On free α-labelings of cubic bipartite graphs

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Abstract

It is known that an α-labeling of a bipartite graph G with n edges can be used to obtain a cyclic G-decomposition of K_{2nx+1} for every positive integer x. It is also known that if two graphs G and H admit a free α-labeling, then their vertex-disjoint union also admits a free α-labeling. We show that if G is a bipartite prism, a bipartite Möbius ladder or a connected cubic bipartite graph of order at most 14, then G admits a free α-labeling. We conjecture that every bipartite cubic graph admits a free α-labeling.

1 Introduction

Let N denote the set of nonnegative integers, and denote the set of integers $\{m, m+1, \ldots, n\}$ by $[m, n]$. For any graph G we call an injective function $h : V(G) \to N$ a *labeling* (or a *valuation*) of G. If $v \in V(G)$, we call $h(v)$ the *label* of v. If $W \subseteq V(G)$, we let $h(W) = \{h(v) : v \in W)\}$. If h is a labeling of G, we define a function $\bar{h} : E(G) \to \mathbb{Z}^+$ by $\bar{h}(e) = |f(u) - f(v)|$, where $e = \{u, v\} \in E(G)$. The number $|h(u) - h(v)|$ is called the *label* of the edge $\{u, v\}$. If $F \subseteq E(G)$, then $\bar{h}(F) = \{\bar{h}(e) : e \in F\}$. For convenience, we will often let $h(G)$ denote the labeled graph G; that is, $h(G)$ is the

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graph with vertex set $h(V(G))$ and edge set $\{\{h(u), h(v)\}; \{u, v\} \in E(G)\}$. Rosa [14] called a labeling h of a graph G with q edges a *β-valuation* of G if $h(V(G)) \subseteq [0, q]$ and $\bar{h}(E(G)) = [1, q]$. A β-valuation is now more commonly called a *graceful labeling*. An *α-labeling* is a graceful labeling having the additional property that there exists an integer λ such that if $\{u, v\} \in E(G)$, then $\{u, v\} = \{a, b\}$, where $h(a) \leq \lambda < h(b)$. The number λ, which is unique, is called the *critical value* of the α-labeling. Note that necessarily 0, λ, $\lambda + 1$, and $|E(G)|$ are in $h(V(G))$. Moreover, G must be bipartite.

If h is an α-labeling of G, then $h' = |E(G)| - h$ is also an α-labeling of G with critical value $\lambda' = |E(G)| - (\lambda + 1)$. We shall refer to h' as the *complementary labeling* of h.

Numerous large classes of bipartite graphs have α-labelings; examples include complete bipartite graphs, caterpillars, d-cubes, bipartite prisms, and cycles of length $4k$ (see Gallian [11] for a survey). Labelings of graphs are particularly interesting because of their applications to graph decompositions. It is well known that if a graph G with q edges admits an α-labeling, then the edge-sets of K_{2qx+1}, $K_{2qx+2} - I$ (where I is a 1-factor), and $K_{qx,qy}$ can be partitioned into subgraphs isomorphic to G for all positive integers x and y (see [14] and [5]). One may not be able to obtain these same results with the the less restrictive graceful labelings of G.

Let G be a bipartite graph on p vertices and with q edges. We list some known necessary conditions for G to admit an α-labeling:

I) *The Order-Size Condition:* $q \geq p - 1$.

II) *The Parity Condition:* If every vertex of G has even degree, then we must have $q \equiv 0 \pmod 4$.

III) *Wu's Condition:* If $d_1, d_2, \ldots, d_p$ is the degree sequence of G, then we must have $\gcd(d_1, d_2, \ldots, d_p, q)$ divides $q(q-1)/2$.

This last condition is unpublished and is due to Wu according to Gallian (see [11]). Numerous graphs satisfy these conditions, but do not admit α-labelings.

The conditions above can help decide which regular bipartite graphs G might admit α-labelings. For example, rK_2 admits an α-labeling if and only if $r = 1$ (since $|V(rK_2)| > |E(rK_2)| + 1$ for $r \geq 2$). Half of the 2-regular bipartite graphs cannot admit an α-labeling by the parity condition. The graph $3C_4$ is the only 2-regular bipartite graph that satisfies the parity condition and is known not to admit an α-labeling. All other 2-regular bipartite graphs that satisfy the parity condition and have at most 3 components admit α-labelings (see [2] and [8]). Moreover, Abrham and Kotzig [1] proved that rC_4 admits an α-labeling for all $r \neq 3$.

As for 3-regular graphs, it is known that $K_{3,3}$, the 3-cube, and all bipartite prisms [10, 9] and bipartite Möbius ladders [12] admit α-labelings. We note that none of the three forbidding conditions above apply to 3-regular graphs. The parity condition does not apply to 4-regular bipartite graphs; however, Wu's condition does. For example, the graph $K_{5,5} - I$, where I is a 1-factor does not admit an α-labeling.

In [6], El-Zanati and Vanden Eynden introduced the concept of a free α-labeling as follows. Let G be a graph with α-labeling h and critical value λ. We say that h is *free* if $\lambda > 2$, and neither 1 nor $\lambda - 1$ is in $h(V(G))$. They showed that if both G_1 and G_2 admit free α-labelings, then the vertex-disjoint unions of G_1 and G_2 also admits a free α-labeling.

Theorem 1 *Let G_i be a graph with a free α-labeling h_i and critical value λ_i for $i = 1, 2$. Then the vertex-disjoint union $G_1 \bigcup G_2$ is a graph with a free α-labeling h with critical value $\lambda_1 + \lambda_2 - 1$.*

We illustrate how Theorem 1 works by showing how the labeling of $G = G_1 \cup G_2$ is obtained. Let $V(G_i) = X_i \cup Y_i$, where if $v \in X_i$, then $h_i(v) \leq \lambda_i$, and if $v \in Y_i$, then $h_i(v) > \lambda_i$, $i = 1, 2$. Define h on $V(G)$ to be h_1 on X_1, $h_2 + \lambda_1 - 1$ on $X_2 \cup Y_2$, and $h_1 + |E(G_2)|$ on Y_1. Then h is an α-valuation for G with critical value $\lambda = \lambda_1 + \lambda_2 - 1 > 2$.

In this article, we show that if G is a bipartite prism, a bipartite Möbius ladder or a connected cubic bipartite graph of order at most 14, then G admits a free α-labeling. We also conjecture that every bipartite cubic graph admits a free α-labeling.

2 Additional Notation and Terminology

We denote the path with vertices $x_0, x_1, \ldots, x_k$, where x_i is adjacent to x_{i+1}, $0 \leq i \leq k-1$, by $(x_0, x_1, \ldots, x_k)$. In using this notation, we are thinking of traversing the path from x_0 to x_k so that x_0 is the first vertex, x_1 is the second vertex, and so on. Let $G_1 = (x_0, x_1, \ldots, x_j)$ and $G_2 = (y_0, y_1, \ldots, y_k)$. If G_1 and G_2 are vertex-disjoint except for $x_j = y_0$, then by $G_1 + G_2$ we mean the path $(x_0, x_1, \ldots, x_j, y_1, y_2, \ldots, y_k)$. If the only vertices they have in common are $x_0 = y_k$ and $x_j = y_0$, then by $G_1 + G_2$ we mean the cycle $(x_0, x_1, \ldots, x_j, y_1, y_2, \ldots, y_{k-1}, x_0)$.

Let $P(2k)$ be the path with $2k$ edges and $2k + 1$ vertices $0, 1, \ldots, 2k$ given by $(0, 2k, 1, 2k - 1, 2, 2k - 2, \ldots, k - 1, k + 1, k)$. Note that the set of vertices of this graph is $A \cup B$, where $A = [0, k]$, $B = [k + 1, 2k]$, and every edge joins a vertex from A to one from B. Furthermore the set of labels of the edges of $P(2k)$ is $[1, 2k]$.

Let a and b be nonnegative integers and k, d_1, and d_2 be positive integers such that $a + kd_1 < b$. Let $\hat{P}(2k, d_1, d_2, a, b)$ be the path with $2k$ edges and

$2k+1$ vertices given by $(a, b+(k-1)d_2, a+d_1, b+(k-2)d_2, a+2d_1, \ldots, a+(k-1)d_1, b, a+kd_1)$. Note that $\hat{P}(2k, 1, 1, 0, k+1)$ is the graph $P(2k)$. Note that this graph $\hat{P}(2k, d_1, d_2, a, b)$ has the following properties:

P1: $\hat{P}(2k, d_1, d_2, a, b)$ is a path with first vertex a, second vertex $b+(k-1)d_2$, and last vertex $a+kd_1$.

P2: Each edge of $\hat{P}(2k, d_1, d_2, a, b)$ joins a vertex from $A = \{a + id_1 : 0 \leq i \leq k\}$ to a vertex with a larger label from $B = \{b+id_2 : 0 \leq i \leq k-1\}$.

P3: The set of edge labels of $\hat{P}(2k, d_1, d_2, a, b)$ is $\{b-a-kd_1+i(d_1+d_2) : 0 \leq i \leq k-1\} \cup \{b-a-(k-1)d_1+i(d_1+d_2) : 0 \leq i \leq k-1\}$.

The path $\hat{P}(10, 2, 4, 14, 40)$ is shown in Figure 1 below.

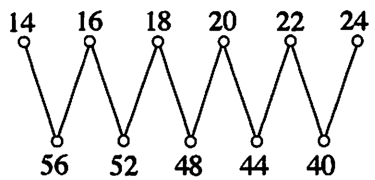


Figure 1: The path $\hat{P}(10, 2, 4, 14, 40)$.

3 Free α-labelings of Some Cubic Graphs

We will show that bipartite prisms, bipartite Möbius ladders, and bipartite cubic graphs of order at most 14 admit free α-labelings.

3.1 Free α-labelings of Bipartite Prisms

By a *prism* D_n $(n \geq 4)$ we mean the cartesian product $C_n \times P_2$ of a cycle with n vertices and a path with 2 vertices. For convenience, we let $D_n = C_n \cup C'_n \cup F$, where $C_n = (v_1, v_2, \ldots, v_n, v_1)$, $C'_n = (v'_1, v'_2, \ldots, v'_n, v'_1)$, and $F = \{\{v_i, v'_i\} : 1 \leq i \leq n\}$. We shall refer to C_n as the *outer* cycle, to C'_n as the *inner* cycle, and to F as the *spokes.* Figure 2 shows the prism D_8. We note that $D_{2n}, n \geq 2$, is necessarily bipartite with bipartition $(O \cup W', W \cup O')$, where $O = \{v_{2i-1} : 1 \leq i \leq n\}$, $W' = \{v'_{2i} : 1 \leq i \leq n\}$, $W = \{v_{2i} : 1 \leq i \leq n\}$, and $O' = \{v'_{2i-1} : 1 \leq i \leq n\}$. We will show that D_n admits a free α-labeling for all even integers $n \geq 4$.

Theorem 2 *The prism D_n admits a free α-labeling for all even $n \geq 4$.*

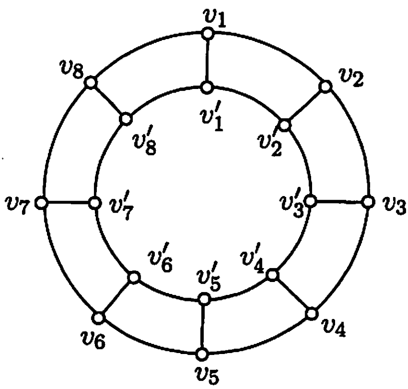


Figure 2: The prism D_8.

Proof. We separate the proof into 3 cases. In each case, we give an α-labeling f such that the complementary labeling of f is free.

Case 1 $n \equiv 0 \pmod 6$.

Let $n = 6t$. Thus, $|V(D_n)| = 12t$ and $|E(D_n)| = 18t$. A free α-labeling of D_6 is given in Table 1 (graph Bc7). For $t \geq 2$, define a one-to-one function $f : V(D_{6t}) \to [0, 18t]$ as follows:

$$
\begin{aligned}
f(v_i) &= i - 1, && v_i \in O = \{v_i : i \text{ odd},\ 1 \leq i < 6t\},\\
f(v_2) &= 18t, &&\\
f(v_i) &= 18t - 2i + 2, && v_i \in W_1 = \{v_i : i \text{ even},\ 2 < i \leq 2t\},\\
f(v_i) &= 18t - 2i, && v_i \in W_2 = \{v_i : i \text{ even},\ 2t < i \leq 4t\},\\
f(v_i) &= 18t - 2i - 2, && v_i \in W_3 = \{v_i : i \text{ even},\ 4t < i < 6t\},\\
f(v_{6t}) &= 12t - 3, &&\\
f(v'_1) &= 18t - 3, &&\\
f(v'_i) &= i - 1, && v_i \in W' = \{v_i : i \text{ even},\ 1 < i \leq 6t\},\\
f(v'_i) &= 18t - 2i + 2, && v_i \in O'_1 = \{v_i : i \text{ odd}, 1 < i \leq 2t + 1\},\\
f(v'_i) &= 18t - 2i, && v_i \in O'_2 = \{v_i : i \text{ odd}, 2t + 1 < i \leq 4t - 1\},\\
f(v'_i) &= 18t - 2i - 2, && v_i \in O'_3 = \{v_i : i \text{ odd}, 4t - 1 < i < 6t\}.
\end{aligned}
$$

Note that $W = \{v_2\} \cup W_1 \cup W_2 \cup W_3 \cup \{v_{6t}\}$ and $O' = \{v'_1\} \cup O'_1 \cup O'_2 \cup O'_3$. Thus the domain of f is indeed $V(D_{6t})$.

Next, we confirm that f is one-to-one. We compute

$$\begin{aligned}
f(O) &= \{0, 2, \ldots, 6t-2\}, \\
f(W_1) &= \{18t-6, 18t-10, \ldots, 14t+2\}, \\
f(W_2) &= \{14t-4, 14t-8, \ldots, 10t\}, \\
f(W_3) &= \{10t-6, 10t-10, \ldots, 6t+2\}, \\
f(W') &= \{1, 3, \ldots, 6t-1\}, \\
f(O_1') &= \{18t-4, 18t-8, \ldots, 14t\}, \\
f(O_2') &= \{14t-6, 14t-10, \ldots, 10t+2\}, \\
f(O_3') &= \{10t-4, 10t-8, \ldots, 6t\}.
\end{aligned}$$

Note that f is piecewise strictly increasing by 2 or strictly decreasing by 4 and that all labels are distinct. Thus f is one-to-one. Moreover, $f(O \cup W') = [0, 6t-1]$ and for $v_i \in W \cup O'$, $6t \le f(v_i) \le 18t$.

To help compute the edge labels, we will describe $f(D_{6t})$ in terms of the paths $\hat{P}(2k, d_1, d_2, a, b)$. For convenience, we will identify the vertices of C_{6t} and C'_{6t} with their labels. We have $f(C_{6t}) = G_1 + G_2 + G_3 + (6t-2, 12t-3, 0, 18t, 2)$, where

$$\begin{aligned}
G_1 &= \hat{P}(2(t-1), 2, 4, 2, 14t+2), \\
G_2 &= \hat{P}(2t, 2, 4, 2t, 10t), \\
G_3 &= \hat{P}(2(t-1), 2, 4, 4t, 6t+2).
\end{aligned}$$

By P3, the resulting edge label sets are:

$$\begin{aligned}
\bar{f}(E(G_1)) &= \{12t+2+6i : 0 \le i \le t-2\} \cup \{12t+4+6i : 0 \le i \le t-2\} \\
&= \{\ell \equiv 2 \pmod 6 : 12t+2 \le \ell \le 18t-10\} \\
&\quad \cup \{\ell \equiv 4 \pmod 6 : 12t+4 \le \ell \le 18t-8\}, \\
\bar{f}(E(G_2)) &= \{6t+6i : 0 \le i \le t-1\} \cup \{6t+2+6i : 0 \le i \le t-1\} \\
&= \{\ell \equiv 0 \pmod 6 : 6t \le \ell \le 12t-6\} \\
&\quad \cup \{\ell \equiv 2 \pmod 6 : 6t+2 \le \ell \le 12t-4\}, \\
\bar{f}(E(G_3)) &= \{4+6i : 0 \le i \le t-2\} \cup \{6+6i : 0 \le i \le t-2\} \\
&= \{\ell \equiv 4 \pmod 6 : 4 \le \ell \le 6t-8\} \\
&\quad \cup \{\ell \equiv 0 \pmod 6 : 6 \le \ell \le 6t-6\}.
\end{aligned}$$

Moreover, the edge labels $6t-1, 12t-3, 18t$, and $18t-2$ occur on the path $(6t-2, 12t-3, 0, 18t, 2)$.

Similarly, we have $f(C'_{6t}) = G'_1 + G'_2 + G'_3 + (6t-1, 18t-3, 1)$, where

$$\begin{aligned} G'_1 &= \hat{P}(2t, 2, 4, 1, 14t), \\ G'_2 &= \hat{P}(2(t-1), 2, 4, 2t+1, 10t+2), \\ G'_3 &= \hat{P}(2t, 2, 4, 4t-1, 6t). \end{aligned}$$

By P3, the resulting edge label sets are:

$$\begin{aligned} \bar{f}(E(G'_1)) &= \{12t-1+6i : 0 \le i \le t-1\} \cup \{12t+1+6i : 0 \le i \le t-1\} \\ &= \{\ell \equiv 5 \pmod 6 : 12t-1 \le \ell \le 18t-7\} \\ &\quad \cup \{\ell \equiv 1 \pmod 6 : 12t+1 \le \ell \le 18t-5\}, \\ \bar{f}(E(G'_2)) &= \{6t+3+6i : 0 \le i \le t-2\} \cup \{6t+5+6i : 0 \le i \le t-2\} \\ &= \{\ell \equiv 3 \pmod 6 : 6t+3 \le \ell \le 12t-9\} \\ &\quad \cup \{\ell \equiv 5 \pmod 6 : 6t+5 \le \ell \le 12t-7\}, \\ \bar{f}(E(G'_3)) &= \{1+6i : 0 \le i \le t-1\} \cup \{3+6i : 0 \le i \le t-1\} \\ &= \{\ell \equiv 1 \pmod 6 : 1 \le \ell \le 6t-5\} \\ &\quad \cup \{\ell \equiv 3 \pmod 6 : 3 \le \ell \le 6t-3\}. \end{aligned}$$

Moreover, the edge labels $12t-2$ and $18t-4$ occur on the path $(6t-1, 18t-3, 1)$.

For each spoke $\{v_i, v'_i\}$, the edge label is given by $f(v_i) - f(v'_i)$ if i is even and by $f(v'_i) - f(v_i)$ if i is odd. Thus the labels on the spokes are given by

$$\bar{f}(\{v_i, v'_i\}) = \begin{cases} 18t-3 & \text{for } i = 1, \\ 18t-1 & \text{for } i = 2, \\ 18t-3i+3 & \text{for } 3 \le i \le 2t+1, \\ 18t-3i+1 & \text{for } 2t+2 \le i \le 4t, \\ 18t-3i-1 & \text{for } 4t+1 \le i \le 6t-1, \\ 6t-2 & \text{for } i = 6t. \end{cases}$$

Thus the set of edge labels on the spokes is:

$$\begin{aligned} \bar{f}(E(F)) = &\{\ell \equiv 0 \pmod 3 : 12t \le \ell \le 18t-6\} \\ &\cup \{\ell \equiv 1 \pmod 3 : 6t+1 \le \ell \le 12t-5\} \\ &\cup \{\ell \equiv 2 \pmod 3 : 2 \le \ell \le 6t-4\} \\ &\cup \{6t-2, 18t-3, 18t-1\}. \end{aligned}$$

It is easy to verify that each label $\ell \in [0, 18t]$ occurs on exactly one edge in D_{6t}. Thus f is an α-labeling of D_{6t} with critical value $\lambda = 6t-1$.

Now, let f' be the complementary labeling of f. The critical value of f' is $\lambda' = 18t - (\lambda + 1) = 12t$. Thus, $\lambda' > 2$. Moreover, since neither $18t - 1$ nor $6t + 1$ is a vertex label in $f(V(D_{6t}))$, neither 1 nor $\lambda' - 1$ is a label in $f'(V(D_{6t}))$. Therefore f' is a free α-labeling of D_{6t}. Figure 3 shows the resulting free α-labeling of D_{12}.

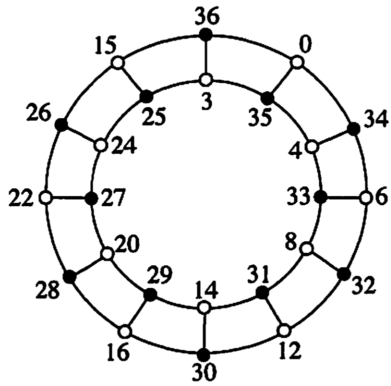


Figure 3: A free α-labeling of D_{12}.

Case 2 $n \equiv 2 \pmod 6$.
Let $n = 6t + 2$. Thus, $|V(D_n)| = 12t + 4$ and $|E(D_n)| = 18t + 6$. If $t = 1$, we let $(0, 22, 10, 20, 14, 18, 3, 24, 0)$ denote the vertex labels of the outer cycle and let $(23, 5, 21, 12, 19, 16, 17, 4, 23)$ denote the vertex labels of the inner cycle with spokes $\{0, 23\}, \{22, 5\}, \dots, \{24, 4\}$. It is easy to verify that yields a free α-labeling of D_8. For $t \geq 2$, we define a one-to-one function $f : V(D_{6t+2}) \to [0, 18t + 6]$ as follows:

$$\begin{aligned}
f(v_i) &= i - 1, & v_i &\in O = \{v_i : i \text{ odd},\ 1 \leq i < 6t + 2\},\\
f(v_2) &= 18t + 6, & &\\
f(v_i) &= 18t - 2i + 7, & v_i &\in W_1 = \{v_i : i \text{ even},\ 2 < i \leq 2t\},\\
f(v_i) &= 18t - 2i + 4, & v_i &\in W_2 = \{v_i : i \text{ even},\ 2t < i < 6t + 2\},\\
f(v_{6t+2}) &= 18t + 3, & &\\
f(v_1') &= 18t + 2, & &\\
f(v_i') &= i - 1, & v_i &\in W' = \{v_i : i \text{ even},\ 1 < i \leq 6t + 2\},\\
f(v_i') &= 18t - 2i + 7, & v_i &\in O_1' = \{v_i : i \text{ odd}, 1 < i \leq 2t + 1\},\\
f(v_i') &= 18t - 2i + 4, & v_i &\in O_2' = \{v_i : i \text{ odd}, 2t + 1 < i < 6t + 2\}.
\end{aligned}$$

Note that $W = \{v_2\} \cup W_1 \cup W_2 \cup \{v_{6t+2}\}$ and $O' = \{v_1'\} \cup O_1' \cup O_2'$. Thus the domain of f is indeed $V(D_{6t+2})$.

Next, we confirm that f is one-to-one. We compute

$$\begin{aligned} f(O) &= \{0, 2, \ldots, 6t\}, \\ f(W_1) &= \{18t-1, 18t-5, \ldots, 14t+7\}, \\ f(W_2) &= \{14t, 14t-4, \ldots, 6t+4\}, \\ f(W') &= \{1, 3, \ldots, 6t+1\}, \\ f(O_1') &= \{18t+1, 18t-3, \ldots, 14t+5\}, \\ f(O_2') &= \{14t-2, 14t-6, \ldots, 6t+2\}. \end{aligned}$$

Note that f is piecewise strictly increasing by 2 or strictly decreasing by 4 and that all labels are distinct. Thus f is one-to-one. Moreover, $f(O \cup W') = [0, 6t+1]$ and for $v_i \in W \cup O'$, $6t+2 \leq f(v_i) \leq 18t+6$.

To help compute the edge labels, we will describe $f(D_{6t+2})$ in terms of the paths $\hat{P}(2k, d_1, d_2, a, b)$. For convenience, we will identify the vertices of C_{6t+2} and C'_{6t+2} with their labels. We have $f(C_{6t+2}) = G_1 + G_2 + (6t, 18t+3, 0, 18t+6, 2)$, where

$$\begin{aligned} G_1 &= \hat{P}(2(t-1), 2, 4, 2, 14t+7), \\ G_2 &= \hat{P}(2(2t), 2, 4, 2t, 6t+4). \end{aligned}$$

By P3, the resulting edge label sets are:

$$\begin{aligned} \bar{f}(E(G_1)) &= \{12t+7+6i : 0 \leq i \leq t-2\} \cup \{12t+9+6i : 0 \leq i \leq t-2\} \\ &= \{\ell \equiv 1 \pmod 6 : 12t+7 \leq \ell \leq 18t-5\} \\ &\quad \cup \{\ell \equiv 3 \pmod 6 : 12t+9 \leq \ell \leq 18t-3\}, \\ \bar{f}(E(G_2)) &= \{4+6i : 0 \leq i \leq 2t-1\} \cup \{6+6i : 0 \leq i \leq 2t-1\} \\ &= \{\ell \equiv 4 \pmod 6 : 4 \leq \ell \leq 12t-2\} \\ &\quad \cup \{\ell \equiv 0 \pmod 6 : 6 \leq \ell \leq 12t\}. \end{aligned}$$

Moreover, the edge labels $12t+3, 18t+3, 18t+6$ and $18t+4$ occur on the path $(6t, 18t+3, 0, 18t+6, 2)$.

Similarly, we have $f(C'_{6t+2}) = G'_1 + G'_2 + (6t+1, 18t+2, 1)$, where

$$\begin{aligned} G'_1 &= \hat{P}(2t, 2, 4, 1, 14t+5), \\ G'_2 &= \hat{P}(2(2t), 2, 4, 2t+1, 6t+2). \end{aligned}$$

By P3, the resulting edge label sets are:

$$\begin{aligned}\bar{f}(E(G_1')) &= \{12t+4+6i : 0 \le i \le t-1\} \cup \{12t+6+6i : 0 \le i \le t-1\} \\ &= \{\ell \equiv 4 \pmod 6 : 12t+4 \le \ell \le 18t-2\} \\ &\quad \cup \{\ell \equiv 0 \pmod 6 : 12t+6 \le \ell \le 18t\}, \\ \bar{f}(E(G_2')) &= \{1+6i : 0 \le i \le 2t-1\} \cup \{3+6i : 0 \le i \le 2t-1\} \\ &= \{\ell \equiv 1 \pmod 6 : 1 \le \ell \le 12t-5\} \\ &\quad \cup \{\ell \equiv 3 \pmod 6 : 3 \le \ell \le 12t-3\}.\end{aligned}$$

Moreover, the edge labels $12t+1$, and $18t+1$ occur on the path $(6t+1, 18t+2, 1)$.

For each spoke $\{v_i, v_i'\}$, the edge label is given by $f(v_i) - f(v_i')$ if i is even and by $f(v_i') - f(v_i)$ if i is odd. Thus the labels on the spokes are given by

$$\bar{f}(\{v_i, v_i'\}) = \begin{cases} 18t+2 & \text{for } i = 1, \\ 18t+5 & \text{for } i = 2, \\ 18t-3i+8 & \text{for } 3 \le i \le 2t+1, \\ 18t-3i+5 & \text{for } 2t+2 \le i \le 6t+1, \\ 12t+2 & \text{for } i = 6t+2. \end{cases}$$

Thus the set of edge labels on the spokes is:

$$\begin{aligned}\bar{f}(E(F)) &= \{\ell \equiv 2 \pmod 3 : 12t+5 \le \ell \le 18t-1\} \\ &\quad \cup \{\ell \equiv 2 \pmod 3 : 2 \le \ell \le 12t-1\} \\ &\quad \cup \{12t+2, 18t+2, 18t+5\}.\end{aligned}$$

It is easy to verify that each label $\ell \in [0, 18t+6]$ occurs on exactly one edge in D_{6t+2}. Thus f is an α-labeling of D_{6t+2}. Although f is not free, it is easy to check that its complementary labeling f' is free. Figure 4 shows the resulting free α-labeling of D_{14}.

Case 3 $n \equiv 4 \pmod 6$.
Let $n = 6t-2$. Thus, $|V(D_n)| = 12t-4$ and $|E(D_n)| = 18t-6$. A free α-labeling of D_4 is given in Table 1 (graph Bc2). For $t \ge 2$, define a

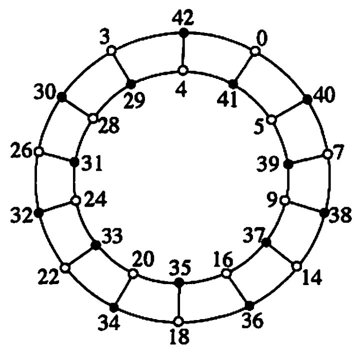


Figure 4: A free α-labeling of D_{14}.

one-to-one function $f : V(D_{6t-2}) \to [0, 18t-6]$ as follows:

$$\begin{aligned}
f(v_i) &= i-1, & v_i &\in O = \{v_i : i \text{ odd},\ 1 \le i < 6t-2\},\\
f(v_i) &= 18t-2i-5, & v_i &\in W_1 = \{v_i : i \text{ even},\ 1 < i \le 2t-2\},\\
f(v_i) &= 18t-2i-8, & v_i &\in W_2 = \{v_i : i \text{ even},\ 2t-2 < i < 6t-2\},\\
f(v_{6t-2}) &= 18t-8, & &\\
f(v_1') &= 18t-6, & &\\
f(v_i') &= i-1, & v_i &\in W' = \{v_i : i \text{ even},\ 1 < i \le 6t-2\},\\
f(v_i') &= 18t-2i-5, & v_i &\in O_1' = \{v_i : i \text{ odd}, 1 < i \le 2t-1\},\\
f(v_i') &= 18t-2i-8, & v_i &\in O_2' = \{v_i : i \text{ odd}, 2t-1 < i < 6t-2\}.
\end{aligned}$$

Note that $W = W_1 \cup W_2 \cup \{v_{6t-2}\}$ and $O' = \{v_1'\} \cup O_1' \cup O_2'$. Thus the domain of f is indeed $V(D_{6t-2})$.

Next, we confirm that f is one-to-one. We compute

$$\begin{aligned}
f(O) &= \{0, 2, \ldots, 6t-4\},\\
f(W_1) &= \{18t-9, 18t-13, \ldots, 14t-1\},\\
f(W_2) &= \{14t-4, 14t-8, \ldots, 6t\},\\
f(W') &= \{1, 3, \ldots, 6t-3\},\\
f(O_1') &= \{18t-11, 18t-15, \ldots, 14t-3\},\\
f(O_2') &= \{14t-10, 14t-14, \ldots, 6t-2\}.
\end{aligned}$$

Note that f is piecewise strictly increasing by 2 or strictly decreasing by 4 and that all labels are distinct. Thus f is one-to-one. Moreover, $f(O \cup W') = [0, 6t-3]$ and for $v_i \in W \cup O'$, $6t-2 \le f(v_i) \le 18t-6$.

To help compute the edge labels, we will describe $f(D_{6t-2})$ in terms of the paths $\hat{P}(2k, d_1, d_2, a, b)$. For convenience, we will identify the vertices of C_{6t-2} and C'_{6t-2} with their labels. We have $f(C_{6t-2}) = G_1 + G_2 + (6t-4, 18t-8, 0)$, where

$$G_1 = \hat{P}(2(t-1), 2, 4, 0, 14t-1),$$
$$G_2 = \hat{P}(2(2t-1), 2, 4, 2t-2, 6t).$$

By P3, the resulting edge label sets are:

$$\begin{aligned}\bar{f}(E(G_1)) &= \{12t+1+6i : 0 \le i \le t-2\} \cup \{12t+3+6i : 0 \le i \le t-2\} \\ &= \{\ell \equiv 1 \pmod 6 : 12t+1 \le \ell \le 18t-11\} \\ &\quad \cup \{\ell \equiv 3 \pmod 6 : 12t+3 \le \ell \le 18t-9\}, \\ \bar{f}(E(G_2)) &= \{4+6i : 0 \le i \le 2t-2\} \cup \{6+6i : 0 \le i \le 2t-2\} \\ &= \{\ell \equiv 4 \pmod 6 : 4 \le \ell \le 12t-8\} \\ &\quad \cup \{\ell \equiv 0 \pmod 6 : 6 \le \ell \le 12t-6\}.\end{aligned}$$

Moreover, the edge labels $12t-4$ and $18t-8$ occur on the path $(6t-4, 18t-8, 0)$.

Similarly, we have $f(C'_{6t-2}) = G'_1 + G'_2 + (6t-3, 18t-6, 1)$, where

$$G'_1 = \hat{P}(2(t-1), 2, 4, 1, 14t-3),$$
$$G'_2 = \hat{P}(2(2t-1), 2, 4, 2t-1, 6t-2).$$

By P3, the resulting edge label sets are:

$$\begin{aligned}\bar{f}(E(G'_1)) &= \{12t-2+6i : 0 \le i \le t-2\} \cup \{12t+6i : 0 \le i \le t-2\} \\ &= \{\ell \equiv 2 \pmod 6 : 12t-2 \le \ell \le 18t-14\} \\ &\quad \cup \{\ell \equiv 0 \pmod 6 : 12t \le \ell \le 18t-12\}, \\ \bar{f}(E(G'_2)) &= \{1+6i : 0 \le i \le 2t-2\} \cup \{3+6i : 0 \le i \le 2t-2\} \\ &= \{\ell \equiv 1 \pmod 6 : 1 \le \ell \le 12t-11\} \\ &\quad \cup \{\ell \equiv 3 \pmod 6 : 3 \le \ell \le 12t-9\}.\end{aligned}$$

Moreover, the edge labels $12t-3$ and $18t-7$ occur on the path $(6t-3, 18t-6, 1)$.

For each spoke $\{v_i, v'_i\}$, the edge label is given by $f(v_i) - f(v'_i)$ if i is even and by $f(v'_i) - f(v_i)$ if i is odd. Thus the labels on the spokes are given by

$$\bar{f}(\{v_i, v'_i\}) = \begin{cases} 18t-6 & \text{for } i = 1, \\ 18t-3i-4 & \text{for } 2 \le i \le 2t-1, \\ 18t-3i-7 & \text{for } 2t \le i \le 6t-3, \\ 12t-5 & \text{for } i = 6t-2. \end{cases}$$

Thus the set of edge labels on the spokes is:

$$\begin{aligned}\bar{f}(E(F)) = \{\ell \equiv 2 \pmod 3) : 12t-1 \le \ell \le 18t-10\} \\ \cup \{\ell \equiv 2 \pmod 3) : 2 \le \ell \le 12t-7\} \\ \cup \{12t-5, 18t-6\}.\end{aligned}$$

It is easy to verify that each label $\ell \in [0, 18t-6]$ occurs on exactly one edge in D_{6t-2}. Thus f is an α-labeling of D_{6t-2}. Although f is not free, it is easy to check that its complementary labeling f' is free. Figure 5 shows the resulting free α-labeling of D_{10}. ∎

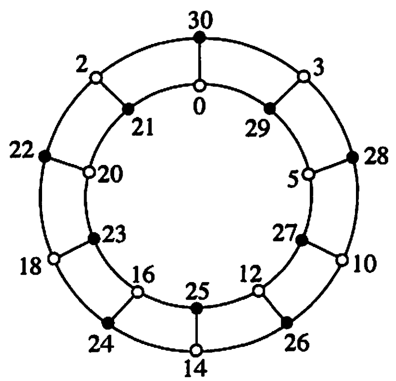


Figure 5: A free α-labeling of D_{10}.

3.2 Free α-labelings of Bipartite Möbius Ladders

For $n \ge 3$, let $v_1, v_2, \ldots, v_n$ and $v'_1, v'_2, \ldots, v'_n$ denote the consecutive vertices of two disjoint paths with n vertices. We obtain the *Möbius ladder* M_n by joining v_i to v'_i for $i = 1, 2, \ldots, n$ and by joining v_1 to v'_n and v_n to v'_1. For convenience, we let $M_n = P_n \cup P'_n \cup F \cup H$, where $P_n = (v_1, v_2, \ldots, v_n)$, $P'_n = (v'_1, v'_2, \ldots, v'_n)$, $F = \{\{v_i, v'_i\} : 1 \le i \le n\}$ and $H = \{\{v_1, v'_n\}, \{v_n, v'_1\}\}$. We shall refer to P_n as the *outer* path, to P'_n as the *inner* path, and to F as the *spokes*. Figure 6 shows the Möbius ladder M_9. We note that $M_{2n+1}, n \ge 1$, is necessarily bipartite with bipartition $(O \cup W', W \cup O')$, where $O = \{v_{2i+1} : 1 \le i \le n\}$, $W' = \{v'_{2i} : 1 \le i \le n\}$, $W = \{v_{2i} : 1 \le i \le n\}$, and $O' = \{v'_{2i+1} : 1 \le i \le n\}$). We will show that M_n admits a free α-labeling for all odd integers $n \ge 3$.

Theorem 3 *The Möbius ladder M_n admits a free α-labeling for all odd $n \ge 3$.*

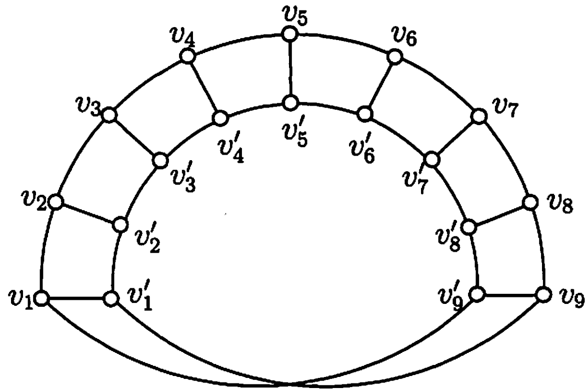


Figure 6: The Möbius ladder M_9.

Proof. We separate the proof into 3 cases. In each case, we give an α-labeling f such that the complementary labeling of f is free.

Case 1 $n \equiv 1 \pmod 6$.

Let $n = 6t+1$. Thus, $|V(M_n)| = 12t+2$ and $|E(M_n)| = 18t+3$, where $t \geq 1$. Define a one-to-one function $f : V(M_{6t+1}) \to [0, 18t+3]$ as follows:

$$\begin{aligned}
f(v_i) &= i-1, & v_i \in O = \{v_i : i \text{ odd},\ 1 \leq i \leq 6t+1\},\\
f(v_i) &= 18t - \tfrac{i}{2} + 1, & v_i \in W_1 = \{v_i : i \text{ even},\ 2 \leq i \leq 4t-2\},\\
f(v_i) &= 18t - \tfrac{i}{2} - 2, & v_i \in W_2 = \{v_i : i \text{ even},\ 4t-2 < i < 6t\},\\
f(v_{6t}) &= 6t+1, &\\
f(v_1') &= 18t+3, &\\
f(v_i') &= i-1, & v_i \in W' = \{v_i : i \text{ even},\ 1 < i \leq 6t\},\\
f(v_i') &= 9t - \tfrac{i-1}{2} + 2, & v_i \in O_1' = \{v_i : i \text{ odd}, 1 < i < 6t+1\},\\
f(v_{6t+1}') &= 18t+1. &
\end{aligned}$$

Note that $W = W_1 \cup W_2 \cup \{v_{6t}\}$ and $O' = \{v_1'\} \cup O_1' \cup \{v_{6t+1}'\}$. Thus the domain of f is indeed $V(M_{6t+1})$.

Next, we confirm that f is one-to-one. We compute

$$\begin{aligned}
f(O) &= \{0, 2, \ldots, 6t\},\\
f(W_1) &= \{18t, 18t-1, \ldots, 16t+2\},\\
f(W_2) &= \{16t-2, 16t-3, \ldots, 15t-1\},\\
f(W') &= \{1, 3, \ldots, 6t-1\},\\
f(O_1') &= \{9t+1, 9t, \ldots, 6t+3\}.
\end{aligned}$$

Note that f is piecewise strictly increasing by 2 or strictly decreasing by 1 and that all labels are distinct. Thus f is one-to-one. Moreover, $f(O \cup W') = [0, 6t]$ and for $v_i \in W \cup O'$, $6t + 1 \leq f(v_i) \leq 18t + 3$.
To help compute the edge labels, we will describe $f(M_{6t+1})$ in terms of the paths $\hat{P}(2k, d_1, d_2, a, b)$. For convenience, we will identify the vertices of P_{6t+1} and P'_{6t+1} with their labels. We have $f(P_{6t+1}) = G_1 + G_2 + (6t - 2, 6t + 1, 6t)$, where

$$G_1 = \hat{P}(2(2t-1), 2, 1, 0, 16t + 2),$$
$$G_2 = \hat{P}(2t, 2, 1, 4t - 2, 15t - 1).$$

By P3, the resulting edge label sets are:

$$\begin{aligned}
\bar{f}(E(G_1)) &= \{12t + 4 + 3i : 0 \leq i \leq 2t - 2\} \cup \{12t + 6 + 3i : 0 \leq i \leq 2t - 2\} \\
&= \{\ell \equiv 1 \pmod 3 : 12t + 4 \leq \ell \leq 18t - 2\} \\
&\quad \cup \{\ell \equiv 0 \pmod 3 : 12t + 6 \leq \ell \leq 18t\}, \\
\bar{f}(E(G_2)) &= \{9t + 1 + 3i : 0 \leq i \leq t - 1\} \cup \{9t + 3 + 3i : 0 \leq i \leq t - 1\} \\
&= \{\ell \equiv 1 \pmod 3 : 9t + 1 \leq \ell \leq 12t - 2\} \\
&\quad \cup \{\ell \equiv 0 \pmod 3 : 9t + 3 \leq \ell \leq 12t\}.
\end{aligned}$$

Moreover, the edge labels 3 and 1 occur on the path $(6t - 2, 6t + 1, 6t)$.

Similarly, we have $f(P'_{6t+1}) = (18t + 3, 1) + G'_1 + (6t - 1, 18t + 1)$, where

$$G'_1 = \hat{P}(2(3t-1), 2, 1, 1, 6t + 3).$$

By P3, the resulting edge label sets are:

$$\begin{aligned}
\bar{f}(E(G'_1)) &= \{3i + 4 : 0 \leq i \leq 3t - 2\} \cup \{3i + 6 : 0 \leq i \leq 3t - 2\} \\
&= \{\ell \equiv 1 \pmod 3 : 4 \leq \ell \leq 9t - 2\} \\
&\quad \cup \{\ell \equiv 0 \pmod 3 : 6 \leq \ell \leq 9t\}.
\end{aligned}$$

Moreover, the edge labels $18t + 2$, and $12t + 2$ occur on the edge $\{18t + 3, 1\}$ and $\{6t - 1, 18t + 1\}$.

For each spoke $\{v_i, v'_i\}$, the edge label is given by $f(v_i) - f(v'_i)$ if i is even and by $f(v'_i) - f(v_i)$ if i is odd. Thus the labels on the spokes are given by

$$\bar{f}(\{v_i, v'_i\}) = \begin{cases} 18t + 3 & \text{for } i = 1, \\ 18t - \frac{3i}{2} + 2 & \text{for } i \text{ even}, 2 \leq i \leq 4t - 2, \\ 18t - \frac{3i}{2} - 1 & \text{for } i \text{ even}, 4t \leq i \leq 6t - 2, \\ 9t - \frac{3(i-1)}{2} + 2 & \text{for } i \text{ odd}, 3 \leq i \leq 6t - 1, \\ 2 & \text{for } i = 6t, \\ 12t + 1 & \text{for } i = 6t + 1. \end{cases}$$

Thus the set of edge labels on the spokes is:

$$\begin{aligned}\bar{f}(E(F)) = &\{\ell \equiv 2 \pmod 3 : 12t+5 \le \ell \le 18t-1\} \\ &\cup \{\ell \equiv 2 \pmod 3 : 9t+2 \le \ell \le 12t-1\} \\ &\cup \{\ell \equiv 2 \pmod 3 : 5 \le \ell \le 9t-1\} \\ &\cup \{2, 12t+1, 18t+3\}.\end{aligned}$$

Moreover, the edge labels $18t+1$, and $12t+3$ occur on the edge $\{v_1, v'_{6t+1}\}$ and $\{v'_1, v_{6t+1}\}$.

It is easy to verify that each label $\ell \in [0, 18t+3]$ occurs on exactly one edge in M_{6t+1}. Thus f is an α-labeling of M_{6t+1}. Although f is not free, it is easy to check that its complementary labeling f' is free. Figure 7 shows the resulting free α-labeling of M_7.

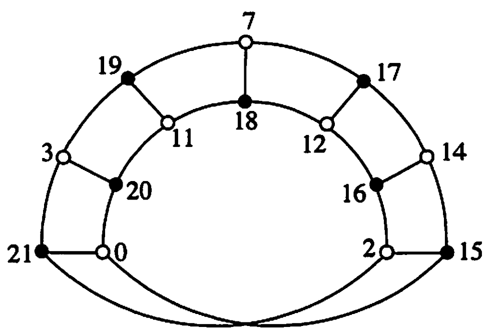


Figure 7: A free α-labeling of M_7.

Case 2 $n \equiv 3 \pmod 6$.
Let $n = 6t-3$. Thus, $|V(M_n)| = 12t-6$ and $|E(M_n)| = 18t-9$. A free α-labeling of M_3 (which is isomorphic to $K_{3,3}$) is given in Table 1 (graph Bc1). For $t \ge 2$, define a one-to-one function $f : V(M_{6t-3}) \to [0, 18t-9]$

as follows:

$$
\begin{aligned}
f(v_i) &= i-1, & v_i \in O &= \{v_i : i \text{ odd},\ 1 \le i \le 6t-3\},\\
f(v_i) &= 18t - \tfrac{i}{2} - 10, & v_i \in W_1 &= \{v_i : i \text{ even},\ 2 \le i < 4t-2\},\\
f(v_i) &= 18t - \tfrac{i}{2} - 12, & v_i \in W_2 &= \{v_i : i \text{ even},\ 4t-2 \le i < 6t-4\},\\
f(v_{6t-4}) &= 6t-3, & &\\
f(v'_1) &= 18t-9, & &\\
f(v'_i) &= i-1, & v_i \in W' &= \{v_i : i \text{ even},\ 2 \le i \le 6t-4\},\\
f(v'_i) &= 9t - \tfrac{i-1}{2} - 2, & v_i \in O'_1 &= \{v_i : i \text{ odd}, 1 < i \le 2t-1\},\\
f(v'_i) &= 9t - \tfrac{i-1}{2} - 4, & v_i \in O'_2 &= \{v_i : i \text{ odd}, 2t-1 < i < 6t-3\},\\
f(v'_{6t-3}) &= 12t-6. & &
\end{aligned}
$$

Note that $W = W_1 \cup W_2 \cup \{v_{6t-4}\}$ and $O' = \{v'_1\} \cup O'_1 \cup O'_2 \cup \{v'_{6t-3}\}$. Thus the domain of f is indeed $V(M_{6t-3})$.

Next, we confirm that f is one-to-one. We compute

$$
\begin{aligned}
f(O) &= \{0, 2, \ldots, 6t-4\},\\
f(W_1) &= \{18t-11, 18t-12, \ldots, 16t-8\},\\
f(W_2) &= \{16t-11, 16t-12, \ldots, 15t-9\},\\
f(W') &= \{1, 3, \ldots, 6t-5\},\\
f(O'_1) &= \{9t-3, 9t-4, \ldots, 8t-1\},\\
f(O'_2) &= \{8t-4, 8t-5, \ldots, 6t-1\}.
\end{aligned}
$$

Note that f is piecewise strictly increasing by 2 or strictly decreasing by 1 and that all labels are distinct. Thus f is one-to-one. Moreover, $f(O \cup W') = [0, 6t-4]$ and for $v_i \in W \cup O'$, $6t-3 \le f(v_i) \le 18t-9$.

To help compute the edge labels, we will describe $f(M_{6t-3})$ in terms of the paths $\hat{P}(2k, d_1, d_2, a, b)$. For convenience, we will identify the vertices of P_{6t-3} and P'_{6t-3} with their labels. We have $f(P_{6t-3}) = G_1 + G_2 + (6t-6, 6t-3, 6t-4)$, where

$$
\begin{aligned}
G_1 &= \hat{P}(2(2t-2), 2, 1, 0, 16t-8),\\
G_2 &= \hat{P}(2(t-1), 2, 1, 4t-4, 15t-9).
\end{aligned}
$$

By P3, the resulting edge label sets are:

$$\begin{aligned}\bar{f}(E(G_1)) &= \{12t-4+3i : 0 \le i \le 2t-3\} \\ &\quad \cup \{12t-2+3i : 0 \le i \le 2t-3\} \\ &= \{\ell \equiv 2 \pmod 3 : 12t-4 \le \ell \le 18t-13\} \\ &\quad \cup \{\ell \equiv 1 \pmod 3 : 12t-2 \le \ell \le 18t-11\}, \\ \bar{f}(E(G_2)) &= \{9t-3+3i : 0 \le i \le t-2\} \cup \{9t-1+3i : 0 \le i \le t-2\} \\ &= \{\ell \equiv 0 \pmod 3 : 9t-3 \le \ell \le 12t-9\} \\ &\quad \cup \{\ell \equiv 2 \pmod 3 : 9t-1 \le \ell \le 12t-7\}.\end{aligned}$$

Moreover, the edge labels 3 and 1 occur on the path $(6t-6, 6t-3, 6t-4)$.

Similarly, we have $f(P'_{6t-3}) = \{18t-9, 1\} + G'_1 + G'_2 + \{6t-5, 12t-6\}$, where

$$G'_1 = \hat{P}(2(t-1), 2, 1, 1, 8t-1),$$
$$G'_2 = \hat{P}(2(2t-2), 2, 1, 2t-1, 6t-1).$$

By P3, the resulting edge label sets are:

$$\begin{aligned}\bar{f}(E(G'_1)) &= \{6t+3i : 0 \le i \le t-2\} \cup \{6t+2+3i : 0 \le i \le t-2\} \\ &= \{\ell \equiv 0 \pmod 3 : 6t \le \ell \le 9t-6\} \\ &\quad \cup \{\ell \equiv 2 \pmod 3 : 6t+2 \le \ell \le 9t-4\}, \\ \bar{f}(E(G'_2)) &= \{4+3i : 0 \le i \le 2t-3\} \cup \{6+3i : 0 \le i \le 2t-3\} \\ &= \{\ell \equiv 1 \pmod 3 : 4 \le \ell \le 6t-5\} \\ &\quad \cup \{\ell \equiv 0 \pmod 3 : 6 \le \ell \le 6t-3\}.\end{aligned}$$

Moreover, the edge labels $18t-10$, and $6t-1$ occur on the edge $\{18t-9, 1\}$ and $\{6t-5, 12t-6\}$.

For each spoke $\{v_i, v'_i\}$, the edge label is given by $f(v_i) - f(v'_i)$ if i is even and by $f(v'_i) - f(v_i)$ if i is odd. Thus the labels on the spokes are given by

$$\bar{f}(\{v_i, v'_i\}) = \begin{cases} 18t-9 & \text{for } i=1, \\ 18t-\frac{3i}{2}-9 & \text{for } i \text{ even, } 2 \le i \le 4t-4, \\ 18t-\frac{3i}{2}-11 & \text{for } i \text{ even, } 4t-2 \le i \le 6t-6, \\ 9t-\frac{3(i-1)}{2}-2 & \text{for } i \text{ odd, } 3 \le i \le 2t-1, \\ 9t-\frac{3(i-1)}{2}-4 & \text{for } i \text{ odd, } 2t+1 \le i \le 6t-5, \\ 2 & \text{for } i=6t-4, \\ 6t-2 & \text{for } i=6t-3. \end{cases}$$

Thus the set of edge labels on the spokes is:

$$\begin{aligned}\bar{f}(E(F)) = &\{\ell \equiv 0 \pmod 3 : 12t-3 \le \ell \le 18t-12\}\\ &\cup \{\ell \equiv 1 \pmod 3 : 9t-2 \le \ell \le 12t-8\}\\ &\cup \{\ell \equiv 1 \pmod 3 : 6t+1 \le \ell \le 9t-5\}\\ &\cup \{\ell \equiv 2 \pmod 3 : 5 \le \ell \le 6t-4\}\\ &\cup \{2, 6t-2, 18t-9\}.\end{aligned}$$

Moreover, the edge labels $12t-6$, and $12t-5$ occur on the edge $\{v_1, v'_{6t-3}\}$ and $\{v'_1, v_{6t-3}\}$.

It is easy to verify that each label $\ell \in [0, 18t-9]$ occurs on exactly one edge in M_{6t-3}. Thus f is an α-labeling of M_{6t-3}. Although f is not free, it is easy to check that its complementary labeling f' is free. Figure 8 shows the resulting free α-labeling of M_9.

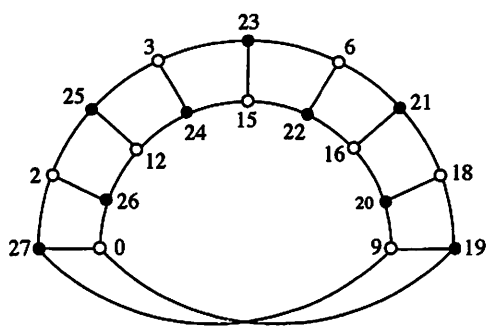


Figure 8: A free α-labeling of M_9.

Case 3 $n \equiv 5 \pmod 6$.
Let $n = 6t-1$. Thus, $|V(M_n)| = 12t-2$ and $|E(M_n)| = 18t-3$. A free α-labeling of M_5 is given in Table 1 (graph Bc3). For $t \ge 2$, define a

one-to-one function $f : V(M_{6t-1}) \to [0, 18t-3]$ as follows:

$$\begin{aligned}
f(v_i) &= i-1, & v_i \in O &= \{v_i : i \text{ odd},\ 1 \le i \le 6t-1\},\\
f(v_2) &= 18t-3, & &\\
f(v_i) &= 9t - \tfrac{i}{2} + 1, & v_i \in W_1 &= \{v_i : i \text{ even},\ 4 \le i \le 6t-4\},\\
f(v_{6t-2}) &= 6t-1, & &\\
f(v_i') &= 18t - \tfrac{i-1}{2} - 7, & v_i \in O_1' &= \{v_i : i \text{ odd}, 1 \le i \le 4t-3\},\\
f(v_i') &= 18t - \tfrac{i-1}{2} - 10, & v_i \in O_2' &= \{v_i : i \text{ odd}, 4t-3 < i \le 6t-5\},\\
f(v_i') &= i-1, & v_i \in W' &= \{v_i : i \text{ even},\ 2 \le i \le 6t-2\},\\
f(v_{6t-3}') &= 6t+1, & &\\
f(v_{6t-1}') &= 18t-6. & &
\end{aligned}$$

Note that $W = \{v_2\} \cup W_1 \cup \{v_{6t-2}\}$ and $O' = O_1' \cup O_2' \cup \{v_{6t-3}'\} \cup \{v_{6t-1}'\}$. Thus the domain of f is indeed $V(M_{6t-1})$.

Next, we confirm that f is one-to-one. We compute

$$\begin{aligned}
f(O) &= \{0, 2, \ldots, 6t-2\},\\
f(W_1) &= \{9t-1, 9t-2, \ldots, 6t+3\},\\
f(W') &= \{1, 3, \ldots, 6t-3\},\\
f(O_1') &= \{18t-7, 18t-8, \ldots, 16t-5\},\\
f(O_2') &= \{16t-9, 16t-10, \ldots, 15t-7\}.
\end{aligned}$$

Note that f is piecewise strictly increasing by 2 or strictly decreasing by 1 and that all labels are distinct. Thus f is one-to-one. Moreover, $f(O \cup W') = [0, 6t-2]$ and for $v_i \in W \cup O'$, $6t-1 \le f(v_i) \le 18t-3$.

To help compute the edge labels, we will describe $f(M_{6t-1})$ in terms of the paths $\hat{P}(2k, d_1, d_2, a, b)$. For convenience, we will identify the vertices of P_{6t-1} and P_{6t-1}' with their labels. We have $f(P_{6t-1}) = (0, 18t-3, 2) + G_1 + (6t-4, 6t-1, 6t-2)$, where

$$G_1 = \hat{P}(2(3t-3), 2, 1, 2, 6t+3).$$

By P3, the resulting edge label sets are:

$$\begin{aligned}
\bar{f}(E(G_1)) &= \{7 + 3i : 0 \le i \le 3t-4\} \cup \{9 + 3i : 0 \le i \le 3t-4\}\\
&= \{\ell \equiv 1 \pmod 3) : 7 \le \ell \le 9t-5\}\\
&\quad \cup \{\ell \equiv 0 \pmod 3) : 9 \le \ell \le 9t-3\}.
\end{aligned}$$

Moreover, the edge labels $18t-3$ and $18t-5$ occur on the path $(0, 18t-3, 2)$ and the edge labels 3 and 1 occur on the path $(6t-4, 6t-1, 6t-2)$.

Similarly, we have $f(P'_{6t-1}) = (18t-7, 1) + G'_1 + G'_2 + (6t-5, 6t+1, 6t-3, 18t-6)$, where

$$G'_1 = \hat{P}(2(2t-2), 2, 1, 1, 16t-5),$$
$$G'_2 = \hat{P}(2(t-1), 2, 1, 4t-3, 15t-7).$$

By P3, the resulting edge label sets are:

$$\begin{aligned}\bar{f}(E(G'_1)) &= \{12t-2+3i : 0 \le i \le 2t-3\} \cup \{12t+3i : 0 \le i \le 2t-3\} \\ &= \{\ell \equiv 1 \pmod 3 : 12t-2 \le \ell \le 18t-11\} \\ &\quad \cup \{\ell \equiv 0 \pmod 3 : 12t \le \ell \le 18t-9\}, \\ \bar{f}(E(G'_2)) &= \{9t-2+3i : 0 \le i \le t-2\} \cup \{9t+3i : 0 \le i \le t-2\} \\ &= \{\ell \equiv 1 \pmod 3 : 9t-2 \le \ell \le 12t-8\} \\ &\quad \cup \{\ell \equiv 0 \pmod 3 : 9t \le \ell \le 12t-6\}.\end{aligned}$$

Moreover, the edge labels $18t-8$ occur on the edge $\{18t-9, 1\}$ and the edge labels $6, 4, 12t-3$ occur on path $(6t-5, 6t+1, 6t-3, 18t-6)$.

For each spoke $\{v_i, v'_i\}$, the edge label is given by $f(v_i) - f(v'_i)$ if i is even and by $f(v'_i) - f(v_i)$ if i is odd. Thus the labels on the spokes are given by

$$\bar{f}(\{v_i, v'_i\}) = \begin{cases} 18t - \frac{3(i-1)}{2} - 7 & \text{for } i \text{ odd}, 1 \le i \le 4t-3, \\ 18t - \frac{3(i-1)}{2} - 10 & \text{for } i \text{ odd}, 4t-1 \le i \le 6t-5, \\ 5 & \text{for } i = 6t-3, \\ 12t-4 & \text{for } i = 6t-1, \\ 18t-4 & \text{for } i = 2, \\ 9t - \frac{3i}{2} + 2 & \text{for } i \text{ even}, 4 \le i \le 6t-4, \\ 2 & \text{for } i = 6t-2. \end{cases}$$

Thus the set of edge labels on the spokes is:

$$\begin{aligned}\bar{f}(E(F)) = \{\ell \equiv 2 \pmod 3 &: 12t-1 \le \ell \le 18t-7\} \\ \cup \{\ell \equiv 2 \pmod 3 &: 9t-1 \le \ell \le 12t-7\} \\ \cup \{\ell \equiv 2 \pmod 3 &: 8 \le \ell \le 9t-4\} \cup \{2, 5, 12t-4, 18t-4\}.\end{aligned}$$

Moreover, the edge labels $18t-6$, and $12t-5$ occur on the edge $\{v_1, v'_{6t-1}\}$ and $\{v'_1, v_{6t-1}\}$.

It is easy to verify that each label $\ell \in [0, 18t-3]$ occurs on exactly one edge in M_{6t-1}. Thus f is an α-labeling of M_{6t-1}. Although f is not free, it is easy to check that its complementary labeling f' is free. Figure 9 shows the resulting free α-labeling of M_{11}. ∎

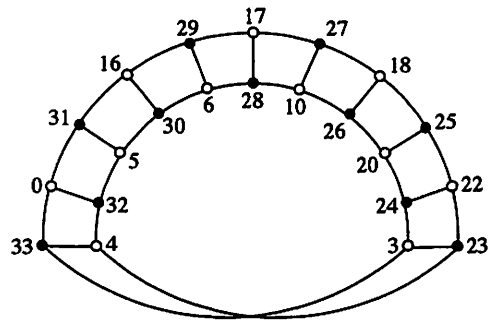


Figure 9: A free α-labeling of M_{11}.

3.3 Free α-labelings of Bipartite Cubic Graphs of Small Order

According to the book *An Atlas of Graphs* [13], there are 22 connected bipartite cubic graphs of order at most 14. Each of these graphs admits a free α-labeling (see Table 1). We referenced these graphs in the same way they are referenced in [13]. Thus we have the following.

Theorem 4 *Every bipartite cubic graph of order at most 14 admits a free α-labeling.*

Corollary 5 *Let G be a cubic bipartite graph such that each component of G is either a prism, a Möbius ladder, or has order at most 14. Then G admits a free α-labeling.*

4 Concluding Remarks

Based on our investigation, we believe that all bipartite cubic graphs admit free α-labelings.

Conjecture 6 *Every bipartite cubic graph admits a free α-labeling.*

Table 1: Connected bipartite cubic graphs: 6–14 vertices

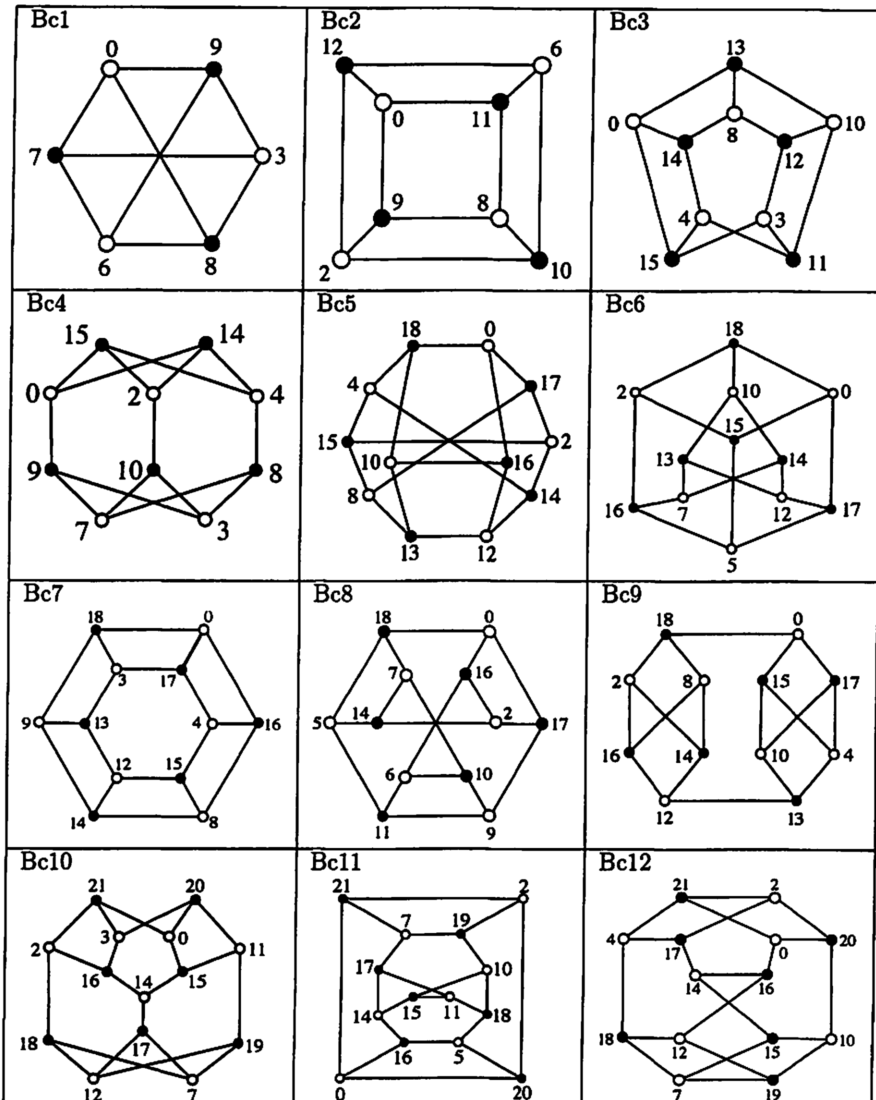
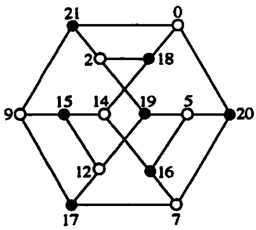
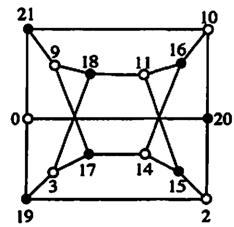


Table 1 (cont.): Connected bipartite cubic graphs: 6–14 vertices

Bc13



Bc14



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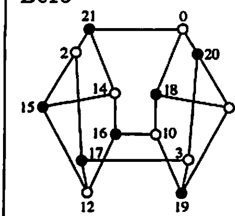
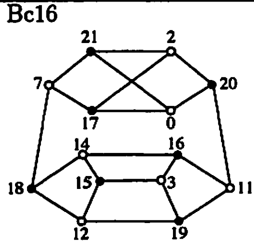
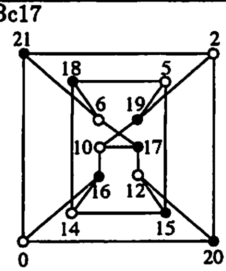


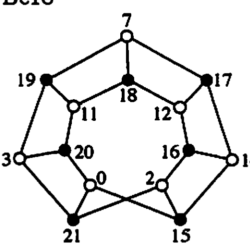
Bc16



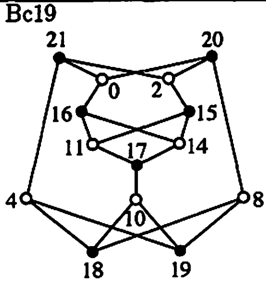
Bc17



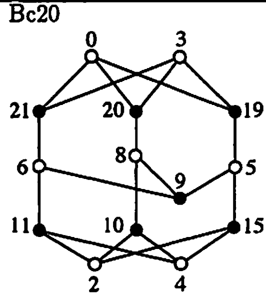
Bc18



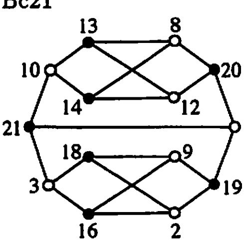
Bc19



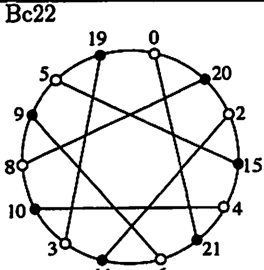
Bc20



Bc21



Bc22



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