Maximal Sets of Hamilton Cycles in Complete Multipartite Graphs III

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This paper is dedicated to Roger Eggleton, a lifelong colleague and friend of the second author.

Abstract

A set of S edge-disjoint hamilton cycles in a graph G is said to be *maximal* if the hamilton cycles in S form a subgraph of G such that *G-E(S)* has no hamilton cycle. The set of integers m for which a graph G contains a maximal set of m edge-disjoint hamilton cycles has previously been determined whenever G is a complete graph, a complete bipartite graph, and in many cases when G is a complete multipartite graph. In this paper, we solve half of the remaining open cases regarding complete multipartite graphs.

1 Introduction

A *complete multipartite graph*, K_n^p, has p parts of size n such that an edge exists between vertices u and v if and only if u and v are in different parts. A *hamilton cycle* in a graph G is a spanning cycle of G. If S is a set of hamilton cycles, then let $G(S)$ be the graph induced by the edges in cycles of S. We denote the edges in this graph by $E(G(S))$ or $E(S)$. The set S is maximal if $G - E(S)$ has no hamilton cycle.

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Considerable research has come before this paper to find maximal sets of hamilton cycles in certain graphs. Hoffman, Rodger, and Rosa [5] found that there exists a maximal set S of m edge-disjoint hamilton cycles in K_n if and only if $m \in \left\{\left\lfloor\frac{n+3}{4}\right\rfloor, \left\lfloor\frac{n+3}{4}\right\rfloor + 1, ..., \left\lfloor\frac{n-1}{2}\right\rfloor\right\}$. It was later shown by Bryant, El-Zanati, and Rodger [1] that there exists a maximal set of S of m edge-disjoint hamilton cycles in $K_{n,n}$ if and only if $\frac{n}{4} < m \leq \frac{n}{2}$. Daven, MacDougall and Rodger [2] extended the results to complete multipartite graphs, showing that there exists a maximal set of m hamilton cycles in K_n^p if and only if (a) $\left\lceil\frac{n(p-1)}{4}\right\rceil \leq m \leq \left\lfloor\frac{n(p-1)}{2}\right\rfloor$ and (b) $m > \frac{n(p-1)}{4}$ if either n is odd and $p \equiv 1 \pmod 4$ or $p = 2$, $n = 1$, except possibly for the undecided case when $n \geq 3$ is odd, p is odd and $m \leq \frac{(n+1)(p-1)}{4}$. Jarrell and Rodger [6] solved the open cases when $p \geq 5$, and removed all but the smallest possible values when $n = 3$, showing that a maximal set of hamilton cycles of size m exists when $n = 3$ and $\left\lceil\frac{n(p-1)}{4}\right\rceil + 1 \leq m \leq \left\lfloor\frac{(n+1)(p-1)-2}{4}\right\rfloor$. Together, these results mean that for each value of p, exactly one value of m remains in doubt (namely $\left\lfloor\frac{n(p-1)}{4}\right\rfloor + 1$) and even that is only in doubt in the case where $n = 3$ and p is odd. Naturally, each remaining case becomes more and more difficult. Indeed, for some time it was unclear whether the remaining values would be in S or not.

Our aim in this paper is to solve half of the remaining open cases; specifically the case when $m = \left\lceil\frac{n(p-1)}{4}\right\rceil + 1$ and $p \equiv 1 \pmod 4$. The case when $m = \left\lceil\frac{n(p-1)}{4}\right\rceil$ and $p \equiv 3 \pmod 4$ is still open.

Putting the results above together produces the following state of knowledge:

Theorem 1.1 *There exists a maximal set of m hamilton cycles in K_n^p (p parts of size n) if and only if*

1. $\left\lceil\frac{n(p-1)}{4}\right\rceil \leq m \leq \left\lfloor\frac{n(p-1)}{2}\right\rfloor$ *and*
2. $m > \frac{n(p-1)}{4}$ *if*
   (a) *n is odd and $p \equiv 1$ (mod 4), or*
   (b) $p = 2$, $n = 1$

except possibly when $m = \left\lceil\frac{n(p-1)}{4}\right\rceil$ and $p \equiv 3 \pmod 4$.

In the following, edge-colorings are used to represent the hamilton cycles, so let $G(i)$ denote the subgraph of G induced by the edges colored i.

2 The Technique of Amalgamations

The approach used to prove the main theorem of the paper is that of amalgamations. An *amalgamation* of a graph G is a graph H formed by a homomorphism $f : V(G) \to V(H)$. So for each $v \in V(H)$, the vertices of $f^{-1}(v)$ can be thought of as being amalgamated into the single vertex v in H; for each $v \in V(H)$, $\eta(v) = |f^{-1}(v)|$ is known as the amalgamation number of v. G is said to be a *disentanglement* of H.

In the proof of Theorem 3.1, an amalgamated graph is constructed in which each color class is connected and each vertex v is incident with $2\eta(v)$ edges of each color, thus looking like what would be obtained by amalgamating a graph in which each color class is a hamilton cycle. For our purposes, the two following results will be essential. The first result describes properties of a graph formed by amalgamating K_n. The second will be used to show that the amalgamated graph we construct can be disentangled to form a subgraph of K_3^p that has a hamilton decomposition.

Lemma 2.1 *[8] Let $H \cong K_n$ be an l-edge-colored graph, and let $f : V(G) \to V(H)$ be an amalgamating function with amalgamation numbers given by the function $\eta : V(H) \to N$. Then H satisfies the following conditions for any vertices $w, v \in V(G)$:*

1. $d(w) = \eta(w)(n-1)$,
2. $m(w, v) = \eta(w)\eta(v)$ *if* $w \neq v$,
3. w *is incident with* $\binom{\eta(w)}{2}$ *loops, and*
4. $d_{G(i)}(w) = \sum_{u \in f^{-1}(w)} d_{H(i)}(u)$ *for* $1 \leq i \leq l$.

Theorem 2.2 *[8] Let H be an l-edge-colored graph satisfying conditions (1)-(4) of Lemma 2.1 for the function $\eta : V(H) \to N$. Then there exists a disentanglement G of H that satisfies*

1. $H \cong K_n$,
2. *for any* $z \in V(G), |d_{H(i)}(v) - d_{H(i)}(u)| \leq 1$ *for* $1 \leq i \leq l$ *and all* $u, v \in f^{-1}(z)$,
3. *if* $\frac{d_{G(i)}(z)}{\eta(z)}$ *is an even integer for all* $z \in V(G)$, *then* $\omega(G(i)) = \omega(H(i))$.

Another important result that is invaluable in the main proof is the following theorem proved by Hilton [4]. A k-edge-coloring of G is said to be *evenly equitable* if $|d_i(v) - d_j(v)| \leq 2$ for $1 \leq i < j \leq k$ and $d_i(v)$ is even for $1 \leq i \leq k$, where $d_i(v)$ is the degree of v in the subgraph induced by the edges colored i.

Theorem 2.3 *[4] For each $k \geq 1$, each finite Eulerian graph has an evenly equitable edge-coloring with k colors.*

3 The $p \equiv 1 \pmod 4$ Case

Theorem 3.1 *For the complete multipartite graph K_n^p, let $p = 4x + 1$ for some integer $x \geq 2$ and let $n = 3$. Then there exists a maximal set of $m = \left\lceil \frac{3(p-1)}{4} \right\rceil + 1 = 3x + 1$ edge disjoint hamilton cycles in K_n^p.*

Proof We define the hamilton cycles on the vertex set $\mathbb{Z}_p \times \mathbb{Z}_3$ in which the parts are $P_i = i \times \mathbb{Z}_3$ for each $i \in \mathbb{Z}_p$. As we look at this problem, it is helpful to think of the parts of the graph arranged in p vertical columns with three vertices in each column; so each part has a top, a middle, and a bottom vertex. Our goal is to choose the edges for our set S of hamilton cycles wisely so that we ensure that our set is maximal. In each case S is shown to be maximal because $K_n^p - E(S) = G$ has a cut vertex. We do this by splitting $V(G)$ into 3 sections. We denote by G_1 the subgraph induced by vertices in the first $2x$ parts (part 0 to part $2x - 1$) together with the top vertex of the center part, which we call u. The subgraph induced by vertices in the last $2x$ parts (part $2x + 1$ to part $4x$) together with the bottom vertex of the center part, which we will call w, is denoted by G_2. Finally, the middle vertex of the center part will be called v. The vertex v will serve as a cut vertex in $G(S)$.

The edges we choose to make our set of hamilton cycles fall into the following three types:

Type 1: All edges in K_3^p that join vertices in G_1 to vertices in G_2 occur in $E(S)$.

Type 2: Precisely $2m$ edges joining vertices in $V(G_1 \bigcup G_2)$ to v occur in $E(S)$. (Approximately half of these edges are incident with vertices in G_1, while the others are incident with vertices in G_2.)

Type 3: Certain edges between two vertices in G_1 or two vertices in G_2 are finally chosen to make $G(S)$ $2m$-regular.

As we select Type 2 edges, we note that v is not yet adjacent to any other vertices. Thus to produce $m = 3x + 1$ hamilton cycles, we need the degree of v to be $2(3x+1) = 6x+2$. If m is even, then $3x+1$ of the edges are chosen to join vertex v to vertices in G_1, while the other $3x+1$ are chosen to join vertex v and vertices in G_2. If m is odd, then $3x$ of the edges join vertex v and vertices in G_1, while the other $3x+2$ join vertex v and vertices in G_2. For the Type 3 edges, we carefully pick edges between two vertices in G_1 or two vertices in G_2 so that we build each vertex to degree $6x+2$. It turns out that the edges of Types 2 and 3 need to be precisely chosen if we use amalgamations to produce G. Instead we define H and let Theorem 2.2 produce G.

The method used to construct the hamilton cycles is that of amalgamations. This has been used successfully in this setting (see [6] for example). The amalgamation used here is the graph homomorphism $f : V(G) \to V(H) = (\mathbb{Z}_p \setminus \{2x\}) \bigcup (\{2x\} \times \mathbb{Z}_3)$ defined as follows. For each $i \in \mathbb{Z}_p \setminus 2x$ and for each $j \in \mathbb{Z}_3$, let $f((i,j)) = i$, and for each $j \in \mathbb{Z}_3$ let $f(2x, j) = (2x, j) = u, v,$ or w if $x = 0, 1,$ or 2 respectively. So, except for the part containing v, f amalgamates the vertices in each part into a single vertex in H with amalgamation number 3. The vertices in the part containing v are not amalgamated by f, so each vertex z has amalgamation number $\eta(z) = 1$. So we will require that $d_H(i) = 6m = 6(3x+1)$ for each $i \in \mathbb{Z}_p \setminus \mathbb{Z}_x$.

The subgraph B of H induced by the edges joining vertices in $\mathbb{Z}_{2x}$ to vertices in $\mathbb{Z}_{4x+1} \setminus \mathbb{Z}_{2x+1}$ is isomorphic to $9K_{2x,2x}$. Let $\varepsilon = 1$ or 0 if m is odd or even respectively. Join v to vertices in $\mathbb{Z}_{x+1-\varepsilon}$ and $\mathbb{Z}_{2x} \setminus \mathbb{Z}_{x+1-\varepsilon}$ with 2 and 1 edges respectively, and join v to vertices in $\mathbb{Z}_{3x+1+\varepsilon} \setminus \mathbb{Z}_{2x+1}$ and $\mathbb{Z}_{4x+1} \setminus \mathbb{Z}_{3x+1+\varepsilon}$ with 2 and 1 edges respectively; these produce Type 2 edges in G. Pair the vertices in $\mathbb{Z}_{2x} \setminus \mathbb{Z}_{x+1-\varepsilon}$ and pair the vertices in $\mathbb{Z}_{4x+1} \setminus \mathbb{Z}_{3x+1+\varepsilon}$, and join each such pair with an edge; these produce the Type 3 edges in G. (Notice that each set has an even size by definition of ε.)

Color the edges of H as follows. Since there exists a hamilton decomposition of $K_{2x,2x}$, the edges of B can be partitioned into $9x$ sets, each of which induces a hamilton cycle of H. Let B_0, ..., B_{3x} be $3x+1$ of these $9x$ sets, and color the edges in B_i with i for each $i \in \mathbb{Z}_m$. Let $H_1 = H - \bigcup_{i \in \mathbb{Z}_{3x+1}} B_i$. Then $d_{H_1}(i) = 4m$ and $d_{H_1}(i,j) = 2m$.

We now give the subgraph H_1 of H an evenly equitable edge coloring with the $3x+1$ colors in $\mathbb{Z}_{3x+1}$. Such a coloring exists by Theorem 2.3. Thus in H_1 each color appears 4 times at each vertex z with $\eta(z) = 3$ and twice at each vertex z where $\eta(z) = 1$. So in H each color now appears 6 times at each vertex where $\eta = 3$ and once where $\eta = 1$. We are now assured that our color classes are connected and that each color appears on the appropriate number of edges, namely $2\eta(z)$, at each vertex z.

The aim now is to disentangle our graph so that we can pick out our maximal set of hamilton cycles. To be able to apply Lemma 2.1 we still must add more edges to H to form H^+ so that H^+ satisfies properties (1-4) of Lemma 2.1 (i.e. so that it is an amalgamation of K_{3p}). So add edges to H so that between each pair of vertices x and y there are: exactly nine edges if $|\{x, y\} \cap \{u, v, w\}| = 0$; exactly three edges if $|\{x, y\} \cap \{u, v, w\}| = 1$; and exactly one edge if $|\{x, y\} \cap \{u, v, w\}| = 2$. Finally, add three loops to each vertex not in $\{u, v, w\}$. All these additional edges and loops are colored 0. It is straightforward to check that H^+ satisfies properties (1-4) of Lemma 2.1, so we can now apply Theorem 2.2 to H^+ to produce G^+, and edge-colored copy of K_{3p}. Removing all edges in G^+ corresponding to loops in H^+ produces K_n^p, and then removing all remaining edges colored 0 produces G. Each color class in G is 2-regular by property (2) of Theorem 2.2, and is connected by property (3), so is a hamilton cycle. Removing the edges in these hamilton cycles from K_n^p in particular means that all Type 1 edges are removed, so produces a graph in which v is a cut vertex (it is actually the graph induced by the edges (not loops) colored 0 in G^+). So the required maximal set of hamilton cycles has been produced. □

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