On Modular Chromatic Indexes of Graphs

Futaba Fujie-Okamoto
Mathematics Department, University of Wisconsin-La Crosse
1725 State St. La Crosse, WI 54601 USA

Ryan Jones, Kyle Kolasinski, Ping Zhang
Department of Mathematics, Western Michigan University
Kalamazoo, MI 49008-5248 USA

Abstract

For a connected graph G of order 3 or more and an edge coloring $c : E(G) \to \mathbb{Z}_k$ ($k \geq 2$) where adjacent edges may be colored the same, the color sum $s(v)$ of a vertex v of G is the sum in $\mathbb{Z}_k$ of the colors of the edges incident with v. The edge coloring c is a modular k-edge coloring of G if $s(u) \neq s(v)$ in $\mathbb{Z}_k$ for all pairs u, v of adjacent vertices in G. The modular chromatic index $\chi'_m(G)$ of G is the minimum k for which G has a modular k-edge coloring. It is shown that $\chi(G) \leq \chi'_m(G) \leq \chi(G)+1$ for every connected graph G of order at least 3, where $\chi(G)$ is the chromatic number of G. Furthermore, it is shown that $\chi'_m(G) = \chi(G) + 1$ if and only if $\chi(G) \equiv 2 \pmod 4$ and every proper $\chi(G)$-coloring of G results in color classes of odd size.

1 Introduction

There have been numerous studies using a variety of methods for the purpose of uniquely distinguishing every two adjacent vertices of a graph. Many of these methods have involved graph colorings. The most studied colorings are proper vertex colorings and proper edge colorings. A *proper vertex coloring* of a graph G is an assignment of colors to the vertices of G such that adjacent vertices are assigned distinct colors and the minimum number of colors in a proper vertex coloring of G is the *chromatic number* $\chi(G)$ of G. A *proper edge coloring* of a graph G is an assignment of colors to the edges of G such that adjacent edges are assigned distinct colors and the minimum number of colors in a proper edge coloring of G is the *chromatic index* $\chi'(G)$ of G.

A coloring that provides a method of distinguishing every two adjacent vertices is said to be *neighbor-distinguishing.* Thus a proper vertex coloring of a graph is itself neighbor-distinguishing. A number of neighbor-distinguishing vertex colorings other than the standard proper colorings have been introduced (see [4, 5, 6, 7], for example). Furthermore, edge colorings (proper or nonproper) have also been introduced to distinguish every pair of adjacent vertices in a graph (see [1, 2, 9, 12] or [8, p. 385], for example).

Another neighbor-distinguishing vertex coloring was introduced in [11]. For a vertex v of a graph G, let $N(v)$ denote the neighborhood of v (the set of vertices adjacent to v). For a graph G without isolated vertices let $c : V(G) \to \mathbb{Z}_k$ $(k \geq 2)$ be a vertex coloring of G where adjacent vertices may be colored the same. The *color sum* of a vertex v of G is defined as the sum in $\mathbb{Z}_k$ of the colors of the vertices in $N(v)$. The coloring c is called a *modular k-coloring* of G if every pair of adjacent vertices of G have different color sums in $\mathbb{Z}_k$. The *modular chromatic number* of G is the minimum k for which G has a modular k-coloring.

A neighbor-distinguishing edge coloring that is closely related to the modular vertex colorings was introduced in [10]. For a graph G without isolated vertices, let $c : E(G) \to \mathbb{Z}_k$ $(k \geq 2)$ be an edge coloring of G where adjacent edges may be colored the same. The *color sum* $s(v)$ of a vertex v of G is defined as the sum in $\mathbb{Z}_k$ of the colors of the edges incident with v, that is, if E_v is the set of edges incident with v in G, then

$$s(v) = \sum_{e \in E_v} c(e).$$

An edge coloring c is a *modular k-edge coloring* of G if $s(u) \neq s(v)$ in $\mathbb{Z}_k$ for all pairs u, v of adjacent vertices of G. An edge coloring c is a *modular edge coloring* if c is a modular k-edge coloring for some integer $k \geq 2$. The *modular chromatic index* $\chi'_m(G)$ of G is the minimum k for which G has a modular k-edge coloring. Note that the modular chromatic index $\chi'_m(G)$ of a graph G exists only when G contains no components isomorphic to K_2. Hence, we only consider connected graphs of order 3 or more in this work. Among the results obtained in [10] are the following.

Theorem 1.1 *For each integer $n \geq 3$,*

$$\chi'_m(K_n) = \begin{cases} n+1 & \text{if } n \equiv 2 \pmod 4 \\ n & \text{otherwise.} \end{cases}$$

$$\chi'_m(C_n) = \begin{cases} 2 & \text{if } n \equiv 0 \pmod 4 \\ 3 & \text{otherwise.} \end{cases}$$

Theorem 1.2 *If G is a connected bipartite graph of order at least 3 with partite sets U and W, then*

$$\chi'_m(G) = \begin{cases} 2 & \text{if at least one of } |U| \text{ and } |W| \text{ is even} \\ 3 & \text{if both } |U| \text{ and } |W| \text{ are odd.} \end{cases}$$

Theorem 1.3 *For every connected graph G of order at least 3, $\chi'_m(G) \geq \chi(G)$. Furthermore, if $\chi(G) \equiv 2 \pmod 4$ and each color class in every proper $\chi(G)$-coloring of G consists of an odd number of vertices, then $\chi'_m(G) > \chi(G)$.*

The following questions were posted in [10].

Question 1.4 *For a connected graph G of order at least 3, is it true that $\chi'_m(G) \leq \chi(G) + 1$?*

Question 1.5 *Let G be a connected graph of order $n \geq 3$. If $\chi'_m(G) = \chi(G) + 1$, then is it always the case that n is even, $\chi(G) \equiv 2 \pmod 4$ and every proper $\chi(G)$-coloring of G results in color classes of odd size?*

By Theorems 1.1 and 1.2, it follows that Questions 1.4 and 1.5 have an affirmative answer when G is a complete graph, a cycle or a bipartite graph. In this paper, we show that each of these two questions has an affirmative answer for all connected graphs of order at least 3. In Section 2, we show that $\chi'_m(G) \leq \chi(G) + 1$ for all connected graphs G of order at least 3 and $\chi'_m(G) = \chi(G)$ if $\chi(G)$ is odd; while in Section 3, we characterize all connected graphs G of order at least 3 for which $\chi'_m(G) = \chi(G) + 1$. We refer to the books [3, 8] for graph theory notation and terminology not described in this paper.

2 Chromatic Number and Modular Chromatic Index

In this section, we present an affirmative answer to Question 1.4. By Theorem 1.2, if G is a connected bipartite graph of order at least 3, then $\chi'_m(G) \leq \chi(G) + 1$. Thus we need only consider connected graphs that are not bipartite. For an integer $k \geq 2$, a graph G is *modular k-edge colorable* if there is a modular k-edge coloring of G. It is clear that if G is a k-chromatic graph of order n, then a proper k-coloring of G can induce a proper k'-coloring of G for each integer k' with $k \leq k' \leq n$ by introducing a new color to a vertex of G.

Theorem 2.1 *Let G be a nontrivial connected graph that is not bipartite. For a positive integer k, if G is $(2k+1)$-colorable, then G is modular $(2k+1)$-edge colorable. Furthermore, for a given proper $(2k+1)$-vertex coloring $c' : V(G) \to \{1, 2, \ldots, 2k+1\}$, there is a modular $(2k+1)$-edge coloring $c : E(G) \to \mathbb{Z}_{2k+1}$ such that $s_c(v) = c'(v)$ for every $v \in V(G)$.*

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ $(n \geq 3)$ and let

$$c' : V(G) \to \{1, 2, \ldots, 2k+1\}$$

be a proper $(2k+1)$-vertex coloring of G. We define recursively a sequence of $n+1$ edge colorings $c_0, c_1, \ldots, c_n$, where $c_i : E(G) \to \mathbb{Z}_{2k+1}$ for $0 \leq i \leq n$ such that (i) $s_{c_i}(v_j) = c'(v_j)$ if $1 \leq j \leq i$ and (ii) $s_{c_i}(v_j) = 0$ if $i+1 \leq j \leq n$. This will imply that $c = c_n$ is a modular $(2k+1)$-edge coloring with $s_c(v) = c'(v)$ for every $v \in V(G)$.

First, we define the edge coloring $c_0 : E(G) \to \mathbb{Z}_{2k+1}$ by $c_0(e) = 0$ for every $e \in E(G)$. Thus $s_{c_0}(v) = 0$ for every $v \in V(G)$. Next, we define the edge coloring $c_1 : E(G) \to \mathbb{Z}_{2k+1}$ of G from c_0 such that $s_{c_1}(v_1) = c'(v_1)$ and $s_{c_1}(v) = 0$ for $v \in V(G) - \{v_1\}$ as follows. Suppose that $c'(v_1) = a$. Since $\gcd(2, 2k+1) = 1$, it follows that $2 \mid a$ in $\mathbb{Z}_{2k+1}$ and so $a = 2b$ for some $b \in \mathbb{Z}_{2k+1}$. Since G is not bipartite, G contains an odd cycle $C = (u_1, u_2, \ldots, u_p, u_{p+1} = u_1)$, where $p \geq 3$ is an odd integer. We consider two cases.

Case 1. *v_1 belongs to C.* Without loss of generality, assume that $v_1 = u_1$. The coloring $c_1 : E(G) \to \mathbb{Z}_{2k+1}$ is then defined by

$$c_1(e) = \begin{cases} c_0(e) & \text{if } e \notin E(C) \\ c_0(e) + b & \text{if } e = u_i u_{i+1},\ i \text{ is odd and } 1 \leq i \leq p \\ c_0(e) - b & \text{if } e = u_i u_{i+1},\ i \text{ is even and } 2 \leq i \leq p-1. \end{cases} \quad (1)$$

Observe that $s_{c_1}(v_1) = 2b = a = c'(v_1)$ and $s_{c_1}(v_i) = s_{c_0}(v_i) = 0$ for $2 \leq i \leq n$.

Case 2. *v_1 does not belong to C.* Since G is connected, there is a path P connecting v_1 and a vertex on C, say $P = (v_1 = w_1, w_2, \ldots, w_q = u_1)$ is a $v_1 - u_1$ path, where $q \geq 2$. We consider two subcases, according to whether q is even or q is odd.

Subcase 2.1. *q is even.* The coloring $c_1 : E(G) \to \mathbb{Z}_{2k+1}$ is defined by

$$c_1(e) = \begin{cases} c_0(e) & \text{if } e \notin E(C) \cup E(P) \\ c_0(e) + a & \text{if } e = w_i w_{i+1},\ i \text{ is odd and } 1 \leq i \leq q-1 \\ c_0(e) - a & \text{if } e = w_i w_{i+1},\ i \text{ is even and } 2 \leq i \leq q-2 \\ c_0(e) - b & \text{if } e = u_i u_{i+1},\ i \text{ is odd and } 1 \leq i \leq p \\ c_0(e) + b & \text{if } e = u_i u_{i+1},\ i \text{ is even and } 2 \leq i \leq p-1. \end{cases} \quad (2)$$

Then $s_{c_1}(v_1) = a = c'(v_1)$ and $s_{c_1}(v_i) = s_{c_0}(v_i) = 0$ for $2 \le i \le n$.

Subcase 2.2. *q is odd.* The coloring $c_1 : E(G) \to \mathbb{Z}_{2k+1}$ is defined by

$$c_1(e) = \begin{cases} c_0(e) & \text{if } e \notin E(C) \cup E(P) \\ c_0(e) + a & \text{if } e = w_i w_{i+1},\ i \text{ is odd and } 1 \le i \le q-1 \\ c_0(e) - a & \text{if } e = w_i w_{i+1},\ i \text{ is even and } 2 \le i \le q-2 \\ c_0(e) + b & \text{if } e = u_i u_{i+1},\ i \text{ is odd and } 1 \le i \le p \\ c_0(e) - b & \text{if } e = u_i u_{i+1},\ i \text{ is even and } 2 \le i \le p-1. \end{cases} \quad (3)$$

Then $s_{c_1}(v_1) = a = c'(v_1)$ and $s_{c_1}(v_i) = s_{c_0}(v_i) = 0$ for $2 \le i \le n$.

In each case, $s_{c_1}(v_1) = c'(v_1)$ and $s_{c_1}(v) = s_{c_0}(v) = 0$ for all $v \in V(G) - \{v_1\}$. (The coloring c_1 is neither a proper edge coloring nor a modular edge coloring of G.) In general, for an integer i with $1 \le i \le n-1$, suppose that the coloring $c_i : E(G) \to \mathbb{Z}_{2k+1}$ is defined such that $s_{c_i}(v_j) = c'(v_j)$ for $1 \le j \le i$ and $s_{c_i}(v_j) = 0$ for $i+1 \le j \le n$. Then the coloring $c_{i+1} : E(G) \to \mathbb{Z}_{2k+1}$ is defined from c_i in the same fashion as described in (1) – (3), namely by replacing c_0 and c_1 in (1) – (3) by c_i and c_{i+1}, respectively. An argument similar to the one used in the case dealing with c_0 and c_1 shows that $s_{c_{i+1}}(v_j) = s_{c_i}(v_j) = c'(v_j)$ for $1 \le j \le i$, $s_{c_{i+1}}(v_{i+1}) = c'(v_{i+1})$ and $s_{c_{i+1}}(v_j) = 0$ for $i+2 \le j \le n$. In particular, $c_n : E(G) \to \mathbb{Z}_{2k+1}$ has the property that $s_{c_n}(v_i) = s_{c_{n-1}}(v_i) = c'(v_i)$ for $1 \le i \le n-1$ and $s_{c_n}(v_n) = c'(v_n)$. Therefore, c_n is a modular $(2k+1)$-edge coloring of G and so G is modular $(2k+1)$-edge colorable. ∎

The following corollaries are consequences of Theorems 1.3 and 2.1.

Corollary 2.2 *If G is a connected graph of order at least 3, then*

$$\chi(G) \le \chi'_m(G) \le \chi(G) + 1.$$

Furthermore, if $\chi(G)$ is odd, then $\chi'_m(G) = \chi(G)$.

Proof. We have seen that $\chi'_m(G) \ge \chi(G)$ in Theorem 1.3. For the upper bound, let G be a connected graph of order $n \ge 3$. If G is bipartite, then the result follows by Theorem 1.2 and so assume that G is not bipartite. If $\chi = \chi(G)$ is even, then G is $(\chi+1)$-colorable and so G is modular $(\chi+1)$-edge colorable by Theorem 2.1. Thus $\chi'_m(G) \le \chi(G) + 1$. If χ is odd, then G is modular χ-edge colorable again by Theorem 2.1. Therefore, $\chi'_m(G) \le \chi(G)$ and so $\chi'_m(G) = \chi(G)$. ∎

By Corollary 2.2, if G is a connected graph of order at least 3 such that $\chi'_m(G) = \chi(G) + 1$, then $\chi(G)$ is even and so either $\chi(G) \equiv 0 \pmod 4$ or $\chi(G) \equiv 2 \pmod 4$. By Theorem 1.3 and Corollary 2.2, we have the following result for connected graphs G with $\chi(G) \equiv 2 \pmod 4$.

Corollary 2.3 *Let G be a connected graph of order at least 3 such that $\chi(G) \equiv 2 \pmod 4$. If each color class in every proper $\chi(G)$-coloring of G consists of an odd number of vertices, then $\chi'_m(G) = \chi(G) + 1$.*

3 A Characterization of Type 1 Graphs

As a consequence of Corollary 2.2, the modular chromatic index $\chi'_m(G)$ of a graph G is either $\chi(G)$ or $\chi(G)+1$. Graphs G for which $\chi'_m(G) = \chi(G)$ are called *type* 0 *graphs* and graphs G for which $\chi'_m(G) = \chi(G) + 1$ are called *type* 1 *graphs*. So every connected graph of order at least 3 is either type 0 or type 1. This gives rise to a natural question: Which graphs are type 0 and which graphs are type 1? By Theorems 1.1 and 1.2 and Corollary 2.2, if $G = K_n$ or $G = C_n$, then G is type 1 if and only if $n \equiv 2 \pmod 4$, while if G is bipartite, then G is type 1 if and only if each partite set of G has an odd number of vertices. Furthermore, if G is type 1, then $\chi(G)$ must be even. In this section, we characterize all connected type 1 graphs. In order to do this, we first determine all type 1 complete multipartite graphs.

3.1 Complete Multipartite Graphs

For positive integers $n_1, n_2, \ldots, n_\ell$ ($\ell \geq 2$), let $G = K_{n_1,n_2,\ldots,n_\ell}$ be a complete ℓ-partite graph of order $n_1 + n_2 + \cdots + n_\ell$ whose partite sets are $V_1, V_2, \ldots, V_\ell$ where $|V_i| = n_i$ for $1 \leq i \leq \ell$. If $\ell = 2$ or ℓ is odd, then $\chi'_m(G)$ is determined by Theorem 1.2 and Corollary 2.2. Thus, we may assume that $\ell \geq 4$ is an even integer. For even integers $\ell \geq 4$, we first determine a class of complete ℓ-partite graphs G for which $\chi'_m(G) = \chi(G)$.

Theorem 3.1 *Let $G = K_{n_1,n_2,\ldots,n_\ell}$ be a complete ℓ-partite graph where $\ell \geq 4$ is even. If there exists a set $S \subseteq \{n_1, n_2, \ldots, n_\ell\}$ such that $|S| = \ell/2$ and the sum of the integers in S is even, then $\chi'_m(G) = \chi(G)$.*

Proof. Let $V_1, V_2, \ldots, V_\ell$ be the partite sets of G where $|V_i| = n_i$ for $1 \leq i \leq \ell$. We may assume that $S = \{n_1, n_2, \ldots, n_{\ell/2}\}$ and $n_1 + n_2 + \cdots + n_{\ell/2}$ is even. Furthermore, let $V = V_1 \cup V_2 \cup \cdots \cup V_{\ell/2}$. Let $u \in V_\ell$ and define an edge coloring $c_0 : E(G) \to \mathbb{Z}_\ell$ by

$$c_0(e) = \begin{cases} 1 & \text{if } e = uv \text{ where } v \in V \\ 0 & \text{otherwise.} \end{cases}$$

It follows that $s_{c_0}(v) = 1$ if $v \in V$, $s_{c_0}(v) = 0$ if $v \in V(G) - (V \cup \{u\})$ and $s_{c_0}(u) \equiv 0 \pmod 2$.

Let $v_1, v_2, \ldots, v_n = u$ be an ordering of the vertices of G. We define a sequence $c_1, c_2, \ldots, c_n$ of edge colorings of G recursively such that for each

i with $1 \le i \le n$ the edge coloring $c_i : E(G) \to \mathbb{Z}_\ell$ induces a vertex coloring $s_{c_i} : V(G) \to \mathbb{Z}_\ell$ for which

$$s_{c_i}(v_i) = \begin{cases} 2j-1 & \text{if } v_i \in V_j \text{ and } V_j \subseteq V \\ 2j-2 & \text{if } v_i \in V_j \text{ and } V_j \subseteq V(G)-V \end{cases} \tag{4}$$

$$s_{c_i}(v) = s_{c_{i-1}}(v) \quad \text{if } v \in V(G) - \{v_i\}. \tag{5}$$

We begin with the coloring c_1. Suppose that $v_1 \in V_j$ for some j with $1 \le j \le \ell$. Since $\ell \ge 4$, the vertex v_1 lies on a triangle C in G, say $C = (v_1, u_1, w_1, v_1)$. Define $c_1 : E(G) \to \mathbb{Z}_\ell$ by

$$c_1(e) = \begin{cases} c_0(e) & \text{if } e \notin E(C) \\ c_0(e) + (j-1) & \text{if } e \in \{v_1u_1, v_1w_1\} \\ c_0(e) - (j-1) & \text{if } e = u_1w_1. \end{cases}$$

If $v_1 \in V$, then $s_{c_0}(v_1) = 1$; while if $v_1 \in V(G) - V$, then $s_{c_0}(v_1) = 0$. This implies that

$$s_{c_1}(v_1) = \begin{cases} 1 + 2(j-1) = 2j-1 & \text{if } v_1 \in V_j \text{ and } V_j \subseteq V \\ 0 + 2(j-1) = 2j-2 & \text{if } v_1 \in V_j \text{ and } V_j \subseteq V(G)-V \end{cases}$$

$$s_{c_1}(v) = s_{c_0}(v) \quad \text{if } v \in V(G) - \{v_1\}.$$

Thus s_{c_1} satisfies (4) and (5).

For an integer i with $1 \le i \le n-1$, suppose that the edge colorings $c_1, c_2, \ldots, c_i$ have been defined, all of which satisfy (4) and (5). We now define the coloring c_{i+1} from c_i in the same fashion as we defined c_1 from c_0. More precisely, assume that $v_{i+1} \in V_j$ for some j with $1 \le j \le \ell$ and that v_{i+1} lies on a triangle $C = (v_{i+1}, u_{i+1}, w_{i+1}, v_{i+1})$. Then define $c_{i+1} : E(G) \to \mathbb{Z}_\ell$ by

$$c_{i+1}(e) = \begin{cases} c_i(e) & \text{if } e \notin E(C) \\ c_i(e) + (j-1) & \text{if } e \in \{v_{i+1}u_{i+1}, v_{i+1}w_{i+1}\} \\ c_i(e) - (j-1) & \text{if } e = u_{i+1}w_{i+1}. \end{cases}$$

We now consider the induced vertex coloring $s_{c_i} : V(G) \to \mathbb{Z}_\ell$. Since

$$s_{c_i}(v_{i+1}) = s_{c_{i-1}}(v_{i+1}) = \cdots = s_{c_0}(v_{i+1}),$$

it follows that $s_{c_i}(v_{i+1}) = 1$ if $v_{i+1} \in V$; while $s_{c_i}(v_{i+1}) = 0$ if $v_{i+1} \in V(G) - V$. Therefore,

$$s_{c_{i+1}}(v_{i+1}) = \begin{cases} 1 + 2(j-1) = 2j-1 & \text{if } v_{i+1} \in V_j \text{ and } V_j \subseteq V \\ 0 + 2(j-1) = 2j-2 & \text{if } v_{i+1} \in V_j \text{ and } V_j \subseteq V(G)-V \end{cases}$$

$$s_{c_{i+1}}(v) = s_{c_0}(v) \quad \text{if } v \in V(G) - \{v_{i+1}\}.$$

Thus $s_{c_{i+1}}$ satisfies (4) and (5). Continuing in this manner, we obtain the edge coloring $c_n : E(G) \to \mathbb{Z}_\ell$ that induces a vertex coloring $s_{c_n} : V(G) \to \mathbb{Z}_\ell$ such that

$$s_{c_n}(v_i) = \begin{cases} 2j-1 & \text{if } v_{i+1} \in V_j \text{ and } V_j \subseteq V \\ 2j-2 & \text{if } v_{i+1} \in V_j \text{ and } V_j \subseteq V(G) - V. \end{cases}$$

This implies that s_{c_n} is a proper vertex ℓ-coloring of G using the ℓ colors in $\mathbb{Z}_\ell$. Thus c_n is a modular ℓ-edge coloring of G. Therefore, $\chi'_m(G) \leq \ell$ and so $\chi'_m(G) = \ell = \chi(G)$. ■

We are now prepared to determine all complete multipartite graphs that are type 1.

Theorem 3.2 *Let $G = K_{n_1,n_2,\ldots,n_\ell}$ be a complete ℓ-partite graph where $\ell \geq 2$. Then $\chi'_m(G) = \chi(G) + 1$ if and only if $\ell \equiv 2 \pmod 4$ and each n_i is odd for $1 \leq i \leq \ell$.*

Proof. If $\ell \equiv 2 \pmod 4$ and each n_i is odd for $1 \leq i \leq \ell$, then $\chi'_m(G) = \chi(G) + 1$ by Corollary 2.3. It remains to verify the converse. If ℓ is odd, then $\chi'_m(G) = \chi(G)$ by Corollary 2.2. Thus, we may assume that ℓ is even and so either $\ell \equiv 0 \pmod 4$ or $\ell \equiv 2 \pmod 4$. Consider the set $N = \{n_1, n_2, \ldots, n_\ell\}$. If N contains an even integer or $\ell \equiv 0 \pmod 4$, then there is a subset $S \subseteq N$ with $|S| = \ell/2$ such that the sum of the integers in S is even. It then follows by Theorem 3.1 that $\chi'_m(G) = \chi(G)$. ■

3.2 Type 1 Graphs

If G is a connected graph with $\chi(G) = \ell$, then the vertex set of G can be partitioned into ℓ independent sets $V_1, V_2, \ldots, V_\ell$, where say $|V_i| = n_i$ for $1 \leq i \leq \ell$. Thus, G is a subgraph of $K_{n_1,n_2,\ldots,n_\ell}$.

Lemma 3.3 *For an integer $\ell \geq 4$, let $K_{n_1,n_2,\ldots,n_\ell}$ be a complete ℓ-partite graph whose partite sets are $V_1, V_2, \ldots, V_\ell$ with $|V_i| = n_i$ for $1 \leq i \leq \ell$. If G is a connected graph with $\chi(G) = \ell \geq 4$ such that*

(i) *$G \subseteq K_{n_1,n_2,\ldots,n_\ell}$ and*

(ii) *for each pair u, v of vertices of G where $u \in V_i$, $v \in V_j$ and $i \neq j$, we have $uv \in E(G)$ whenever G contains a $u - v$ path of odd length,*

then $G = K_{n_1,n_2,\ldots,n_\ell}$.

Proof. For each vertex v of G, where say $v \in V_i$ for some i with $1 \leq i \leq \ell$, we show that v is adjacent to every vertex in $V(G) - V_i$. Without loss of generality, suppose that $v \in V_1$.

We first claim that $d_G(u,v) \leq 2$ for every $u \in V(G) - V_1$. If this is not the case, then suppose that $u \in V_2$ and $d_G(v,u) = d \geq 3$. Let $(u = v_0, v_1, v_2, \ldots, v_d = v)$ be a $u - v$ geodesic in G. If $v_3 \notin V_2$, then there is a $u - v_3$ path of length 3 and so $uv_3 \in E(G)$, creating a $u - v$ path of length $d-2$. Hence, $d \geq 4$ and $u, v_3 \in V_2$ while $v_1, v_2 \notin V_2$. If $d \geq 5$, then it can be similarly shown that $v_5 \in V_2$ since $uv_5 \notin E(G)$. However then, $v_2v_5 \in E(G)$, which cannot occur. Therefore, $d = 4$ and $v_1, v \in V_1$. Assume further that $v_2 \in V_3$, say. Now consider the partite set V_4. If $w \in V_4$ and is adjacent to v_2, then w must be adjacent to both u and v since there is a path of length 3 from w to each of u and v. However, this creates a $u - v$ path of length 2, which is impossible. Therefore, we may assume that there is a vertex $w' \in V_3 - \{v_2\}$ such that $ww' \in E(G)$ or otherwise $V_3 \cup V_4$ is independent, which contradicts the fact that $\chi(G) = \ell$. Now, since G is connected, let P be a $u - w$ path of length t. If t is odd, then $uw \in E(G)$ and so there is a $v - w$ path of length 5. On the other hand, if t is even, then there exists a $u - w'$ path of length either $t-1$ or $t+1$ and so $uw' \in E(G)$, which in turn implies that there is a $v - w'$ path of length 5. However then, either w or w' is adjacent to both u and v, contradicting the assumption that $d_G(u,v) = 4$. Thus $d_G(u,v) \leq 2$, as claimed.

Let

$$X = \{x \in V(G) - V_1 : d_G(v,x) = 1\} = N(v)$$
$$Y = \{y \in V(G) - V_1 : d_G(v,y) = 2\}.$$

We show that $Y = \emptyset$. Assume, to the contrary, that $Y \neq \emptyset$. If Y is not independent, say $y, y' \in Y$ and $yy' \in E(G)$, then there is either a $v - y$ path or a $v - y'$ path of length 3, which cannot occur since $vy, vy' \notin E(G)$. Therefore, Y is an independent set. Then X cannot be independent since $\{V_1, X, Y\}$ is a partition of $V(G)$ and $\chi(G) \geq 4$. Let $x, x' \in X$ and $xx' \in E(G)$. Without loss of generality, we may assume that $x \in V_2$ and $x' \in V_3$. Let $y \in Y$. If y is adjacent to either x or x', then there is a $v - y$ path of length 3, which is again a contradiction. Therefore, $yx, yx' \notin E(G)$. Then there exists a vertex $x'' \in X - \{x, x'\}$ so that (v, x'', y) is a $v - y$ geodesic. However, this implies that $xy \in E(G)$ if $y \notin V_2$ and $x'y \in E(G)$ otherwise, neither of which can occur. Thus $Y = \emptyset$ and v is adjacent to every vertex in $V(G) - V_1$. This completes the proof. ∎

Recall that for an integer $k \geq 2$, a graph G is modular k-edge colorable if G has a modular k-edge coloring.

Lemma 3.4 *Let G be a connected graph of order at least 3 containing two nonadjacent vertices u and v that are connected by a path of odd length. Let $k \geq 2$ be an integer. Then $G + uv$ is modular k-edge colorable if and only if there is a modular k-edge coloring of G with respect to which $s(u) \neq s(v)$.*

Proof. If there is a modular k-edge coloring $c : E(G) \to \mathbb{Z}_k$ such that $s_c(u) \neq s_c(v)$, then the coloring $c' : E(G+uv) \to \mathbb{Z}_k$ defined by $c'(uv) = 0$ and $c'(e) = c(e)$ for $e \in E(G)$ is a modular k-edge coloring of $G+uv$. Thus $G+uv$ is modular k-edge colorable.

For the converse, assume that $G+uv$ is modular k-edge colorable and let c be a modular k-edge coloring of $G+uv$. Suppose that P is a $u-v$ path of odd length in G, say $P = (u = v_1, v_2, \ldots, v_p = v)$ where $p \geq 4$ is even. Then the k-edge coloring c' of G defined by

$$c'(e) = \begin{cases} c(e) & \text{if } e \notin E(P) \\ c(e) + c(uv) & \text{if } e = v_i v_{i+1},\ 1 \leq i \leq p-1 \text{ and } i \text{ is odd} \\ c(e) - c(uv) & \text{if } e = v_i v_{i+1},\ 2 \leq i \leq p-2 \text{ and } i \text{ is even} \end{cases}$$

is a modular k-edge coloring of G with the property that $s_{c'}(u) \neq s_{c'}(v)$. ■

The following is an immediate consequence of Lemma 3.4.

Corollary 3.5 *Let G be a connected graph of order at least 3 containing two nonadjacent vertices u and v that are connected by a path of odd length. If $G+uv$ is modular k-edge colorable for some integer $k \geq 2$, then G is also modular k-edge colorable.*

By Corollary 3.5, if G is a connected graph of order at least 3 containing two nonadjacent vertices u and v that are connected by a path of odd length, then the fact that $G+uv$ is modular k-edge colorable implies that G is modular k-edge colorable. This motivates the next definition. Let G be a connected graph with $\chi(G) = \ell \geq 4$. Suppose that the vertex set of G can be partitioned into ℓ independent sets $V_1, V_2, \ldots, V_\ell$ with $|V_i| = n_i$ for $1 \leq i \leq \ell$. Then G is a subgraph of a complete ℓ-partite graph $K = K_{n_1, n_2, \ldots, n_\ell}$. Define the *odd path closure of G with respect to K*, denoted by $C(G;K)$, to be the graph obtained from G by recursively joining pairs of nonadjacent vertices that belong to different partite sets in K and are connected by a path of odd length in G. Thus, we have the following result as a consequence of Lemma 3.3 and Corollary 3.5.

Corollary 3.6 *Let G be a connected graph of order at least 3 with $\chi(G) = \ell \geq 4$. Then $G \subseteq K$ for some complete ℓ-partite graph K and $\chi'_m(G) \leq \chi'_m(C(G;K)) = \chi'_m(K)$.*

We are now prepared to present a characterization of type 1 graphs, which provides an affirmative answer to Question 1.5.

Theorem 3.7 *Let G be a connected graph of order at least 3. Then $\chi'_m(G) = \chi(G) + 1$ if and only if $\chi(G) \equiv 2 \pmod 4$ and every proper $\chi(G)$-coloring of G results in color classes of odd size.*

Proof. By Corollary 2.3, if $\chi(G) \equiv 2 \pmod 4$ and every proper $\chi(G)$-coloring of G results in color classes of odd size, then $\chi'_m(G) = \chi(G) + 1$. Thus, it remains to verify the converse. Let $\chi(G) = \ell$ and suppose that $\chi'_m(G) = \ell + 1$. Consider a proper ℓ-coloring of G and let $V_1, V_2, \ldots, V_\ell$ be the resulting color classes, where say $|V_i| = n_i$ for $1 \le i \le \ell$. Then G is a subgraph of $K_{n_1,n_2,\ldots,n_\ell}$ and so

$$\ell + 1 = \chi'_m(G) \le \chi'_m(K_{n_1,n_2,\ldots,n_\ell}) \le \ell + 1$$

by Corollaries 2.2 and 3.6. The result now follows by Theorem 3.2. ■

References

[1] L. Addario-Berry, R. E. L. Aldred, K. Dalal and B. A. Reed, Vertex colouring edge partitions. *J. Combin. Theory Ser. B* **94** (2005) 237-244.

[2] P. N. Balister, E. Györi, J.Lehel and R. H. Schelp, Adjacent vertex distinguishing edge-colorings. *SIAM J. Discrete Math.* **21** (2007) 237-250.

[3] G. Chartrand, L. Lesniak and P. Zhang, *Graphs & Digraphs, Fifth Edition.* Chapman & Hall/CRC, Boca Raton, FL (2011).

[4] G. Chartrand, F. Okamoto, C. W. Rasmussen and P. Zhang, The set chromatic number of a graph. *Discuss. Math. Graph Theory.* **29** (2009) 545-561.

[5] G. Chartrand, F. Okamoto, E. Salehi and P. Zhang, The multiset chromatic number of a graph. *Math. Bohem.* **134** (2009) 191-209.

[6] G. Chartrand, F. Okamoto and P. Zhang, The metric chromatic number of a graph. *Australas. J. Combin.* **44** (2009) 273-286.

[7] G. Chartrand, F. Okamoto and P. Zhang, The sigma chromatic number of a graph. *Graphs and Combin.* **26** (2010) 755-773.

[8] G. Chartrand and P. Zhang, *Chromatic Graph Theory.* Chapman & Hall/CRC Press, Boca Raton (2009).

[9] E. Györi, M. Horňàk, C. Palmer and M. Woźniak, General neighbor-distinguishing index of a graph. *Discrete Math.* **308** (2008) 827-831.

[10] R. Jones, K. Kolasinski, F. Okamoto and P. Zhang, Modular neighbor-distinguishing edge colorings of graphs. *J. Combin. Math. Combin. Comput.* **76** (2011) 159-175.