On Edge-Balance Index Sets of L-product of Cycles with Stars, Part II

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Abstract

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$, and let $\mathbb{Z}_2 = \{0,1\}$. Any edge labeling f induces a partial vertex labeling $f^+ : V(G) \to \mathbb{Z}_2$ assigning 0 or 1 to $f^+(v)$, v being an element of $V(G)$, depending on whether there are more 0-edges or 1-edges incident with v, and no label is given to $f^+(v)$ otherwise. For each $i \in \mathbb{Z}_2$, let $v_f(i) = |\{v \in V(G) : f^+(v) = i\}|$ and let $e_f(i) = |\{e \in E(G) : f(e) = i\}|$. An edge-labeling f of G is said to be edge-friendly if $|e_f(0) - e_f(1)| \leq 1$. The edge-balance index set of the graph G is defined as $EBI(G) = \{|v_f(0) - v_f(1)| : f \text{ is edge-friendly.}\}$. In this paper, exact values of the edge-balance index sets of L-product of cycles with stars, $C_n \times_L (\mathrm{St}(m), c))$, where m is even, and c is the center of the star graph are presented.

1 Introduction

In [6], Kong and second author considered a new labeling problem of graph theory. Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$, and let $\mathbb{Z}_2 = \{0,1\}$. An edge labeling $f : E(G) \to \mathbb{Z}_2$ induces a vertex partial labeling $f^+ : V(G) \to \mathbb{Z}_2$ defined by $f^+(v) = 0$ if the edges labeled 0 incident on v is more than the number of edges labeled 1 incident on v, and $f^+(v) = 1$ if the edges labeled 1 incident on v is more than the number of edges labeled 0 incident on v. $f^+(v)$ is not defined if the number of edges labeled by 0 is equal to the number of edges labeled 1. For $i \in \mathbb{Z}_2$, let $v_f(i) = |\{v \in V(G) : f^+(v) = i\}|$, and let $e_f(i) = |\{e \in E(G) : f(e) = i\}|$.

With these notations, we now introduce the notion of an edge-balanced graph.

Definition 1. An edge labeling f of a graph G is said to be ***edge-friendly*** if $|e_f(0) - e_f(1)| \leq 1$. A graph G is said to be an ***edge-balanced*** graph if there is an edge-friendly labeling f of G satisfying $|v_f(0) - v_f(1)| \leq 1$.

Chen, Lee, et al in [1] proved that all connected simple graphs except the star $K_{1,2k+1}$, where $k \geq 0$ are edge-balanced.

Definition 2. The ***edge-balance index set*** of the graph G, EBI(G), is defined as $\{|v_f(0) - v_f(1)| :$ the edge labeling f is edge-friendly.$\}$.

We will use $v(0)$, $v(1)$, $e(0)$, $e(1)$ instead of $v_f(0)$, $v_f(1)$, $e_f(0)$, $e_f(1)$, provided there is no ambiguity.

Example 1. EBI(nK_2) is $\{0\}$ if n is even and $\{2\}$ if n is odd.

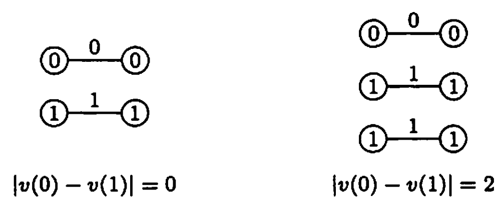

Figure 1: The edge-balance index set of $2K_2$ and $3K_2$

For any $n \geq 1$, we denote the tree with $n+1$ vertices of diameter two by St(n). The star has a center c and n appended edges from c.

Example 2. The edge-balance index set of the star St(n) is

$$\text{EBI}(\text{St}(n)) = \begin{cases} \{0\} & \text{if } n \text{ is even,} \\ \{2\} & \text{if } n \text{ is odd.} \end{cases}$$

Example 3. In [14], Lee, Lo and Tao showed that

$$\text{EBI}(P_n) = \begin{cases} \{2\} & \text{if } n \text{ is } 2, \\ \{0\} & \text{if } n \text{ is } 3, \\ \{1,2\} & \text{if } n \text{ is } 4, \\ \{0,1\} & \text{if } n \text{ is odd and greater than } 3, \\ \{0,1,2\} & \text{if } n \text{ is even and greater than } 4. \end{cases}$$

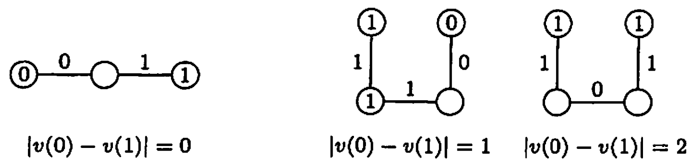

Figure 2: The edge-balance index set of P_3 and P_4

Figure 2 shows the EBI of P_3 and P_4.

Example 4. Figure 3 shows that the edge-balance index set of a tree with six vertices is $\{0, 1, 2\}$.

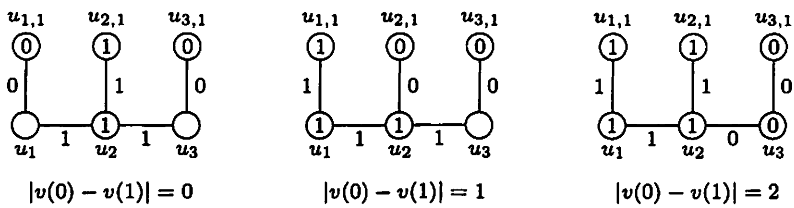

Figure 3: The edge-balance index set of a tree with six vertices

The edge-balance index sets can be viewed as the dual of balance index sets. The balance index sets of graphs were considered in [7, 9, 10, 11, 12, 13, 15, 17, 18]. Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$, and let $\mathbb{Z}_2 = \{0, 1\}$. A labeling $f : V(G) \to \mathbb{Z}_2$ induces an edge partial labeling $f^* : E(G) \to A$ defined by $f^*(vw) = f(v)$, if and only if $f(v) = f(w)$ for each edge $vw \in E(G)$. For $i \in \mathbb{Z}_2$, let $v_f(i) = \text{card}\{v \in V(G) : f(v) = i\}$ and $e_{f^*}(i) = \text{card}\{e \in E(G) : f^*(e) = i\}$. A labeling f of a graph G is said to be ***friendly*** if $|v_f(0) - v_f(1)| \leq 1$. If $|e_f(0) - e_f(1)| \leq 1$ then G is said to be ***balanced***. The ***balance index set*** of the graph G, $\text{BI}(G)$, is defined as $\{|e_f(0) - e_f(1)| : \text{the vertex labeling } f \text{ is friendly}\}$.

Edge-balance index sets of trees, flower graphs and $(p, p+1)$-graphs were considered in [8, 14, 16].

Let H be a connected graph with a distinguished vertex s. Construct a new graph $G \times_L (H, s)$ as follows: take $|V(G)|$ copies of (H, s) and identify each vertex of G with s of a single copy of H. We call the resulting graph the ***L-product*** of G and (H, s). In [4], Chou et al. investigated the edge-balance index sets of L-product of cycles with stars, $C_n \times_L (\text{St}(m), c))$, where c when m is odd. In this paper, exact values of the edge-balance index sets of L-product of cycles with stars, $C_n \times_L (\text{St}(m), c))$, where m is even, and c is the center of the star graph are presented.

2 Summary of the Results from Part I

In the previous paper [4] of the series, we partially determined the edge-balance sets of $C_n \times_L (\mathrm{St}(m), c))$ where m is odd or 2. We recall the results here:

Proposition 2.1. *The edge-balance index set of $C_n \times_L (St(1), c))$ is $\{0, 2, \cdots, 2k\}$ for $n = 3k$, $3k+1$ and $3k+2$.*

Proposition 2.2. *The edge-balance index set of $C_n \times_L (St(2), c))$ is*

1. $\{1, 2, \cdots, 3k\}$ *if* $n = 4k$;
2. $\{2, 3, \cdots, 3k+2\}$ *if* $n = 4k+1$;
3. $\{1, 2, \cdots, 3k+1\}$ *if* $n = 4k+2$;
4. $\{2, 3, \cdots, 3k+3\}$ *if* $n = 4k+3$.

Proposition 2.3. *Consider $C_n \times_L (St(m), c)$ where m is odd and greater or equal to three. Let $n = mk + r$ where $0 \leq r < m$ and $r = 2t$ if r is even or $r = 2t+1$ if r is odd. We also define k_0, k_1, r_0, r_1 such that $\frac{n(m+1)}{2} = mk_0 + r_0$ and $\frac{n(m-1)}{2} = mk_1 + r_1$ where $0 \leq r_0, r_1 < m$. Let $T = r_0 - r_1 - 2$. The edge-balance index set of $C_n \times_L (St(m), c)$ is*

1. $\{0, 2, 4, 6, \cdots, n+k\}$ *if* $r = 0$;
2. $\{0, 2, 4, 6, \cdots, n+k\}$ *if* $T < 0$ *and* r *is even;*
3. $\{0, 2, 4, 6, \cdots, n+k-1\}$ *if* $T < 0$ *and* r *is odd;*
4. $\{0, 2, 4, 6, \cdots, n+k+2\}$ *if* $T > 0$ *and* r *is even;*
5. $\{0, 2, 4, 6, \cdots, n+k+1\}$ *if* $T > 0$ *and* r *is odd.*

3 The Highest Edge-balance Index

Starting from this section, we use the notation $C_n \times_L \mathrm{St}(m)$ instead of $C_n \times_L (\mathrm{St}(m), c))$ when the context is clear.

In a $C_n \times_L \mathrm{St}(m)$ graph where m is *even*, and m is greater than 2, we notice that all of the vertices are grouped into n packages, where a package consists of m degree 1 vertices and 1 degree $m+2$ vertex. For convenience, we name those degree 1 vertices star vertices and degree $m+2$ vertices cycle vertices. We also name the edges within C_n cycle edges and the edge incident to a star vertex star edge.

Figure 4: Packages

With an edge-friendly labeling, every degree 1 vertex must be labeled the same as its incident edge. However, in $C_n \times_L \mathrm{St}(m)$, cycles vertices in each package can be either labeled or unlabeled, because m being even implies that $m+2$ is even, so it is possible to have the same amount of 0-edges as 1-edges.

In order to create the highest edge-balance index for $C_n \times_L \mathrm{St}(m)$, one can try to maximize $v(0)$ and minimize $v(1)$ at the same time. Since all star vertices must be labeled, they can contribute to $v(0)$ very efficiently by using only one 0-edge. At the same time, cycle vertices require several 1-edges to become labeled 1. Therefore, we can label all cycle edges 1 and then label the remaining 0 and 1's into star edges to generate the highest edge-balance index.

Due to the symmetry of the roles of 0 and 1 in an edge-balance index, we can assume that $e(0) \geq e(1)$ without loss of generality.

We note here that, for a $C_n \times_L \mathrm{St}(m)$ graph, there are $n(m+1)$ vertices and $n(m+1)$ edges.

To generate the highest edge-balance index, we need to label the cycle vertices 0 as many as possible using as few 0-edges as possible. In each package, we need at least $\frac{m+2}{2}+1$ 0-edges to label its cycle vertex 0. By using the division algorithm, we have

$$e(0) = \left(\frac{m+2}{2}+1\right)q + r, \tag{1}$$

where $0 \leq r < \frac{m+2}{2}+1$. Here q represents the number of packages with its cycle vertex labeled 0 and r is the number of 0-edges in a package with its cycle vertex not labeled 0. Therefore, under our labeling strategy, we have $e(0)+q$ vertices labeled 0.

Since our label all the cycle edges 1, we have $e(1)-n$ 1-edges left for labeling on star edges. By Equation 1, we have $n-q$ cycle vertices not labeled 0. Notice here that since $0 \leq r < \frac{m+2}{2}+1$, it is possible for r being $\frac{m+2}{2}$ to create at most one unlabeled cycle vertex. Thus, the number of vertices labeled 1 is

$$(e(1)-n)+(n-q)-1+U = e(1)-q-1+U$$

where U equals 0 or 1 depends on whether we have a unlabeled vertex or not, respectively. Note that the only situation to have an unlabeled vertex is when $r = \frac{m+2}{2}$.

We summarize the above discussion as:

Lemma 3.1. *In a $C_n \times_L St(m)$ graph where m is even and m is greater than 2, let*

$$e(0) = \left(\frac{m+2}{2} + 1\right) q + r$$

where $0 \le r < \frac{m+2}{2} + 1$. An edge-friendly labeling of $C_n \times_L St(m)$ generates

$$v(0) = e(0) + q$$

and

$$v(1) = e(1) - q - 1 + U$$

where

$$U = \begin{cases} 0 & , \text{ if } r = \frac{m+2}{2}; \\ 1 & , \text{ otherwise.} \end{cases}$$

Now, we are in a position to find the highest edge-balance index of an edge fridleny labeling of $C_n \times_L \mathrm{St}(m)$.

Theorem 3.2. *In a $C_n \times_L St(m)$ graph where m is even and m is greater than 2, the highest edge balance index is*

$$\begin{cases} 2q+2 & , \text{ if } n \text{ is odd and } (m+4) \mid (3-3n); \\ 2q+1 & , \text{ if } n \text{ is odd and } (m+4) \nmid (3-3n); \\ 2q+1 & , \text{ if } n \text{ is even and } (m+4) \mid (2-3n); \\ 2q & , \text{ if } n \text{ is even and } (m+4) \nmid (2-3n). \end{cases}$$

Proof. When n is odd, an edge-friendly labeling makes $e(0) = \frac{n(m+1)+1}{2}$ and $e(1) = \frac{n(m+1)-1}{2}$. By Lemma 3.1, we have

$$v(0) = \frac{n(m+1)+1}{2} + q$$

and

$$v(1) = \frac{n(m+1)-1}{2} - q - 1 + U.$$

Hence,

$$\begin{aligned} v(0) - v(1) &= \left(\frac{n(m+1)+1}{2} + q\right) - \left(\frac{n(m+1)-1}{2} - q - 1 + U\right) \\ &= 2q + 2 - U. \end{aligned}$$

To determine U, we look at when we have $r = \frac{m+2}{2}$. Now, Equation 1 becomes

$$\frac{n(m+1)+1}{2} = \left(\frac{m+2}{2}+1\right)q + \frac{m+2}{2}.$$

By solving the equation for q, we have

$$q = \frac{(n(m+1)+1)-(m+2)}{m+4} = (n-1) + \frac{3-3n}{m+4}.$$

Since q is an integer, we need $(m+4) \mid (3-3n)$.

Therefore, the highest edge-balance index is $2q+2$ when $(m+4) \mid (3-3n)$ or $2q+1$ when $(m+4) \nmid (3-3n)$.

Similarly, when n is even, an edge-friendly labeling makes $e(0) = e(1) = \frac{n(m+1)}{2}$. By Lemma 3.1, we have

$$v(0) = \frac{n(m+1)}{2} + q$$

and

$$v(1) = \frac{n(m+1)}{2} - q - 1 + U.$$

Hence,

$$\begin{aligned} v(0) - v(1) &= \left(\frac{n(m+1)}{2} + q\right) - \left(\frac{n(m+1)}{2} - q - 1 + U\right) \\ &= 2q + 1 - U. \end{aligned}$$

To determine U, we look at when we have $r = \frac{m+2}{2}$. Now, the Equation 1 becomes

$$\frac{n(m+1)}{2} = \left(\frac{m+2}{2}+1\right)q + \frac{m+2}{2}.$$

By solving the equation for q, we have

$$q = \frac{(n(m+1))-(m+2)}{m+4} = (n-1) + \frac{2-3n}{m+4}.$$

Since q is an integer, we need $(2-3n) \mid (m+4)$.

Therefore, the highest edge-balance index is $2q+1$ when $(m+4) \mid (2-3n)$ or $2q$ when $(m+4) \nmid (2-3n)$. □

4 The Edge-balance Index Set of $C_n \times_L \mathrm{St}(m)$ When m is Even

In order to use the edge-friendly labeling with the highest edge-balance index to generate more lower edge-balance indexes, we need to arrange it in a way that we can easily describe it.

First we name the cycle vertices by $v_1, v_2, \ldots, v_n$. Then, for $i = 1, 2, \ldots, n$, we name the cycle edge (v_i, v_{i+1}) by e_i. Note that, for convenience, we modulo n for the subscript of v's and e's. Also, we name a package P_i if it contains the cycle vertex v_i.

Our labeling strategy for generating the highest edge-balance index puts all 0-edges on star edges. Thus, we label $\frac{m+2}{2}+1$ 0-edges in the package P_i for $i = 1, 2, \ldots, q$ and the remaining r 0-edges in the package P_{q+1}. Then, assign 1 to all other edges.

We call a package with $\frac{m+2}{2}+1$ 0-edges a 0-package. Therefore, the packages, $P_1, P_2, \ldots, P_q$ are 0-packages. We also call the package P_{q+1} the partial package and the remaining $P_{q+2}, \ldots, P_n$ packages 1-packages. Note that, in an 1-package, every edge is labeled 1.

For a 0-package P_i, where $i = 1, 2, \ldots, q$, if we switch the label of e_{i-1} with the label of a 0-star edge, we can see that the label of v_i remains unchanged since the number of 0-edges and 1-edges are not altered. But, this switch turns a 0-star edge into an 1-star edge. Thus, the edge-balance index is reduced by 2. For $i = 2, 3, \ldots, q$, this switch makes the vertex v_{i-1} gain one 0-edge and lose one 1-edge. Since v_{i+1} was labeled 0 before switching, the label of v_{i-1} is unchanged. For $i = 1$, since P_n is an 1-package, gaining one 0-edge and losing one 1-edge does not alter the label of v_n.

Each switch reduces the edge-balance index by 2, and we can perform the switching q times. This leads to

Lemma 4.1. *The edge-balance index set of $C_n \times_L St(m)$ contains*

$$\begin{cases} \{2,4,6,\ldots,2q,2q+2\} & \text{, if } n \text{ is odd and } (m+4) \mid (3-3n); \\ \{1,3,5,\ldots,2q-1,2q+1\} & \text{, if } n \text{ is odd and } (m+4) \nmid (3-3n); \\ \{1,3,5,\ldots,2q-1,2q+1\} & \text{, if } n \text{ is even and } (m+4) \mid (2-3n); \\ \{0,2,4,\ldots,2q-2,2q\} & \text{, if } n \text{ is even and } (m+4) \nmid (2-3n). \end{cases}$$

Let us get back to the edge-friendly labeling with the highest edge-balance index. If we switch one 1-star edge on P_n with a 0-star edge on P_1, then we can see that we have the same number of 0- and 1-star edges. Also, the label of v_n remains unchanged because P_n is an 1-package. But, the label of v_1 becomes unlabeled because gaining one 1-edge and losing one

0-edge makes the number of 0-edges and 1-edges of v_1 both equal to $\frac{m+2}{2}$. Thus, we construct a edge-friendly labeling with the edge-balance index one less than the highest edge-balance index. Note that, in this labeling, we have only $q-1$ 0-packages left.

Similarly, for a 0-package P_i, where $i = 2, \ldots, q$, if we switch the label of e_i with the label of a 0-star edge on P_i, we can see that the label of v_i remains unchanged since the number of 0-edges and 1-edges are not altered. But, this switch turns a 0-star edge into an 1-star edge. Thus, the edge-balance index is reduced by 2. For $i = 2, 3, \ldots, q-1$, this switch makes the vertex v_{i+1} to gain one 0-edge and to lose one 1-edge. Since v_{i+1} is labeled 0 before switching, the label of v_{i+1} remains unchanged.

Each switch, starting from one less than the highest edge-balance index, reduces the edge-balance index by 2, and we can perform the switching $q-2$ times. In addition to this, if we switch the label of e_q, which is labeled 1, with a 0-star edge on P_{q+1}, we can see that the label of v_{q+1} remains unchanged since the number of 0-edges and 1-edges are not altered. But, this switch turns a 0-star edge into an 1-star edge. Thus, the edge-balance index is reduced by 2. This switch makes the vertex v_q to gain one 0-edge and to lose one 1-edge. Since v_q was labeled 0 before switching, the label of v_q remains unchanged. All together, we have

Lemma 4.2. *The edge-balance index set of $C_n \times_L St(m)$ contains*

$$\begin{cases} \{3,5,7,\ldots,2q-1,2q+1\} & \text{, if } n \text{ is odd and } (m+4) \mid (3-3n); \\ \{2,4,6,\ldots,2q-2,2q\} & \text{, if } n \text{ is odd and } (m+4) \nmid (3-3n); \\ \{2,4,6,\ldots,2q-2,2q\} & \text{, if } n \text{ is even and } (m+4) \mid (2-3n); \\ \{1,3,5,\ldots,2q-3,2q-1\} & \text{, if } n \text{ is even and } (m+4) \nmid (2-3n). \end{cases}$$

Now, we can put everything together to obtain the main result.

Theorem 4.3. *The edge-balance index set of $C_n \times_L St(m)$ is*

$$\begin{cases} \{0,1,2,\ldots,2q+1\} & \text{, if } n \text{ is odd and } (m+4) \mid (3-3n); \\ \{0,1,2,\ldots,2q\} & \text{, if } n \text{ is odd and } (m+4) \nmid (3-3n); \\ \{0,1,2,\ldots,2q\} & \text{, if } n \text{ is even and } (m+4) \mid (2-3n); \\ \{0,1,2,\ldots,2q-1\} & \text{, if } n \text{ is even and } (m+4) \nmid (2-3n). \end{cases}$$

Proof. By Lemma 4.1 and Lemma 4.2, we have

$$\begin{cases} \{2,3,4,\ldots,2q+1\} & \text{, if } n \text{ is odd and } (m+4) \mid (3-3n); \\ \{1,2,3,\ldots,2q\} & \text{, if } n \text{ is odd and } (m+4) \nmid (3-3n); \\ \{1,2,3,\ldots,2q\} & \text{, if } n \text{ is even and } (m+4) \mid (2-3n); \\ \{0,1,2,\ldots,2q-1\} & \text{, if } n \text{ is even and } (m+4) \nmid (2-3n). \end{cases}$$

To get the remaining edge-balance indices, we construct a "balanced" edge-friendly labeling. Since m is even, label all m star edges in a package by $\frac{m}{2}$ 0s and $\frac{m}{2}$ 1s. Then, label the cycle edges by $0, 1, 0, 1, \ldots$, alternately, starting from v_1. Obviously, the edge-balance index of this edge-friendly labeling is 0 if n is even and 1 is n is odd.

When n is odd, from the "balanced" edge-friendly labeling, by switching the label of e_2 with the label of a 0-star edge on P_2, we can see that the label of v_2 remains unchanged since the number of 0-edges and 1-edges are not altered. But, this switch turns a 0-star edge into an 1-star edge. Thus, the edge-balance index is reduced by 2. At the same time, the vertex v_3 gains one 0-edge and lose one 1-edge. Since v_3 was unlabeled before switching, the label of v_3 becomes labeled 0. This increases the edge-balance index by 1. Therefore, since the "balanced" edge-friendly labeling has the edge-balance index 1, after switching, the edge-balance index becomes 0.

This completes the proof. □

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