On Color-Connected Graphs

Ryan Jones, Kyle Kolasinski, Chira Lumduanhom and Ping Zhang
Department of Mathematics, Western Michigan University
Kalamazoo, MI 49008-5248 USA

Abstract

For a connected graph G and a positive integer k, the kth power G^k of G is the graph with $V(G^k) = V(G)$ where $uv \in E(G^k)$ if the distance $d_G(u, v)$ between u and v is at most k. The edge coloring of G^k defined by assigning each edge uv of G^k the color $d_G(u, v)$ produces an edge-colored graph G^k called a distance-colored graph. A distance-colored graph is properly p-connected if every two distinct vertices u and v in the graph are connected by p internally disjoint properly colored $u - v$ paths. It is shown that G^2 is properly 2-connected for every 2-connected graph that is not complete, a double star is the only tree T for which T^2 is properly 2-connected and G^3 is properly 2-connected for every connected graph G of diameter at least 3. All pairs k, n of positive integers for which P_n^k is properly k-connected are determined. It is shown that every properly colored graph H with $\chi'(H)$ colors is a subgraph of some distance-colored graph and the question of determining the smallest order of such a graph is studied.

1 Introduction

For a connected graph G and a positive integer k, the *kth power* G^k of G is the graph with $V(G^k) = V(G)$ where $uv \in E(G^k)$ if the distance $d_G(u, v)$ between u and v (the length of a shortest $u - v$ path in G) is at most k. The graph G^2 is called the *square* of G and G^3 is the *cube* of G. Among the best known results concerning powers of graphs are those concerning Hamiltonian properties. In particular, Sekanina [7] proved that the cube of every connected graph is Hamiltonian-connected and is therefore Hamiltonian (if its order is at least 3). Hence for every connected graph G of order $n \geq 3$, there is a Hamiltonian cycle $C = (v_1, v_2, \ldots, v_n, v_{n+1} = v_1)$ of G^3, which implies that $d_G(v_i, v_{i+1}) \in \{1, 2, 3\}$ for $1 \leq i \leq n$. While the square of every connected graph need not be Hamiltonian, Fleischner [4] showed that the square of every 2-connected graph is Hamiltonian.

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For a connected graph G of order n and diameter d and integers i and k with $1 \leq i \leq k \leq d$, the color i is assigned to an edge uv in G^k if $d_G(u, v) = i$. The resulting edge-colored graph G^k is called a *distance-colored graph*. If a distance-colored graph G^k contains a properly colored subgraph isomorphic to a graph H, then G^k is called *H-colored*. Certainly, it is only possible for G^k to be H-colored if $\chi'(H) \leq k$, where $\chi'(H)$ is the chromatic index of H. If $H = C_n$, then G^k is called a *Hamiltonian-colored graph*. The *Hamiltonian coloring exponent* hce(G) of G is defined as the minimum k for which G^k is Hamiltonian-colored. These concepts were introduced and studied in [1] and studied further in [2, 5, 6].

For a connected graph G and a positive integer k, the distance-colored graph G^k is called *properly color-connected*, or simply *color-connected*, if every two vertices u and v in G^k are connected by a properly colored $u - v$ path in G^k. If the diameter of G is d, then G^d is complete and so every two vertices u and v in G^d are connected by the path (u, v) of length 1 in G^d. Thus G^d is color-connected. The smallest integer k for which G^k is color-connected is called the *color-connection exponent* cce(G) of G. In fact, this number is 2 for all non-complete connected graphs (see [6]).

Theorem 1.1 *If G is a non-complete connected graph, then* cce$(G) = 2$.

By Theorem 1.1, the distance-colored graph G^2 is colored-connected and so for every two vertices u and v of G^2, there is at least one properly colored $u - v$ path in G^2. This gives rise to another concept. If G is a connected graph with connectivity $\kappa(G) = \kappa$, then it follows from a well-known theorem of Whitney [8] that for every two distinct vertices u and v of G, the graph G contains κ internally disjoint $u - v$ paths. A graph G is *p-connected* if $\kappa(G) \geq p$. For a connected graph G and an integer $k \geq 2$, the distance-colored graph G^k is *properly colored p-connected* or simply *properly p-connected* if for every two distinct vertices u and v of G^k, there are p internally disjoint properly colored $u - v$ paths in G^k. If G is a complete graph, then G^k is not properly p-connected for all integers k, p with $k \geq 1$ and $p \geq 2$. Thus we consider only non-complete connected graphs. For a connected graph G and an integer $k \geq 2$, the *color-connectivity* of G^k is the maximum positive integer p for which G^k is properly p-connected.

In Section 2, we study the color-connectivities of the square and the cube of a connected graph, while in Section 3 we determine all pairs k, n of positive integers for which P_n^k is properly k-connected. In Section 4, we show that every properly colored graph H with $\chi'(H)$ colors is a subgraph of some distance-colored graph and study the question of determining the smallest order of such a graph. We refer to the book [3] for graph theory notation and terminology not described in this paper. We assume that all graphs under consideration are connected.

2 Properly 2-Connected Graphs

For a connected graph G of order 3 or more, it is well known that G^2 is 2-connected and so every two distinct vertices u and v of G are connected by two internally disjoint $u-v$ paths in G^2. By Theorem 1.1, the distance-colored graph G^2 is colored-connected. However, G^2 need not be properly 2-connected, that is, it is possible that for some pair u, v of vertices of G^2, two internally disjoint properly colored $u-v$ paths do not exist in G^2. For example, if $G = K_{1,n-1}$ where $n \geq 3$, then $G^2 = K_n$ is 2-connected but not properly 2-connected (as any two end-vertices of G are not connected by two internally disjoint properly colored paths in G^2). On the other hand, if G is 2-connected, then G^2 is properly 2-connected. In order to show this, we first present a lemma, which was established in [1].

Lemma 2.1 *For each integer $n \geq 3$, the distance-colored graph P_n^2 contains a properly colored Hamiltonian path.*

Theorem 2.2 *If G is a 2-connected graph that is not complete, then G^2 is properly 2-connected.*

Proof. Let u and v be two distinct vertices of G. Since G is 2-connected, G contains at least two internally disjoint $u-v$ paths. Among all internally disjoint $u-v$ paths in G, let P and P' be two of smallest possible lengths. Let $P = (u = x_1, x_2, \ldots, x_s = v)$ and $P' = (u = y_1, y_2, \ldots, y_t = v)$. By the defining property of P and P', it follows that $d_P(x_i, x_{i+2}) = 2 = d_G(x_i, x_{i+2})$ for $1 \leq i \leq s-2$ and $d_{P'}(y_j, y_{j+2}) = 2 = d_G(y_j, y_{j+2})$ for $1 \leq j \leq t-2$. By Lemma 2.1, there is a properly colored $u-v$ path Q in the square of P and a properly colored $u-v$ path Q' in the square of P'. Since P and P' are internally disjoint, so are Q and Q'. Therefore, G^2 is properly 2-connected. ∎

For a connected graph G, its square can be properly 2-connected without G being 2-connected, however. In the case of trees, for example, we know precisely those trees whose square is properly 2-connected. A *double star* is a tree of diameter 3. The double stars are the only trees T for which T^2 is properly 2-connected.

Theorem 2.3 *Let T be a tree of order at least 3. Then T^2 is properly 2-connected if and only if T is a double star.*

Proof. First, suppose that T is a double star whose central vertices are x and y. Let $X = \{x_1, x_2, \ldots, x_r\}$ and $Y = \{y_1, y_2, \ldots, y_s\}$ be the sets of end-vertices of T such that x is adjacent to every vertex in X and y is adjacent to every vertex in Y, where then $r, s \geq 1$. Let $u, v \in V(T)$. We show that u and v are connected by two internally disjoint properly colored paths. If

$\{u,v\} = \{x,y\}$, say $u = x$ and $v = y$, then (u,v) and (u,x_1,v) are two internally disjoint properly colored $u-v$ paths in T^2. If $\{u,v\}\cap\{x,y\} = \emptyset$, then we may assume that $u \in X$. If $v \in X$, say $v = x_2$, then (u,v) and (u,x,y_1,y,v) are two internally disjoint properly colored $u-v$ paths in T^2. If $v \in Y$, say $v = y_1$, then (u,y,v) and (u,x,v) are two internally disjoint properly colored $u-v$ paths in T^2. If $|\{u,v\}\cap\{x,y\}| = 1$, then we may assume that $u = x$ and $v \neq y$. If $v \in X$, say $v = x_1$, then (u,v) and (u,y,v) are two internally disjoint properly colored $u-v$ paths in T^2. If $v \in Y$, say $v = y_1$, then (u,v) and (u,x_1,y,v) are two internally disjoint properly colored $u-v$ paths in T^2.

For the converse, assume that T is not a double star. Let $d = \text{diam}(T)$ and so $d \neq 3$. If $d = 2$, then T is a star and we saw that T^2 is not properly 2-connected. Thus we may assume that $d \geq 4$. Let u be an end-vertex of T whose eccentricity $e_T(u)$ is d and let v be a vertex of T such that $d_T(u,v) = 4$. We show that there are no two internally disjoint properly colored $u-v$ paths in T^2. Suppose that (u,v_1,v_2,v_3,v) is the $u-v$ path in T. Assume, to the contrary, that T^2 contains two internally disjoint properly colored $u-v$ paths P_1 and P_2. By an extensive case-by-case analysis, it can be shown that each properly colored $u-v$ path in T^2 must contain at least two vertices in $\{v_1,v_2,v_3\}$. This implies that each of P_1 and P_2 must contain at least two vertices from $\{v_1,v_2,v_3\}$, which is impossible. ∎

We have seen that if G is a connected graph, then G^2 is 2-connected. Since $(G^2)^2 = G^4$ for each connected graph G, the following is an immediate consequence of Theorem 2.2.

Corollary 2.4 *If G is a connected graph such that G^2 is not complete, then G^4 is properly 2-connected.*

By Corollary 2.4, if G is a connected graph of diameter at least 3, then G^4 is properly 2-connected. This gives rise to a natural question: For a connected graph G of diameter at least 3, what is the minimum k such that G^k is properly 2-connected? By Theorem 2.2 and Corollary 2.4, either $k = 3$ or $k = 4$. Next, we show that $k = 3$.

Theorem 2.5 *If G is a connected graph of diameter at least 3, then G^3 is properly 2-connected.*

Proof. Since the $\text{diam}(G) \geq 3$, it follows that the order of G is at least 4. Let u and v be two distinct vertices of G. We show that u and v are connected by two internally disjoint properly colored paths in G^3. We consider two cases, according to $1 \leq d_G(u,v) \leq 2$ or $d_G(u,v) \geq 3$.

Case 1. $1 \leq d_G(u,v) \leq 2$. If $N_G(u) \neq N_G(v)$, say $x \in N_G(u) - N_G(v)$, then (u,v) and (u,x,v) are two internally disjoint properly colored

$u - v$ paths in G^3. Thus, we may assume that $N_G(u) = N_G(v)$. Since $\text{diam}(G) \geq 3$, it follows that $\text{rad}(G) \geq 2$ and $e_G(y) \geq 2$ for all $y \in V(G)$. If $e_G(u) \geq 3$ or $e_G(v) \geq 3$, say the former, then there is $w \in V(G)$ such that $d_G(u, w) = 3$. Let (u, x_1, x_2, w) be a $u - w$ geodesic in G. Since $N_G(u) = N_G(v)$, it follows that v is adjacent to x_1 (and v is not adjacent to x_2). Then (u, v) and (u, w, x_1, v) are two internally disjoint properly colored $u - v$ paths in G^3. Thus we may assume that $e_G(u) = e_G(v) = 2$. Then $\text{rad}(G) = 2$ and $3 \leq \text{diam}(G) \leq 4$. Let w and w' be vertices with $d(w, w') = 3$. Clearly, u, v, w, w' are distinct. Then (u, v) and (u, w, w', v) are two internally disjoint properly colored $u - v$ paths in G^3.

Case 2. $d_G(u, v) \geq 3$. Let $d = d_G(u, v)$ and let $P = (u = v_0, v_1, v_2, \ldots, v_d = v)$ be a $u - v$ geodesic in G. Thus $d_P(x, y) = d_G(x, y)$ for all $x, y \in V(P)$. If $d = 3$, then (u, v) and (u, v_2, v) are two internally disjoint properly colored $u - v$ paths in G^3. If $d = 4$, then (u, v_1, v) and (u, v_3, v) are two internally disjoint properly colored $u - v$ paths in G^3. Thus, we may assume that $d \geq 5$. Suppose that $d \equiv i \pmod 4$ where $i = 0, 1, 2, 3$ and let $d = 4k + i$ for some positive integer k. In each case, G^3 contains two internally disjoint properly colored $u - v$ paths Q_1 and Q_2 as follows: For $d = 4k$,

$$\begin{aligned} Q_1 &= (u, v_1, v_4, v_5, v_8, v_9, \ldots, v_{4k-3}, v_{4k}) \\ Q_2 &= (u, v_2, v_3, v_6, v_7, v_{10}, \ldots, v_{4k-2}, v_{4k}). \end{aligned}$$

For $d = 4k + 1$,

$$\begin{aligned} Q_1 &= (u, v_1, v_4, v_5, v_8, v_9, \ldots, v_{4k}, v_{4k+1}) \\ Q_2 &= (u, v_2, v_3, v_6, v_7, v_{10}, \ldots, v_{4k-1}, v_{4k+1}). \end{aligned}$$

For $d = 4k + 2$,

$$\begin{aligned} Q_1 &= (u, v_1, v_4, v_5, v_8, v_9, \ldots, v_{4k}, v_{4k+2}) \\ Q_2 &= (u, v_2, v_3, v_6, v_7, v_{10}, \ldots, v_{4k-1}, v_{4k+2}). \end{aligned}$$

For $d = 4k + 3$,

$$\begin{aligned} Q_1 &= (u, v_1, v_4, v_5, v_8, v_9, \ldots, v_{4k+1}, v_{4k+3}) \\ Q_2 &= (u, v_2, v_3, v_6, v_7, v_{10}, \ldots, v_{4k+2}, v_{4k+3}). \end{aligned}$$

Therefore, G^3 is properly 2-connected. ■

Theorem 2.5 brings up another question: For a connected graph G of diameter at least 3, is G^3 is properly 3-connected? We will see in the next section that this is not true in general.

3 Color-Connectivities of Powers of a Path

We have seen in the proofs of Theorems 2.2 and 2.5 that the color-connectivity of the distance-colored graph P_n^k of a path P_n of order n plays an important role in determining the color-connectivity of a connected graph. Thus in this section, we investigate the color-connectivities of the distance-colored graph P_n^k for integers n and k with $n \geq k+1 \geq 3$. By Theorem 2.3, the distance-colored graph P_n^2 is properly 2-connected if and only if $n = 4$. By Theorem 2.5, P_n^3 is properly 2-connected for all $n \geq 4$. However, P_n^3 is not properly 3-connected in general. In fact, for $n \geq 4$, it can be verified that the distance-colored graph P_n^3 is properly 3-connected if and only if $n = 6$. Next, we determine all pairs k, n of integers with $n \geq k+1 \geq 4$ for which P_n^k is properly k-connected. In order to do this, we first present two lemmas.

Lemma 3.1 *For each even integer $k \geq 2$, the distance-colored graph P_{k+2}^k is properly k-connected.*

Proof. We proceed by induction on even integers $k \geq 2$. By Theorem 2.3, P_4^2 is properly 2-connected and so the statement holds for $k = 2$. Suppose that P_{k+2}^k is properly k-connected for some even integer $k \geq 2$. Let $P_{k+4} = (v_0, v_1, v_2, \ldots, v_{k+3})$ and let u and v be two distinct vertices of P_{k+4}. We show that there are $k+2$ internally disjoint properly colored $u-v$ paths in P_{k+4}^{k+2}. We consider two cases.

Case 1. *u and v are not end-vertices of G.* Then $u, v \in \{v_1, v_2, \ldots, v_{k+2}\}$. Let $P_{k+2} = (v_1, v_2, \ldots, v_{k+2})$. Since P_{k+2}^k is properly k-connected by the induction hypothesis, there are k internally disjoint properly colored $u-v$ paths $Q_1, Q_2, \ldots, Q_k$ in P_{k+2}^k. Let $Q_{k+1} = (u, v_0, v)$ and $Q_{k+2} = (u, v_{k+3}, v)$. Therefore, $Q_1, Q_2, \ldots, Q_{k+2}$ are $k+2$ internally disjoint properly colored $u-v$ paths in P_{k+4}^{k+2}.

Case 2. *At least one of u and v is an end-vertex of G, say $u = v_0$.* If $v = v_{k+3}$, then let $Q_i = (v_0, v_i, v_{k+3})$ for $1 \leq i \leq k+2$, producing $k+2$ internally disjoint properly colored $v_0 - v_{k+3}$ paths in P_{k+4}^{k+2}. Thus, we may assume that $v \neq v_{k+3}$. If $v = v_{2\ell+1}$ for some nonnegative integer ℓ, where $1 \leq 2\ell+1 \leq k+1$, then construct $k+2$ internally disjoint properly colored $v_0 - v_{2\ell+1}$ paths $Q_1, Q_2, \ldots, Q_{k+2}$ in P_{k+4}^{k+2} as follows:

$$Q_i = \begin{cases} (v_0, v_i, v_{2\ell+1}) & \text{if } 1 \leq i \leq 2\ell \\ (v_0, v_{2\ell+1}) & \text{if } i = 2\ell+1 \\ (v_0, v_i, v_{2\ell+1}) & \text{if } 2\ell+2 \leq i \leq k+2. \end{cases}$$

If $v = v_{2\ell}$ for some positive integer ℓ where $2 \leq 2\ell \leq k+2$, then construct $k+2$ internally disjoint properly colored $v_0 - v_{2\ell}$ paths $Q_1, Q_2, \ldots, Q_{k+2}$

in P^{k+2}_{k+4} as follows:

$$Q_i = \begin{cases} (v_0, v_i, v_{2\ell}) & \text{if } 1 \le i \le k+2 \text{ and } i \ne \ell, 2\ell \\ (v_0, v_{2\ell}) & \text{if } i = 2\ell \\ (v_0, v_\ell, v_{k+3}, v_{2\ell}) & \text{if } i = \ell. \end{cases}$$

Therefore, P^{k+2}_{k+4} is properly $(k+2)$-connected. ∎

Lemma 3.2 *For each odd integer $k \ge 3$, the distance-colored graph P^k_{k+3} is properly k-connected.*

Proof. We proceed by induction on odd integers $k \ge 3$. Figure 1 shows that P^3_6 is properly 3-connected.

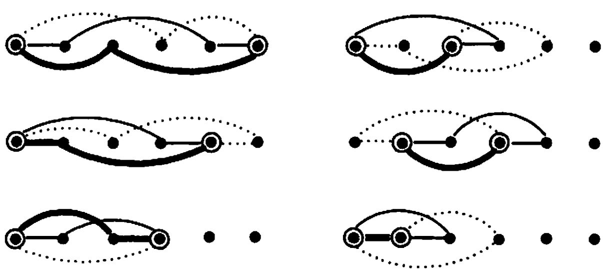

Figure 1: Showing that P^3_6 is properly 3-connected

Assume that P^k_{k+3} is properly k-connected for some odd integer $k \ge 3$. We show that P^{k+2}_{k+5} is properly $(k+2)$-connected. Let $P_{k+5} = (v_0, v_1, \ldots, v_{k+4})$ and let u and v be two distinct vertices of P_{k+5}. We may assume that $u = v_s$ and $v = v_t$ where $0 \le s < t \le k+4$. We consider two cases.

Case 1. *u and v are not end-vertices of P_{k+5}.* Then $u, v \in \{v_1, v_2, \ldots, v_{k+3}\}$. First, suppose that $u \ne v_1$ and $v \ne v_{k+3}$. By the induction hypothesis, there are k internally disjoint properly colored $u-v$ paths $Q_1, Q_2, \ldots, Q_k$ whose vertices belong to $\{v_1, v_2, \ldots, v_{k+3}\}$. Let $Q_{k+1} = (u, v_0, v)$ and $Q_{k+2} = (u, v_{k+4}, v)$. Then $Q_1, Q_2, \ldots, Q_{k+2}$ are $k+2$ internally disjoint properly colored $u-v$ paths in P^{k+2}_{k+5} and so P^{k+2}_{k+5} is properly $(k+2)$-connected.

Next, suppose that either $u = v_1$ or $v = v_{k+3}$. We may assume, without loss of generality, that $u \in \{v_1, v_2, \ldots, v_{k+2}\}$ and $v = v_{k+3}$. Let $d_{P_{k+5}}(u, v) = p$. Suppose that $u = v_i$, where $1 \le i \le k+2$, and so $p = k+3-i$. If p is odd, then define

$$Q_j = \begin{cases} (v_i, v_j, v_{k+3}) & \text{if } j \in \{1, 2, \ldots, k+2\} - \{i\} \\ (v_i, v_{k+3}) & \text{if } j = i. \end{cases}$$

If p is even, then define

$$Q_j = \begin{cases} (v_i, v_j, v_{k+3}) & \text{if } j \in \{1, 2, \ldots, k+2\} - \{i, \frac{i+k+3}{2}\} \\ (v_i, v_{k+3}) & \text{if } j = i \\ (v_i, v_{k+4}, v_{k+3}) & \text{if } j = \frac{i+k+3}{2}. \end{cases}$$

In each case, $Q_1, Q_2, \ldots, Q_{k+2}$ are $k+2$ internally disjoint properly colored $u-v$ paths in P_{k+5}^{k+2} and so P_{k+5}^{k+2} is properly $(k+2)$-connected.

Case 2. At least one of u and v is an end-vertex of P_{k+5}, say $u = v_0$. Observe that the neighborhood of v_0 in P_{k+5}^{k+2} is $\{v_1, v_2, \ldots, v_{k+2}\}$.

- If $v = v_t$ and $1 \le t \le k+2$, then define

$$Q_i = \begin{cases} (v_0, v_i, v_t) & \text{if } i \in \{1, 2, \ldots, k+2\} - \{t, \frac{t}{2}\} \\ (v_0, v_t) & \text{if } i = t \\ (v_0, v_i, v_{k+3}, v_t) & \text{if } i = \frac{t}{2}. \end{cases}$$

- If $v = v_{k+3}$, then define $Q_i = (v_0, v_i, v_{k+3})$ for $1 \le i \le k+2$ and $i \neq \frac{k+3}{2}$ and $Q_{\frac{k+3}{2}} = (v_0, v_{\frac{k+3}{2}}, v_{k+4}, v_{k+3})$;
- If $v = v_{k+4}$, then define $Q_1 = (v_0, v_1, v_{k+3}, v_{k+4})$ and $Q_i = (v_0, v_i, v_{k+4})$ for $2 \le i \le k+2$.

In each case, $Q_1, Q_2, \ldots, Q_{k+2}$ are $k+2$ internally disjoint properly colored $u-v$ paths in P_{k+5}^{k+2} and so P_{k+5}^{k+2} is properly $(k+2)$-connected. ■

We are now prepared to determine all pairs k, n of integers with $n \ge k+1 \ge 4$ for which P_n^k is properly k-connected.

Theorem 3.3 *Let k and n be integers where $n \ge k+1 \ge 4$. Then P_n^k is properly k-connected if and only if n is even and either k is odd and $k+3 \le n \le 2k$ or k is even and $k+2 \le n \le 2k$.*

Proof. First, suppose that n is even and we show that P_n^k is properly k-connected if either k is odd and $k+3 \le n \le 2k$ or k is even and $k+2 \le n \le 2k$. We consider two cases.

Case 1. k is odd. We proceed by a finite induction on integers n to show that, for each odd integer $k \ge 3$, if $k+3 \le n \le 2k$, then P_n^k is properly k-connected. By Lemma 3.2, this statement is true for $n = k+3$. Assume, for each odd integer $k \ge 3$, that P_n^k is properly k-connected for an even integer n with $k+3 \le n \le 2k-2$. We show that P_{n+2}^k is properly k-connected. Let $P_{n+2} = (v_0, v_1, \ldots, v_{n+1})$ and let u and v be two distinct vertices of P_{n+2}. First, suppose that $d_{P_{n+2}}(u, v) < n$. We may assume, without loss of generality, that $u, v \in \{v_0, v_1, \ldots, v_{n-1}\}$. By the induction hypothesis, there are k internally disjoint properly colored $u-v$ paths in P_n^k where $P_n = (v_0, v_1, \ldots, v_{n-1})$ and so in P_{n+2}^k as well. Next, suppose that $n \le d_{P_{n+2}}(u,v) \le n+1$. We may assume that $u = v_0$. If $v = v_n$, then define

$$Q_i = \begin{cases} (v_0, v_i, v_{i+k}, v_n) & \text{if } 1 \le i < n-k \text{ and } i \ne \frac{n}{2} \\ (v_0, v_i, v_n) & \text{if } n-k \le i \le k \text{ and } i \ne \frac{n}{2} \\ (v_0, v_{\frac{n}{2}}, v_{n+1}, v_n) & \text{if } i = \frac{n}{2}. \end{cases}$$

If $v = v_{n+1}$, then define

$$Q_i = \begin{cases} (v_0, v_i, v_{i+k}, v_{n+1}) & \text{if } 1 \le i \le n-k \\ (v_0, v_i, v_{n+1}) & \text{if } n-k < i \le k. \end{cases}$$

In each case, $Q_1, Q_2, \ldots, Q_k$ are k internally disjoint properly colored $u-v$ paths in P_{n+2}^k and so P_{n+2}^k is properly k-connected.

Case 2. *k is even.* We proceed by a finite induction on integers n to show that, for each even integer $k \ge 4$, if $k+2 \le n \le 2k$, then P_n^k is properly k-connected. By Lemma 3.1, this statement is true for $n = k+2$. Assume for each even integer k that P_n^k is properly k-connected for an even integer n with $k+2 \le n \le 2k-2$. An argument similar to the one in Case 1 shows that P_{n+2}^k is properly k-connected.

To verify the converse, we consider two cases, according to n is odd or n is even.

Case 1. *n is odd.* We show that P_n^k is not properly k-connected for all k and n with $k \ge 2$ and $n \ge k+1$. Assume, to the contrary, that for some integer $k \ge 2$ there is an odd integer $n \ge k+1$ such that P_n^k is properly k-connected. By Theorem 2.3, $k \ge 3$. Let $n = 2\ell+1$ for some integer $\ell \ge 2$ and let $P_n = (v_0, v_1, \ldots, v_{2\ell})$. Since P_n^k is properly k-connected, there are k internally disjoint properly colored $v_0 - v_{2\ell}$ paths $Q_1, Q_2, \ldots, Q_k$ in P_n^k. Since $\deg_{P_n^k} v_0 = k$ and v_0 is adjacent to $v_1, v_2, \ldots, v_k$ in P_n^k, we may assume that v_0 is adjacent to v_i in Q_i for $1 \le i \le k$. Since these k paths are internally disjoint, this implies that $v_1 \in V(Q_1)$ must be adjacent to v_{k+1} in Q_1 and so $v_2 \in V(Q_2)$ must be adjacent to v_{k+2} in Q_2. Continuing in this manner, we obtain that $v_i \in V(Q_i)$ must be adjacent to v_{k+i} in Q_i for $1 \le i \le k$. However then, $Q_\ell = (v_0, v_\ell, v_{2\ell})$ is not properly colored, which is a contradiction.

Case 2. *n is even.* In this case, for an odd integer $k \ge 3$, we show that if $n = k+1$ or $n \ge 2k+2$, then P_n^k is not properly k-connected; while for an even integer $k \ge 2$, we show that if $n \ge 2k+2$, then P_n^k is not properly k-connected. Therefore, it suffices to show that if $n = k+1$ or $n \ge 2k+2$, then P_n^k is not properly k-connected. Assume, to the contrary, that P_n^k is properly k-connected for some integers $k \geq 2$ and n such that $n = k+1$ or $n \geq 2k+1$. We consider these two subcases.

Subcase 2.1. $n = k+1$. Let $P_{k+1} = (v_0, v_1, \ldots, v_k)$. Since P_{k+1}^k is properly k-connected, there are k internally disjoint properly colored $v_0 - v_2$ paths $Q_1, Q_2, \ldots, Q_k$ in P_{k+1}^k. Since $\deg_{P_{k+1}^k} v_0 = k$ and v_0 is adjacent to $v_1, v_2, \ldots, v_k$ in P_{k+1}^k, it follows that v_0 is adjacent to v_i $(1 \leq i \leq k)$ in exactly one of these k paths. We may assume that v_0 is adjacent to v_i in Q_i for $1 \leq i \leq k$. However then, $Q_1 = (v_0, v_1, v_2)$ is not properly colored, which is a contradiction.

Subcase 2.2. $n \geq 2k+2$. Let $P_n = (v_0, v_1, \ldots, v_{n-1})$. Since P_n^k is properly k-connected, there are k internally disjoint properly colored $v_0 - v_{2k}$ paths $Q_1, Q_2, \ldots, Q_k$ in P_n^k. Since $\deg_{P_n^k} v_0 = k$ and v_0 is adjacent to $v_1, v_2, \ldots, v_k$ in P_n^k, we may assume that v_0 is adjacent to v_i in Q_i for $1 \leq i \leq k$. Thus, $v_i \in V(Q_i)$ must be adjacent to v_{k+i} in Q_i for $1 \leq i \leq k$. However then, v_0v_k and v_kv_{2k} are adjacent edges in Q_k, which is impossible. ∎

While it follows from Theorem 3.3 that the distance-colored graph P_n^k is *not* properly k-connected whenever $n \geq 2k+1 \geq 7$, the following is believed to be true.

Conjecture 3.4 *For each integer $k \geq 3$, the distance-colored graph P_n^{k+1} is properly k-connected for all $n \geq 2k+1$.*

Conjecture 3.4 is true for $k = 3$, however, as we now verify. First, we introduce some additional definitions and notation. For a path $(x_0, x_1, \ldots, x_{n-1})$, the vertex x_{n-1} is called the *terminal vertex* of this path. For a path Q of a connected graph H and $v \in V(H) - V(Q)$, let $t(Q, v)$ denote the distance between v and the terminal vertex of Q. For a set $\mathcal{Q}$ of paths in a connected graph H, let $V(\mathcal{Q})$ be the set of vertices of H that belong to some path in $\mathcal{Q}$. For each $v \in V(H) - V(\mathcal{Q})$, define

$$T(\mathcal{Q}, v) = \max\{t(Q, v) : Q \in \mathcal{Q}\}.$$

That is, $T(\mathcal{Q}, v)$ is the maximum distance between v and the terminal vertex of a path in $\mathcal{Q}$.

For a connected graph G, let Q be a properly colored $u - v$ path in G^k for some integer $k \geq 2$ and let w be a vertex of G^k that does not belong to Q. Then Q *is extendable to* w if (Q, w) is a properly colored $u - w$ path in G^k. In this case, we can extend Q to w in G^k and the path (Q, w) is then called the *extension of* Q *to* w in G^k.

Let $P_n = (v_0, v_1, \ldots, v_{n-1})$ be a path of order $n \geq 7$. Next, we present an algorithm that produces three internally disjoint properly colored paths

Q_1, Q_2 and Q_3 in P_n^4 with initial vertex v_0 such that the distance between v_{n-1} and the terminal vertex of each path Q_i $(1 \le i \le 3)$ is at most 4. That is, if $\mathcal{Q} = \{Q_1, Q_2, Q_3\}$, then $T(\mathcal{Q}, v_{n-1}) \le 4$.

Algorithm 1 *For $P_n = (v_0, v_1, \ldots, v_{n-1})$ where $n \ge 7$, this algorithm produces three internally disjoint properly colored paths Q_1, Q_2 and Q_3 in P_n^4 with initial vertex v_0 such that the distance between v_{n-1} and the terminal vertex of each path Q_i $(1 \le i \le 3)$ is at most 4.*

Input: *An integer $n \ge 7$ and a path $P_n = (v_0, v_1, \ldots, v_{n-1})$.*

Step 1: *The first path begins at v_0 and moves to v_1, the second path begins at v_0 and moves to v_2 and the third path begins at v_0 and moves to v_4.*

At the end of Step 1, we obtain three paths

$$Q_1^1 = (v_0, v_1),\ Q_2^1 = (v_0, v_2),\ Q_3^1 = (v_0, v_4).$$

Let $\mathcal{Q}^1 = \{Q_1^1, Q_2^1, Q_3^1\}$.

Step 2: *If $T(\mathcal{Q}^1, v_{n-1}) \le 4$, then stop; while if $T(\mathcal{Q}^1, v_{n-1}) > 4$, then do the following:*

Let $Q_p^1 \in \mathcal{Q}^1$ $(1 \le p \le 3)$ such that $t(Q_p^1, v_{n-1}) = T(\mathcal{Q}^1, v_{n-1})$. Suppose that j is the smallest integer such that v_j does not belong to any path in $\mathcal{Q}^1$ and Q_p^1 is extendable to v_j in P_n^4. Let $Q_p^2 = (Q_p^1, v_j)$ and rename the remaining two paths Q_q^1 and $Q_{q'}^1$ (where $q, q' \in \{1,2,3\} - \{p\}$) in $\mathcal{Q}^1$ as Q_q^2 and $Q_{q'}^2$, respectively. Let $\mathcal{Q}^2 = \{Q_1^2, Q_2^2, Q_3^2\}$.

If $T(\mathcal{Q}^2, v_{n-1}) \le 4$, then stop; while if $T(\mathcal{Q}^2, v_{n-1}) > 4$, then repeat this procedure above. In general, at the step i for an integer $i \ge 1$, let $\mathcal{Q}^i = \{Q_1^i, Q_2^i, Q_3^i\}$.

Step $i+1$: *If $T(\mathcal{Q}^i, v_{n-1}) \le 4$, then stop; while if $T(\mathcal{Q}^i, v_{n-1}) > 4$, then do the following:*

Let $Q_p^i \in \mathcal{Q}^i$ $(1 \le p \le 3)$ such that $t(Q_p^i, v_{n-1}) = T(\mathcal{Q}^i, v_{n-1})$. Suppose that j is the smallest integer such that v_j does not belong to any path in $\mathcal{Q}^i$ and Q_p^i is extendable to v_j in P_n^4. Let $Q_p^{i+1} = (Q_p^i, v_j)$ and rename the remaining two paths Q_q^i and $Q_{q'}^i$ (where $q, q' \in \{1,2,3\} - \{p\}$) in $\mathcal{Q}^i$ as Q_q^{i+1} and $Q_{q'}^{i+1}$, respectively. Let $\mathcal{Q}^{i+1} = \{Q_1^{i+1}, Q_2^{i+1}, Q_3^{i+1}\}$.

Output: *Three internally disjoint properly colored paths* Q_1, Q_2 *and* Q_3 *in* P_n^4 *with initial vertex* v_0 *such that* $T(\{Q_1, Q_2, Q_3\}, v_{n-1}) \leq 4$.

For a properly colored path $Q = (x_1, x_2, \ldots, x_\ell)$ in G^k, the *distance sequence* of Q is defined as

$$d(Q): \ d_1, d_2, \ldots, d_{\ell-1}$$

where $d_i = d_G(x_i, x_{i+1})$ for $1 \leq i \leq \ell - 1$.

Lemma 3.5 *For each integer* $n \geq 7$, *let* $P_n = (v_0, v_1, \ldots, v_{n-1})$. *Then the distance-colored graph* P_n^4 *contains three internally disjoint properly colored* $v_0 - v_{n-1}$ *paths.*

Proof. First, suppose that $7 \leq n \leq 9$.

- For $n = 7$, let $Q_1 = (v_0, v_1, v_3, v_6)$, $Q_2 = (v_0, v_2, v_5, v_6)$ and $Q_3 = (v_0, v_4, v_6)$;
- For $n = 8$, let $Q_1 = (v_0, v_1, v_3, v_7)$, $Q_2 = (v_0, v_2, v_5, v_7)$ and $Q_3 = (v_0, v_4, v_7)$;
- For $n = 9$, let $Q_1 = (v_0, v_1, v_3, v_7, v_8)$, $Q_2 = (v_0, v_2, v_6, v_8)$ and $Q_3 = (v_0, v_4, v_5, v_8)$.

We now assume that $n \geq 10$. By Algorithm 1, we obtain three internally disjoint properly colored paths Q_1, Q_2 and Q_3 in P_n^4 for an arbitrarily large integer n such that $T(\{Q_1, Q_2, Q_3\}, v_{n-1}) \leq 4$ and the distance sequences of these three paths are

$$\begin{aligned} d(Q_1) &: \ 1, 2, 3, 2, \underbrace{4, 3, 4, 3, 2}, \underbrace{4, 3, 4, 3, 2}, \ldots \\ d(Q_2) &: \ 2, 3, 4, \underbrace{1, 3, 4, 1, 3, 4}, \underbrace{1, 3, 4, 1, 3, 4}, \ldots \\ d(Q_3) &: \ 4, 3, \underbrace{4, 3, 2, 4, 3}, \underbrace{4, 3, 2, 4, 3}, \ldots \end{aligned}$$

Let

$$s_1 : 4, 3, 4, 3, 2, \ s_2 : 1, 3, 4, 1, 3, 4 \text{ and } s_3 : 4, 3, 2, 4, 3. \tag{1}$$

Then the three internally disjoint properly colored paths Q_1, Q_2 and Q_3 obtained by Algorithm 1 have the distance sequences as follows:

$$\begin{aligned} d(Q_1) &: \ 1, 2, 3, 2, s_1, s_1, , \ldots \\ d(Q_2) &: \ 2, 3, 4, s_2, s_2, \ldots \\ d(Q_3) &: \ 4, 3, s_3, s_3, \ldots \end{aligned}$$

Observe that the sum of integers in s_i in (1) is 16 for $1 \leq i \leq 3$. Let

$$Q_1^0 = (v_0, v_1, v_3, v_6, v_8),\ Q_2^0 = (v_0, v_2, v_5, v_9) \text{ and } Q_3^0 = (v_0, v_4, v_7).$$

Then Q_1^0, Q_2^0 and Q_3^0 are internally disjoint properly colored paths in P_n^4 for a sufficiently large integer n. For each integer $i \geq 0$, the three paths Q_1^{i+1}, Q_2^{i+1} and $Q_3 1^{i+1}$ are constructed from Q_1^i, Q_2^i and Q_3^i, respectively, as follows:

(1) the path Q_1^{i+1} is obtained from Q_1^i and the path

$$X_i = (v_{8+16i}, v_{8+16i+4}, v_{8+16i+7}, v_{8+16i+11}, v_{8+16i+14}, v_{8+16(i+1)})$$

by identifying the the vertex v_{8+16i} in Q_1^i and X_i, respectively.

(2) the path Q_2^{i+1} is obtained from Q_2^i and the path

$$Y_i = (v_{9+16i}, v_{9+16i+1}, v_{9+16i+4}, v_{9+16i+8}, v_{9+16i+9}, v_{9+16i+12}, v_{9+16(i+1)})$$

by identifying the the vertex v_{9+16i} in Q_2^i and Y_i, respectively.

(3) the path Q_3^{i+1} is obtained from Q_3^i and the path

$$Z_i = (v_{7+16i}, v_{7+16i+4}, v_{7+16i+7}, v_{7+16i+9}, v_{7+16i+13}, v_{7+16(i+1)})$$

by identifying the the vertex v_{7+16i} in Q_3^i and Z_i, respectively.

Observe that Q_1^i, Q_2^i and Q_3^i are internally disjoint properly colored paths in P_n^4 for all $i \geq 0$ (where n is sufficiently large). We now verify the following claim.

Claim. For each $n \geq 10$, three internally disjoint properly colored $v_0 - v_{n-1}$ paths Q_1, Q_2 and Q_3 can be constructed from the paths Q_1^i, Q_2^i and Q_3^i for some integer i with $i \geq \lceil \frac{n-9}{16} \rceil$ by an appropriate modification.

Proof of Claim. The distance sequences of Q_1^0, Q_2^0 and Q_3^0 are

$$d(Q_1^0) : 1, 2, 3, 2, \quad d(Q_2^0) : 2, 3, 4, \quad \text{and} \quad d(Q_3^0) : 4, 3.$$

In general, for each integer $i \geq 0$,

$$d(Q_1^{i+1}) : d(Q_1^i), s_1, \quad d(Q_2^{i+1}) : d(Q_2^i), s_2, \quad \text{and} \quad d(Q_3^{i+1}) : d(Q_3^i), s_3$$

where s_i are shown in (1) for $1 \leq i \leq 3$. Since (i) the sum of integers in s_i is 16 for $1 \leq i \leq 3$ and (ii) the terminal terms in $d(Q_1^i)$, $d(Q_2^i)$, $d(Q_3^i)$ are $2, 4, 3$ for all $i \geq 0$, it follows that, to verify the claim, it suffices to show that for each n with $10 \leq n \leq 25$, three internally disjoint properly colored $v_0 - v_{n-1}$ paths Q_1, Q_2 and Q_3 can be obtained from Q_1^1, Q_2^1 and Q_3^1 in P_n^4. This is verified by the following table, where a path $(v_{i_1}, v_{i_2}, \ldots, v_{i_s})$ is denoted by $(i_1, i_2, \ldots, i_s)$

$n = 10$	Q_1: (0, 1, 3, 7, 9) Q_2: (0, 2, 6, 9) Q_3: (0, 4, 5, 9)	$n = 18$	Q_1: (0, 1, 3, 6, 8, 12, 15, 17) Q_2: (0, 2, 5, 9, 10, 13, 17) Q_3: (0, 4, 7, 11, 14, 16, 17)
$n = 11$	Q_1: (0, 1, 3, 6, 10) Q_2: (0, 2, 5, 9, 10) Q_3: (0, 4, 7, 8, 10)	$n = 19$	Q_1: (0, 1, 3, 6, 8, 12, 15, 16, 18) Q_2: (0, 2, 5, 9, 10, 13, 17, 18) Q_3: (0, 4, 7, 11, 14, 18)
$n = 12$	Q_1: (0, 1, 3, 6, 8, 11) Q_2: (0, 2, 5, 9, 11) Q_3: (0, 4, 7, 11)	$n = 20$	Q_1: (0, 1, 3, 6, 8, 12, 15, 19) Q_2: (0, 2, 5, 9, 10, 13, 17, 19) Q_3: (0, 4, 7, 11, 14, 16, 19)
$n = 13$	Q_1: (0, 1, 3, 6, 8, 12) Q_2: (0, 2, 5, 9, 12) Q_3: (0, 4, 7, 11, 12)	$n = 21$	Q_1: (0, 1, 3, 6, 8, 12, 15, 19, 20) Q_2: (0, 2, 5, 9, 10, 13, 17, 20) Q_3: (0, 4, 7, 11, 14, 16, 20)
$n = 14$	Q_1: (0, 1, 3, 6, 8, 12, 13) Q_2: (0, 2, 5, 9, 10, 13) Q_3: (0, 4, 7, 11, 13)	$n = 22$	Q_1: (0, 1, 3, 6, 8, 12, 15, 19, 21) Q_2: (0, 2, 5, 9, 10, 13, 17, 18, 21) Q_3: (0, 4, 7, 11, 14, 16, 20, 21)
$n = 15$	Q_1: (0, 1, 3, 6, 8, 12, 14) Q_2: (0, 2, 5, 9, 10, 13, 14) Q_3: (0, 4, 7, 11, 14)	$n = 23$	Q_1: (0, 1, 3, 6, 8, 12, 15, 19, 22) Q_2: (0, 2, 5, 9, 10, 13, 17, 18, 21, 22) Q_3: (0, 4, 7, 11, 14, 16, 20, 22)
$n = 16$	Q_1: (0, 1, 3, 6, 8, 12, 15) Q_2: (0, 2, 5, 9, 10, 13, 15) Q_3: (0, 4, 7, 11, 14, 15)	$n = 24$	Q_1: (0, 1, 3, 6, 8, 12, 15, 19, 22, 23) Q_2: (0, 2, 5, 9, 10, 13, 17, 18, 21, 23) Q_3: (0, 4, 7, 11, 14, 16, 20, 23)
$n = 17$	Q_1: (0, 1, 3, 6, 8, 12, 15, 16) Q_2: (0, 2, 5, 9, 10, 14, 16) Q_3: (0, 4, 7, 11, 13, 16)	$n = 25$	Q_1: (0, 1, 3, 6, 8, 12, 15, 19, 21, 24) Q_2: (0, 2, 5, 9, 10, 13, 17, 18, 22, 24) Q_3: (0, 4, 7, 11, 14, 16, 20, 23, 24)

The result then follows from the claim. ∎

We are now prepared to show that P_n^4 is properly 3-connected for each $n \geq 7$.

Theorem 3.6 *For each integer $n \geq 7$, the distance-colored graph P_n^4 is properly 3-connected.*

Proof. Let $P_n = (v_0, v_1, \ldots, v_{n-1})$. We proceed by induction on n. For $n = 7$, it is straightforward to verify that P_7^4 is properly 3-connected. Assume that P_k^4 is properly 3-connected for some integer $k \geq 7$. Now let u and v be two distinct vertices of P_{k+1}^4. First, suppose that $\{u, v\} \neq \{v_0, v_k\}$. We may assume, without loss of generality, that $v_k \notin \{u, v\}$. Let $P_k = P_{k+1} - v_k$. Then $u, v \in V(P_k)$ and $d_{P_k}(u, v) = d_{P_{k+1}}(u, v)$. By the induction hypothesis, P_k^4 contains three internally disjoint properly colored $u - v$ paths and these three paths are also properly colored paths in P_{k+1}^4. Next, suppose that $\{u, v\} = \{v_0, v_k\}$. It then follows by Lemma 3.5 that P_{k+1}^4 contains three internally disjoint properly colored $v_0 - v_k$ paths. ∎

4 On H-Colored and H-Chromatic Graphs

In Sections 2 and 3, we investigated connected graphs G for which G^k is properly p-connected for some integers $k, p \geq 2$. If G^k is properly p-connected, then G^k contains a properly colored subdivision of $K_{2,p}$ as a subgraph. In fact, there is no restriction on what properly colored subgraphs that G^k can possess.

Theorem 4.1 *For every connected graph H, there exists a connected graph G and a positive integer k such that the distance-colored graph G^k contains a copy of H and a proper edge coloring of H using $\chi'(H)$ colors.*

Proof. Since the result is obvious if $H = K_2$, we may assume that H has order at least 3. Let $\chi'(H) = \chi \geq 2$ and let c be a proper χ-edge coloring of H using the colors $1, 2, \ldots, \chi$. We consider two cases.

Case 1. $\chi = 2$. For each $e \in E(H)$ such that $c(e) = 2$, subdivide the edge e exactly once. Denote the resulting graph by G. Then the distance-colored graph G^2 contains a copy of H and a proper 2-edge coloring of H using colors 1 and 2.

Case 2. $\chi \geq 3$. Let $M = \chi - 3$. For each $e \in E(H)$, subdivide the edge e a total of $M + c(e) - 1$ times, resulting in a path of length $M + c(e)$. Denote the resulting graph by G. Suppose that e joins u and v in H. Since there is a path of length $M + c(e)$ connecting u and v in G, it follows that $d_G(u, v) \leq M + c(e)$. We claim that $d_G(u, v) = M + c(e)$. Suppose that $d_G(u, v) < M + c(e)$. Then there exists a $u - v$ path in G of length less than $M + c(e)$. From the way in which G is constructed, the length of such a path must be at least $(M + p) + (M + q)$ for some positive integers p and q such that $p + q \geq 3$. Since $(M + p) + (M + q) \geq 2M + 3$, it follows that $2M + 3 < M + c(e) \leq M + \chi$ and so $M < \chi - 3$, which is impossible. Thus $G^{M+\chi}$ contains a copy of H and a proper χ-edge coloring of H using colors $M + 1, M + 2, \ldots, M + \chi$. ∎

Theorem 4.1 raises the question of determining the smallest order of such a graph G. Let H be a given connected graph. A connected graph G is *H-colored* if there is a positive integer k such that the distance-colored graph G^k contains a properly edge-colored copy of H with $\chi'(H)$ colors. If the properly edge-colored copy of H in an H-colored graph G is produced by a $\chi'(H)$-edge coloring c, then G is an *H-colored graph with respect to* c. The minimum order of such a graph G is called the *color-order* of H, denoted by $\text{co}(H)$. An H-colored graph of order $\text{co}(H)$ is called an *H-chromatic graph.* Thus if H is a nontrivial connected graph of order n and G is an H-colored graph of order n', then $n \leq \text{co}(H) \leq n'$.

Let H be a nontrivial connected graph of size m with $E(H) = \{e_1, e_2, \ldots, e_m\}$ and $\chi'(H) = \chi$. For a χ-edge coloring c of H, define the *σ-number* of c by

$$\sigma(c) = \sum_{i=1}^{m} (c(e_i) - 1).$$

Let $C(H)$ be the set of all χ-colorings of H. The *minimum σ-number* (or simply *σ-number*) of H is defined by

$$\sigma(H) = \min \{\sigma(c) : \ c \in C(H)\}.$$

First, we establish bounds for the color-order of a connected graph in terms of its order, size, chromatic index and minimum σ-number.

Theorem 4.2 *If H is a nontrivial connected graph of order n with $\chi'(H) \geq 3$, then*

$$n + \sigma(H) \leq \text{co}(H) \leq n + m(\chi'(H) - 3) + \sigma(H). \qquad (2)$$

Proof. The upper bound for $\text{co}(H)$ is a consequence of the proof of Theorem 4.1, in which the H-colored graph has order $n + m(\chi'(H) - 3) + \sigma(H)$ if we choose a χ-edge coloring whose σ-number is $\sigma(H)$. Thus, it remains to verify the lower bound. Let c be a χ-edge coloring of H using the colors $1, 2, \ldots, \chi$ and let G be an H-colored graph with respect to c. Thus there is a positive integer k such that the distance-colored G^k contains a properly edge-colored copy of H produced by the coloring c. Let $E(H) = \{e_1, e_2, \ldots, e_m\}$. For each i with $1 \leq i \leq m$, let $e_i = x_i y_i$ and $c(e_i) = c_i$. Since $d_G(x_i, y_i) = c_i$, there is an $x_i - y_i$ geodesic Q in G^k. Suppose that $Q = (x_i = v_0, v_1, \ldots, v_{c_i} = y_i)$. Since e_i is an edge of H, each of the vertices $v_1, v_2, \ldots, v_{c_i - 1}$ belongs to $V(G) - V(H)$. This implies that each edge e_i $(1 \leq i \leq m)$ colored c_i contributes a total $c_i - 1$ to the order of G. Since $V(H) \subseteq V(G)$, the order of G is at least

$$n + \sum_{i=1}^{m}(c_i - 1) \geq n + \sigma(H),$$

giving the desired result. ■

Next, we describe a class of connected graphs H for which $\text{co}(H) = n + \sigma(H)$. The *girth* $g(H)$ of a graph H having a cycle is the length of a smallest cycle in H.

Proposition 4.3 *Let H be a nontrivial connected graph of order n. If H is a tree or $\chi'(H) \leq \lfloor 3(g(H) - 1)/2 \rfloor$, then $\text{co}(H) = n + \sigma(H)$.*

Proof. Suppose that $\chi'(H) = \chi$. Let c be a χ-edge coloring of H using the colors $1, 2, \ldots, \chi$ such that $\sigma(c) = \sigma(H)$. For each $e \in E(H)$, subdivide the edge e a total of $c(e) - 1$ times, resulting in a path of length $c(e)$. Denote the resulting graph by G. Thus the order of G is $n + \sigma(c)$. It remains to show that G is H-colored with respect to the coloring c. Suppose that e joins u and v in H. Since there is a $u - v$ path of length $c(e)$ connecting u and v in G, it follows that $d_G(u, v) \leq c(e)$. We claim that $d_G(u, v) = c(e)$.

First, suppose that H is a tree. Since G is a subdivision of a tree, G is a tree as well. Thus the $u - v$ path of length $c(e)$ is the only $u - v$ path in G. Hence $d_G(u, v) = c(e)$ if H is a tree. Next, suppose that H is not a tree. Assume, to the contrary, that $d_G(u, v) < c(e)$. Then there exists a $u - v$ path Q in G of length less than $c(e)$. Let P be a $u - v$ path in H. Since the girth of H is g and $uv \in E(H)$, it follows that the length of P is either 1 or is at least $g - 1$. If the length of P is 1, then $P = (u, v)$ and P gives rise to a $u - v$ path of length $c(e)$ in G. Thus, we may assume that the length of P is ℓ and so $\ell \geq g - 1$. Let $P = (u = x_0, x_1, x_2, \ldots, x_\ell = v)$. Since P is properly colored by c, it follows from the way in which G is constructed that P gives rise to a $u - v$ path in G whose length is at least

$$\sum_{i=0}^{\ell-1} c(x_i x_{i+1}) \geq \left\lceil \frac{g-1}{2} \right\rceil + 2 \left\lfloor \frac{g-1}{2} \right\rfloor = \left\lfloor \frac{3(g-1)}{2} \right\rfloor.$$

This implies that the length of Q is at least $\lfloor 3(g-1)/2 \rfloor$. However then,

$$c(e) > d_G(u, v) \geq \left\lfloor \frac{3(g-1)}{2} \right\rfloor \geq \chi'(H) \geq c(e),$$

which is a contradiction. Therefore, G^χ contains a properly edge-colored copy of H using χ colors and G is H-colored. ∎

The following is an immediate consequence of Theorem 4.2 and Proposition 4.3.

Corollary 4.4 *If H is a nontrivial connected graph of order n such that $\chi'(H) = 2$ or $\chi'(H) = 3$, then* $\text{co}(H) = n + \sigma(H)$.

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