is 28 so that there will be 14 blocks with z inside. We list the blocks: $z12$, zab, $z1a$, $z2a$, $z1b$, $z2b$, $ab1$, $ab2$, $12a$, $12b$, $z1a$, $z2a$, $z1b$, $z2b$, $z1a$, $z2a$, $z1b$, $z2b$, $ab1$, $ab2$, $12a$, $12b$, and arrive at 22 triples. Checking, there are 5 occurrences each of the pairs ab and 12 and 7 occurrences each of the pairs $z1$, $z2$, za,zb, $1a$, $1b$, $2a$ and $2b$. Thus we have our required design.

For any GDD$(2,2,1;\lambda_1,\lambda_2)$, each of its two indices is bounded above by a multiple of the other.

Theorem 7 *For any GDD$(2,2,1;\lambda_1,\lambda_2)$, it is necessary that $\lambda_1 \leq 2\lambda_2 \leq 4\lambda_1$.*

Note that the inequality is more restrictive for this case than is Theorem 4(2). To see these inequalities, first observe that there are λ_1 blocks of the type $\{a,b,*)$ and λ_1 blocks of the type $\{1,2,*\}$, where $*$ is an element from another group. In these two sets of blocks, let t denote the number of blocks with z and let s denote the number without z. We note

$$t + s = 2\lambda_1.$$

The rest of the blocks contain one point from each group. Since there are $4\lambda_2$ pairs with z, there are $4\lambda_2 - 2t$ pairs with z from these blocks. It follows that there are $2\lambda_2 - t$ blocks of this type. The total number of blocks is

$$b = \lambda_1 + \lambda_1 + 2\lambda_2 - t = 2(\lambda_1 + \lambda_2) - t.$$

However, for any GDD with three groups with sizes 2, 2, and 1,

$$b = \tfrac{1}{6}(4\lambda_1 + 16\lambda_2) = (2\lambda_1 + 8\lambda_2)/3.$$

Equating these two expressions and solving for t gives $t = (4\lambda_1 - 2\lambda_2)/3$. As t is non-negative, this implies $0 \leq 4\lambda_1 - 2\lambda_2$, from which we get $\lambda_2 \leq 2\lambda_1$.

There are $2\lambda_1$ blocks of type $\{a, b, *\}$ or $\{1, 2, *\}$. These triples give $4\lambda_1$ *second* associate pairs in the design, which has $16\lambda_2/2$ second associate pairs in all. Thus, $4\lambda_1 \leq 8\lambda_2$ or $\lambda_1 \leq 2\lambda_2$. Combining the two key inequalities gives the desired result:

$$\lambda_1 \leq 2\lambda_2 \leq 4\lambda_1.$$

This inequality also explains the two non-existence examples which began this section.

We next present two small designs which can be used to build examples with large indices.

Example 6 There exist GDD(2, 2, 1; 2, 4) and GDD(2, 2, 1; 4, 2). For $(\lambda_1, \lambda_2) = (2, 4)$ use blocks $\{a, b, 1\}$, $\{a, b, 2\}$, $\{1, 2, a\}$, $\{1, 2, b\}$, $\{a, 1, z\}$, $\{a, 1, z\}$, $\{a, 2, z\}$, $\{a, 2, z\}$, $\{b, 1, z\}$,$\{b, 1, z\}$,$\{b, 2, z\}$, $\{b, 2, z\}$. For $(\lambda_1, \lambda_2) = (4, 2)$, use blocks $\{a, b, z\}$, $\{a, b, z\}$, $\{a, b, 1\}$, $\{a, b, 2\}$, $\{1, 2, z\}$, $\{1, 2, z\}$, $\{1, 2, a\}$, $\{1, 2, b\}$.

Now let x denote a number of copies of the GDD(2, 2, 1, 2, 4) and let y denote a number of copies of the GDD(2, 2, 1, 4, 2). Set

$$2x + 4y = 6s, \text{ and}$$
$$4x + 2y = 6t.$$

Solving for x and y, we get

$$x = 2t - s, \text{ and}$$
$$y = 2s - t.$$

It follows that we get a GDD(2, 2, 1; $6s$, $6t$) using x copies and y copies of the respective GDDs. (It follows from the previous theorem that x and y are nonnegative.)

For a GDD$(2,2,1;6s+2,6t+4)$ use $x = 2t - s + 1$ and $y = 2s - t$ copies of the designs. For a GDD$(2,2,1;6s+4,6t+2)$ use $x = 2t - s$ and $y = 2s - t + 1$ copies of the designs.

We have shown that GDD$(2,2,1;\lambda_1,\lambda_2)$ exist for all possible even indices allowed by Table 3 in the $n \equiv 2 \pmod 6$ case. Now suppose both indices are odd. With x and y as before, let u denote a number of copies of a GDD$(2,2,1;7,5)$ and w denote a number of copies of a GDD$(2,2,1;5,7)$.

First suppose $(\lambda_1,\lambda_2) = (6s+7,6t+5)$. Then we may take $x = 2t - s$ and $y = 2s - t$ and $w = 0$ and $u = 1$.

Next, if $(\lambda_1,\lambda_2) = (6s+5,6t+7)$, let $x = 2t - s$, $y = 2s - t$, and $w = 1$ and $u = 0$. For $(\lambda_1,\lambda_2) = (6s+9,6t+15)$, let $x = 2t - s + 2$ and $y = 2s - t$, and let $u = 0$ and $w = 1$. For $(\lambda_1,\lambda_2) = (6s+15,6t+9)$, let $x = 2t - s$, $y = 2s - t + 2$, and let $w = 0$ and $u = 1$. We have now proven the following theorem since we have constructed examples of all such designs allowed by Theorems 5 and 7.

Theorem 8 *The necessary conditions are sufficient for the existence of GDD* $(2,2,1;\lambda_1,\lambda_2)$.

4.2 Groups With Sizes (3, 2, 1)

Here $n = 3$, and we establish a necessary condition applying Theorem 4 and construct examples which satisfy it. Let $G_1 = \{1,2,3\}$, $G_2 = \{a,b\}$, and $G_3 = \{z\}$.

Example 7 A GDD$(3,2,1;8,2)$. Use blocks $\{1,2,3\}\times 7$, $\{1,2,z\}$, $\{2,3,z\}$, $\{3,1,z\}$. Use two copies each of the triples $\{a,b,1\}$, $\{a,b,2\}$, $\{a,b,3\}$, and $\{a,b,z\}$.

Example 8 Two non-isomorphic GDD$(3,2,1;12,6)$. Use blocks $\{1,2,3\}\times 5$, $\{1,2,a\}\times 2$, $\{1,3,a\}\times 2$, $\{3,2,a\}\times 2$, $\{1,2,b\}\times 2$, $\{1,3,b\}\times 2$, $\{3,2,b\}\times 2$, $\{a,b,1\}\times 2$, $\{a,b,2\}\times 2$, $\{a,b,3\}\times 2$, $\{a,b,z)\times 6$, $\{1,2,z)\times 3$, $\{2,3,z)\times 3$, $\{3,1,z)\times 3$. For the second example, use the blocks of a GDD$(3,2,1;8,2)$, and the blocks of a BIBD$(6,3,4)$.

The construction of the GDD$(3,2,1;8,2)$ provides an extreme example in the sense of the next theorem.

Theorem 9 *There exists a GDD*$(3,2,1;\lambda_1,\lambda_2)$ *only if the indices satisfy the conditions listed below:*

(λ_1, λ_2)	Condition on s, t
$(6s, 6t)$	$1.25s \le 5t \le 8s$
$(6s+2, 6t+2)$	$1.25s \le 5t \le 8s+1$
$(6s+4, 6t+4)$	$1.25s \le 5t \le 8s+2$

The left-hand inequality follows, in each case, from Theorem 6, which

provides an upper bound on $\lambda_1 = (n+1)\lambda_2 = 4\lambda_2$ and this gives a bound for s, t. The right-hand inequalities of the 3 cases develop using Theorem 4, where $(\lambda_1, \lambda_2) = (6s+2w, 6t+2w)$, for $2w = 0, 2, 4$:

$$r_2 - (\lambda_1 + \lambda_2) \le b - \{r_3 + (r_2 - \lambda_2)\},$$
$$5\lambda_2 \le 8\lambda_1,$$
$$5(6t + 2w) \le 8(6s + 2w),$$
$$5t \le 8s + w.$$

For the 3 cases in the preceeding theorem, we will construct example designs for allowable parameters and thereby show these necessary conditions are sufficient. For example, a GDD(3, 2, 1; 8, 2) exists, and if we use the blocks of this design and the blocks of a BIBD(6, 3, 2), we get a GDD(3, 2, 1, 10, 4).

Theorem 10 *A GDD(3, 2, 1;$\lambda_1\lambda_2$) with $4\lambda_2 \ge \lambda_1 > \lambda_2$ may be constructed using x copies of a GDD(3, 2, 1; 8, 2) and y-copies of a BIBD(6, 3, 2), where $x = (\lambda_1 - \lambda_2)/6$ and $y = (4\lambda_2 - \lambda_1)/6$.*

Set $\lambda_1 = 8x + 2y$ and $\lambda_2 = 2x + 2y$ and solve for x and y.

Example 9 A GDD(3, 2, 1; 14, 20). Use blocks $a*\{1,2,3\}\times 5$, $b*\{1,2,3\}\times 5$, and $z*\{1,2,3\}\times 5$ (see Section 1.1 for this notation). Use 6 copies each of $\{z,a,1\}$, $\{z,a,2\}$, $\{z,a,3\}$, $\{z,b,1\}$, $\{z,b,2\}$, and $\{z,b,3\}$. Use $\{z,a,b\}\times 2$. Use 4 copies each of $\{a,b,1\}$, $\{a,b,2\}$, $\{a,b,3\}$.

Example 10 A GDD(3, 2, 1; 10, 16). Use blocks $\{a,z,1\}\times 5$, $\{a,z,2\}\times 5$, $\{a,z,3\}\times 5$, $\{b,z,1\}\times 5$, $\{b,z,2\}\times 5$, $\{b,z,3\}\times 5$. Use three copies of each of the six blocks $\{1,2,a\}$, $\{1,3,a\}$, $\{2,3,a\}$, $\{1,2,b\}$, $\{1,3,b\}$, $\{2,3,b\}$. Use $\{a,b,1\}\times 4$, $\{a,b,2\}\times 3$, and $\{a,b,3\}\times 3$. Use $\{2,3,a\}$, $\{2,3,b\}$, $\{1,3,b\}$, and $\{1,2,a\}$. Finally, use $z*\{1,2,3\}\times 2$, and $\{a,z,3\}$, $\{b,z,2\}$.

It follows from Theorem 10 that a GDD(3, 2, 1; 6, 12) does not exist (since $s = 1$ and $t = 2$ - thus $5t \not\le 8s$). It follows that the GDD(3, 2, 1; 10, 16) example is the smallest example with $\lambda_1 < \lambda_2$.

Example 11 A GDD(3, 2, 1; 12, 18) Use blocks $\{1,2,3\}$, $\{z,1,a\}\times 6$, $\{z,1,b\}\times 6$, $\{z,2,a\}\times 6$, $\{z,2,b\times 6$, $\{z,3,a\}\times 6$, $\{z,3,b\}\times 6$, $\{a,b,1\}\times 4$, $\{a,b,2\}\times 4$, $\{a,b,3\}\times 4$, $\{1,2,a\}\times 4$, $\{1,3,a\}\times 4$, $\{2,3,a\}\times 4$, $\{1,2,b\}\times 4$, $\{1,3,b\}\times 4$, $\{2,3,b\}\times 4$, $\{z,1,2\}\times 3$, $\{z,1,3\}\times 3$, $\{z,2,3\}\times 3$. A different example comes from the blocks of a GDD(3, 2, 1; 10, 16) and a BIBD(6, 3, 2).

Theorem 11 *There exists a GDD$(3,2,1;6s+2w,6t+2w)$, with $w=0,1,2$, if and only if the necessary conditions are satisfied.*

In view of Theorem 10, we only need to show existence for the case $\lambda_1<\lambda_2$. A GDD$(3,2,1;6s+2w,6t+2w)$ with $\lambda_1<\lambda_2$ may be constructed using the blocks of x-copies of a GDD(3, 2, 1; 10, 16) and y-copies of a BIBD(6, 3, 2) where $x=t-s$ and $y=8s-5t+w$.

4.3 Groups With Sizes (4,2,1)

As in the previous subsections, we apply Theorem 4 with $n=4$ to obtain:

Theorem 12 *For any GDD$(4,2,1;\ \lambda_1,\lambda_2)$, it is necessary that $4\lambda_2\le 7\lambda_1\le 35\lambda_2$.*

Example 12 A GDD(4, 2, 1; 8, 2). Use blocks $\{1,2,3\}\times 4$ and $4*\{1,2,3\}\times 3$. Use the blocks $\{a,b,z\}\times 2$, $\{a,b,2\}\times 2$, $\{a,b,3\}\times 2$, $\{a,b,4\}$, $\{a,b,1\}$. Next use $\{1,2,z\}$, $\{1,3,z\}$, $\{2,4,z\}$, $\{3,4,z\}$, $\{2,3,4\}$, $\{1,4,a\}$, and $\{1,4,b\}$.

Observe $\lambda_2=1$ is not possible since, by Theorem 12, $\lambda_1\le(4+1)\lambda_2=5$. However, Theorem 5 requires $\lambda_1\equiv\lambda_2 \pmod 6$ so λ_1 must be at least 7, which is too large. There do exist GDD$(4,2,1;6t,6)$ with $2\le t\le 5$, by Theorem 6. When $\lambda_1<\lambda_2$, Theorem 12 (and the congruence restriction from Theorem 5) tell one that the smallest possible values of the indices are $(\lambda_1,\lambda_2)=(8,14)$. Therefore, if the indices are multiples of 6, then $(\lambda_1,\lambda_2)=(6s+6j,6t+6j)$ where $t/s=7/4$ (the left-hand side of the inequality in Theorem 12).

Example 13 A GDD(4, 2, 1; 8, 14). Decompose 8 copies of K_4 into 24 one-factors, say $F_1,F_2,\ldots,F_{24}$. Use blocks $a+F_i$ for $1\le i\le 8$. Use $b+F_i$ for $9\le i\le 16$. Use $z+F_i$ for $17\le i\le 23$. Use F_{24} with a,b in the blocks $\{a,3,4\}$ and $\{b,1,2\}$. Use $\{a,z,1\}$, $\{a,z,2\}$, $\{b,z,3\}$, and $\{b,z,4\}$. Use 2 copies each of the set of blocks $\{a,b,1\}$, $\{a,b,2\}$, $\{a,b,3\}$, and $\{a,b,4\}$. Use 3 copies of each of the next 8 blocks: $\{a,z,1\}$, $\{a,z,2\}$,

$\{a, z, 3\}$, $\{a, z, 4\}$, $\{b, z, 1\}$, $\{b, z, 2\}$, $\{b, a, 3\}$ and $\{b, z, 4\}$. It is noteworthy that, in this example, we have exact equality in the left-hand side of the inequality and no three pairs of the 4-element group are used to make a block.

Theorem 13 *The necessary conditions are sufficient for the existence of GDD* $(4, 2, 1; \lambda_1, \lambda_2)$.

The proof will be complete if we construct designs which satisfy the necessary conditions. If $\lambda_1 < \lambda_2$, construct a GDD$(4, 2, 1; 6s + +6j + w, 6t + 6j + w)$ by using $(t - s)$-copies of a GDD$(4, 2, 1; 8, 14)$ and copies of a BIBD$(7, 3, 6j + w)$. If $\lambda_2 < \lambda_1$, then $s - t \leq 30$, and one may use a number of copies of a BIBD$(7, 3, 1)$ and a linear combination of GDD$(4, 2, 1;\ \lambda_1, \lambda_2)$ with $(\lambda_1, \lambda_2) \in \{(30, 6), (24, 6)\ , (18, 6), (12, 6), (8, 2)\}$.

4.4 Groups With Sizes (5, 2, 1)

When $n = 5$, both indices must be even (Theorem 5). We first apply Theorem 4 with $n = 5$ to obtain:

Theorem 14 *For any GDD*$(5, 2, 1;\ \lambda_1, \lambda_2)$, *it is necessary that* $\lambda_2 \leq 2\lambda_1 \leq 12\lambda_2$.

We next construct GDD with the smallest parameters allowable by this theorem.

Example 14 A GDD$(5, 2, 1; 2, 4)$. Columns are blocks.

a	a	a	a	b	b	b	b	a	a
1	2	3	4	1	2	3	4	b	b
z	z	z	z	z	z	z	z	5	5

,

5	3	4	3	5	3	1
b	b	b	b	b	b	b
1	2	1	4	4	2	2

,

z	z	z	z	z	z	a	a	a	a	a	a	a
5	5	5	5	1	2	1	1	1	5	5	4	2
1	2	3	4	3	4	2	3	4	2	3	3	4

Example 15 A GDD$(5, 2, 1; 4, 2)$. The necessary blocks, say $\mathcal{B}_2$, are the relative complement of $\mathcal{B}_1$ in $\mathcal{A}_1$, where $\mathcal{B}_1$ denotes the blocks in the GDD$(5, 2, 1; 2, 4)$ just above and $\mathcal{A}_1$ denotes the blocks in the (unique) BIBD$(8, 3, 6)$ with the same 8 points (whose blocks are exactly the set of all triples on the 8 points).

Let $\mathcal{D}_1$ denote the GDD$(5,2,1;36,6)$ from Theorem 6, let $\mathcal{D}_2 =$ GDD$(5,2,1;30,6)$, let $\mathcal{D}_3 =$ GDD$(5,2,1;24,6)$ and let $\mathcal{D}_4 =$ GDD$(5,2,1;18,6)$. Let $\mathcal{D}_5 =$ GDD$(5,2,1;12,6)$, most easily constructed using 3 copies of $\mathcal{B}_2$, above.

Example 16 A GDD$(5,2,1;10,2)$ exists. Denote this design by $\mathcal{D}_6$. Use the blocks in a BIBD$(5,3,9)$ using the points of G_1, and use the blocks in the arrays:

a	a	a	a	a	a	a	a	a	a
b	b	b	b	b	b	b	b	b	b
1	2	3	4	5	z	1	2	3	4

,

a	b	z	z	z	z	5	5
5	5	2	3	2	3	2	4
z	z	4	1	1	4	3	1

Theorem 15 *The necessary conditions are sufficient for the existence of a GDD$(5,2,1;\lambda_1,\lambda_2)$.*

Suppose $\lambda_1 < \lambda_2 \leq 2\lambda_1$. Then use the blocks of x-copies of a GDD $(5,2,1;2,4)$ and y-copies of a BIBD$(8,3,6)$ where x and y may be found as before using $\lambda_1 = 6s_1 + s_2 = 2x + 6y$ and $\lambda_2 = 6t_1 + t_2. = 4x + 6y$. When $6\lambda_2 \geq \lambda_1 > \lambda_2$, a GDD$(5,2,1;\ \lambda_1,\lambda_2)$ may be constructed using a suitable linear combination of the set of designs $\{\mathcal{B}_1, \mathcal{B}_2, \mathcal{D}_1, ..., \mathcal{D}_6, \mathcal{A}_1\}$.

For example, a GDD$(5,2,1;124,68)$ may be constructed using $8 \times \mathcal{D}_5$, $\mathcal{B}_2$, and $2 \times \mathcal{A}_1$.

4.5 Groups With sizes (6,2,1)

When the first index is larger than the second, we must consider $\lambda_2 = 1, 2$

These possibilities will occur, and we illustrate with some important examples.

Example 17 A GDD$(6,2,1;7,1)$. Use blocks $\{a,b,j\}$ for $j \in \{1,2,...,6,z\}$. Use the blocks of a BIBD$(6,3,6)$ and use the blocks in the array

z	z	z	1	2	1	2
1	3	5	3	3	4	5
2	4	6	5	6	6	4

..

Example 18 There exists a GDD$(6,2,1;8,2)$. Use the blocks of a BIBD$(6,3,4)$ with the points $\{1,2,...,6\}$. Use a BIBD$(7,3,1)$ with the points $\{1,2,...,6,a\}$. Use a BIBD$(7,3,1)$ with the points $\{1,2,...,6,b\}$. Use a BIBD$(7,3,2)$ using the points $\{1,2,...,6,z\}$. Finally, use the blocks $\{a,b,z\} \times 2$ and $\{a,b,j\}$ for $j \in \{1,2,...,6\}$.

Theorem 16 *The necessary conditions are sufficient for the existence of GDD* $(6, 2, 1;\ \lambda_1, \lambda_2)$ *with* $\lambda_1 > \lambda_2$.

We may assume $\lambda_2 > 1$. The blocks of a BIBD$(9, 3, 1)$ based on the points $\{1, 2, ..., 6, a, b, z\}$ may be added to the blocks of a GDD$(6t, 2, 1; 8, 2)$ to create a GDD$(6t, 2, 1; 9, 3)$, and all other GDDs with $\lambda_1 - \lambda_2 = 6s$ and with odd indices may be formed by adding the blocks of a BIBD$(6t + 3, 3, 2m + 1)$ to a design or designs already constructed. When both indices are even, then Theorem 6, the BIBD$(9, 3, 1)$ and the GDD$(6, 2, 1; 8, 2)$ just above, and their multiples, give all possible indices.

We consider the case with $\lambda_1 < \lambda_2$, and for convenience we restate part of Theorem 4 in a form useful here.

Lemma 2 *For any GDD*$(n, 2, 1; \lambda_1, \lambda_2)$, *it is necessary that* $(3n - 4)\lambda_2 \leq (n^2 - n + 2)\lambda_1$ *and, when* $n = 6$, *this becomes* $7\lambda_2 \leq 16\lambda_1$.

Note $\lambda_2 = 7 + 6s$ if λ_1 were to equal 1, but even $\lambda_2 = 7$ is too large, applying the lemma. So $\lambda_1 = 1$ is not possible. In fact, the smallest indices possible in this case are $(\lambda_1, \lambda_2) = (5, 11)$ and $(\lambda_1, \lambda_2) = (6, 12)$ and we construct both.

Example 19 A GDD$(6, 2, 1; 6, 12)$. Use 10 one-factors, say F_1,...,F_{10}, from 2 copies of the complete graph K_6 as follows: $a * F_i$ for $i = 1, ..., 5$; $b * F_i$ for $i = 6, ..., 10$. Use the blocks of a resolvable BIBD$(6, 3, 4)$ with 10 resolution classes $R_1, ..., R_{10}$. Use $a * R_i$ $(i = 1, 2)$. Use $b \times R_i$ $(i = 3, 4)$. Use $z * R_i$ $(i = 5, ..., 8)$ The blocks in R_9 and R_{10} remain unaltered. Finally, use the blocks $\{a, b, j\}$ and $\{a, z, j\} \times 2$ and $\{b, z, j\} \times 2$ for $j = 1$,...,6.

By taking multiple copies of this design one may obtain $(\lambda_1, \lambda_2) = (6t, 12t)$ for any t, and in particular $(\lambda_1, \lambda_2) = (42, 84)$ is possible with $t = 7$. However, the lemma allows $(\lambda_1, \lambda_2) = (42, 96)$.

Example 20 A GDD$(6, 2, 1; 5, 11)$ and, using it, a GDD$(6, 2, 1; 9, 15)$. The former is the smallest design allowed by Theorem 5 and Theorem 4. First, decompose five copies of the complete graph K_6 on six points into 23 one-factors, F_1 to F_{23}, and into the particular two-factor which makes the two triangles $\{1, 2, 6\}$ and $\{3, 4, 5\}$. Use the latter triangle in the design but decompose the pairs of the first triangle into $\{a, 1, 6\}$, $\{b, 2, 6\}$ and $\{z, 1, 2\}$. Next, use blocks $a + F_i$ for $i = 1, ...8$; $b + F_i$ for $i = 9, ...16$; $z + F_i$ for $i = 17, ..., 23$. Use the blocks $\{a, z, j\}$ and $\{b, z, j\}$ for $j \in$

$\{1,2,3,4,5,6\}$. Further, use $\{a,z,j\}$ for $j \in \{2,3,4,5,6\}$ and $\{b,z,j\}$ for $j \in \{3,4,5,6,1\}$. The GDD(6, 2, 1; 9, 15) can be constructed from the blocks of this GDD(6, 2, 1; 5, 11) and a BIBD(9, 3, 4).

Theorem 17 *A BIBD*(9, 3, 1) *and a GDD*(6, 2, 1; 5, 11) *can be embedded into a GDD*(6, 2, 1; 6, 12).

This is just a way of saying that the blocks of the two designs give a GDD(6, 2, 1; 6, 12) which is non-isomorphic to the (6, 12) example just above.

We may not yet construct all GDDs for $n = 6$ with $\lambda_2 - \lambda_1 = 6s$. Since a BIBD(9, 3, 1) exists, we may, however, use the blocks of w copies of this BIBD to the blocks of some multiple of the GDD(6, 2, 1; 5, 11) and/or those blocks of a GDD(6, 2, 1; $6s$, $6t$) to create a GDD(6, 2, 1; $6s+w$, $6t+w$), and w may be odd or even. Therefore, to complete the picture for $n = 6$ requires the construction of GDD(6, 2, 1; $6s$, $6t$) for each s up to the largest possible corresponding value of t. Fortunately, it essentially suffices only to consider the largest t-value applying the Lemma above. The array below gives the maximum possible second index corresponding to a given first index. For $(\lambda_1, \lambda_2) = (42, 96)$ exact equality occurs in Lemma 10.

λ_1	6	12	18	24	30	36	42
Max λ_2	12	24	36	54	66	78	96

It is a consequence of Theorem 5 and Lemma 2 that, if $42 < 6s = \lambda_1 < \lambda_2 = 6t$, a $(6s, 6t)$-design, that is a GDD(6, 2, 1; $6s$, $6t$), may be constructed using a linear combination of multiples of smaller designs - from those in the table and those constructed in this section (see below).

When $\lambda_1 = 12$, only two designs are possible, a GDD(6, 2, 1; 12, 18) and a GDD(6, 2, 1; 12, 24). The former may be realized from a GDD(6, 2, 1; 6, 12) and a BIBD(9, 3, 6). The (12, 24)-design may be realized from two copies of the (6, 12)-design already constructed.

The designs with $\lambda = 18$ are equally easy to get in similar fashion. A GDD(6, 2, 1; 24, 54) cannot be constructed from smaller designs. However, a GDD(6, 2, 1; 30, 66) can be obtained from the (24, 54) and (6, 12) designs, and the (36, 78) design can be obtained from the (30, 66) and the (6, 12). Thus, it only remains to construct the two critical designs mentioned.

Example 21 A GDD(6, 2, 1; 24, 54). Use blocks $\{a,z,j\}\times 9$ and $\{b,z,j\}\times 9$ and $\{a,b,j\} \times 4$ for $j \in G_1$. Use ten one-factors, F_1 to F_{10}, from two copies of K_6, and form blocks $a+F_i$ $(i = 1..., 5)$ and $b+F_i$ $(i = 6, ..., 10)$. Use the blocks of 13 copies of a resolvable BIBD(6, 3, 4) which yield 130 resolution classes. We decompose 54 of the classes using 18 with each of a, b, znd z.

Example 22 A GDD(6, 2, 1; 42, 96). In this case, $7\lambda_2 = 16\lambda_1$. Use the blocks $\{a, z, j\} \times 16$ and $\{b, z, j\} \times 16$ for $j \in \{1, 2, ..., 6\}$. In this way points a and b each appear 96 times with z. Use the blocks $\{a, b, j\} \times 7$ for $j \in \{1, 2, ..., 6\}$. Now points a and b appear together in 42 blocks. Points a and b appear in blocks 23 times with each point of $G_1$and point z appears 32 times with points of G_1. Decompose 42 copies of K_6 into 210 one-factors. Use 73 one-factors with point a and 73 with point b to make new blocks. Use the remaining 64 one-factors to make blocks with point z. Notice that no block contains a triple of points from group G_1.

It is straightforward to alter this construction to lower λ_2 to 90 but keep λ_1 at 42. Use blocks $\{a, z, j\} \times 15$, use $\{b, z, j\} \times 15$, and use $\{a, b, j\} \times 7$ for $j \in G_1$. Next use the blocks of a BIBD(6, 3, 2) and of a resolvable BIBD(6, 3, 40), with 100 resolution classes, say R_1,..., R_{100}. We leave two classes alone and decompose 98 of the classes: $a * R_i$ $(i = 1..., 34)$, $b * R_i$ $(i = 35, ..., 68)$, and $z * R_i$ $(i = 69, ..., 98)$. One may continue to lower λ_2 by six in similar fashion. We omit the details, but mention that, instead of a BIBD(6, 3, 2), alternate lowerings require one to use 10 one-factors, five with each of a and b. In this way one may obtain a GDD(6, 2, 1; 42, $6t$) with $8 \le t \le 16$.

We have now constructed all the designs in the table above, and GDDs with slightly lower second index, with first index fixed, are easy to obtain from smaller designs or by direct construction. All odd index designs may be obtained from those constructed and BIBD(9, 3, w). We have thus show that:

Theorem 18 *The necessary conditions are sufficient for the existence of GDD* (6, 2, 1; λ_1, λ_2) *with* $\lambda_1 < \lambda_2$.

4.6 Concluding Remarks

The case for $n = 6$ shows that there are complications as n gets larger, and the most severe complications occur for $\lambda_1 < \lambda_2$ since the relevant inequality (Lemma 10, Theorem 4) is quadratic in n. It is thus much harder to get general constructions (like Theorem 6). The other direction, with $\lambda_1 > \lambda_2$, is restricted only linearly with n. It is possible to predict that success in further understanding, for the group vector $(n, 2, 1)$ or other group vectors, will come from restricting considerations to a particular n

or range of values, or by restricting consideration to $\lambda_i < \lambda_j$, for $i = 1$ or 2. The authors are presently persuing both approaches.

We present two quite general theorems, each for a family of n-values, and each illustrating the value of having designs with small parameters available to give access to families of designs with larger parameters.

Theorem 19 *There exists a GDD*$(6t+5, 2, 1; 4, 2)$ *for all* $t \geq 0$.

Use the blocks of a resolvable BIBD$(6t+3, 3, 4)$ based on points $\{1, 2, ..., 6t+3\}$. Make blocks with points $6t+4$ and $6t+5$ using two resolution classes with each. Use the blocks of a GDD$(2, 2, 1; 4, 2)$ with groups $\{6t+4, 6t+5\}$, $\{a, b\}$ and $\{z\}$.

Theorem 20 *There exists a GDD*$(6t+3, 2, 1; 8, 2)$ *for all* $t \geq 0$.

We may assume $t \geq 1$. We make use of the blocks of two copies of a resolvable BIBD$(6t, 3, 4)$ using the points $\{1, 2, ..., 6t\}$. There are $4(6t-1)$ resolution classes, and we decompose 15 of them (at least 20 are available). Use 4 classes with each of points $6t+1$, $6t+2$, and $6t+3$, and use one with each of a, b, and z. Finally, use the blocks of a GDD$(3, 2, 1; 8, 2)$ with groups $\{6t+1,\ 6t+2,\ 6t+3\}$, $\{a, b\}$, and $\{z\}$.

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